





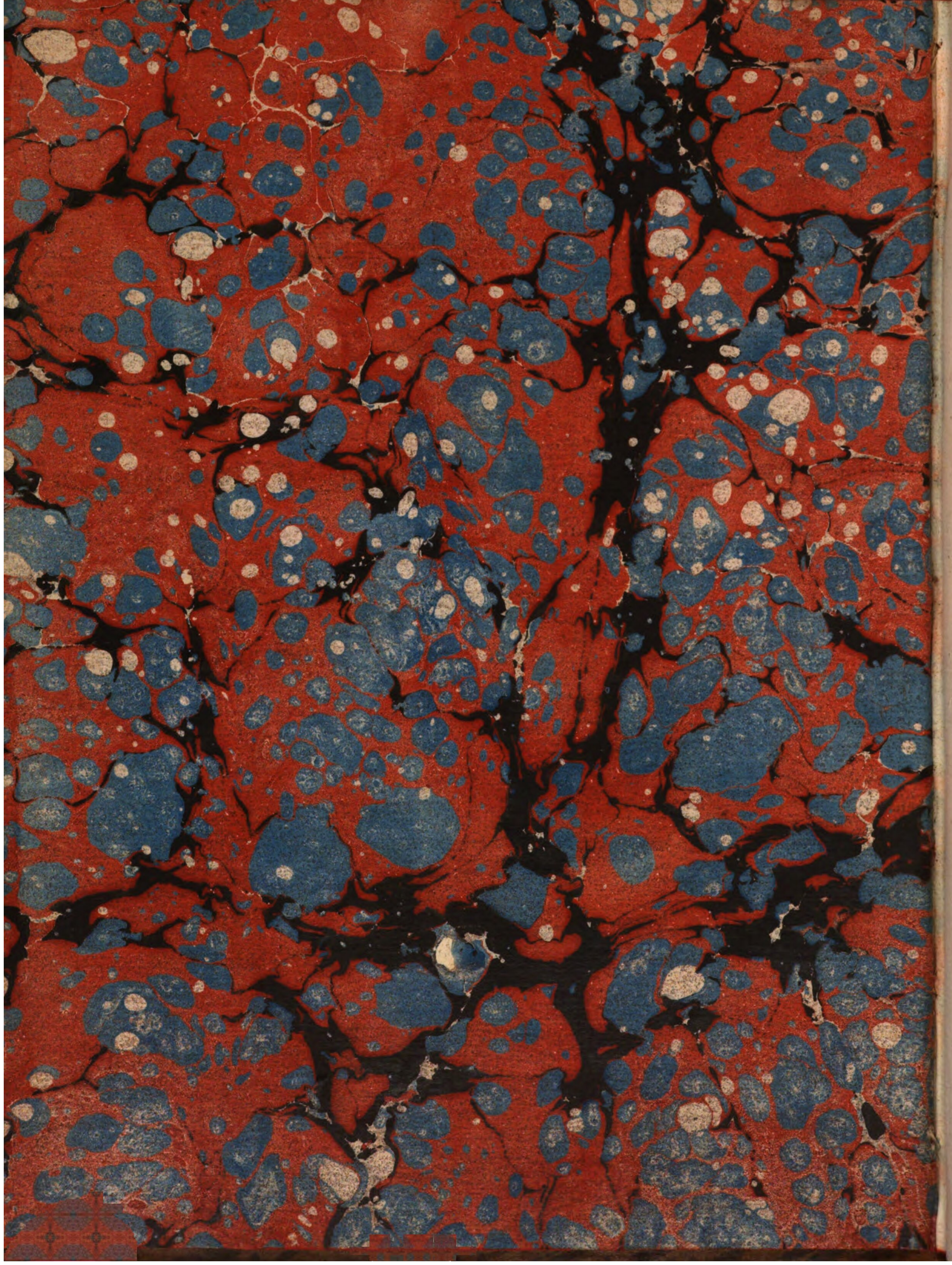
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4^o Math D. 118

Mathefis. Geometria specialis. 236.

Sp. 5343²

R

MAGNITUDINUM EXPONENTIALIUM

LOGARITHMORUM

ET

TRIGONOMETRIAE SUBLIMIS THEORIA

NOVA METHODO PERTRACTATA.

AUCTORE PETRO FERRONIO

R. C. PETRI LEOPOLDI

AUSTRIACI

MAGNI ETRURIAE DUCIS

MATHEMATICO

ET IN REGIIS PISANO AC FLORENTINO LYCEIS
PUBLICO MATHESEOS PROFESSORE.

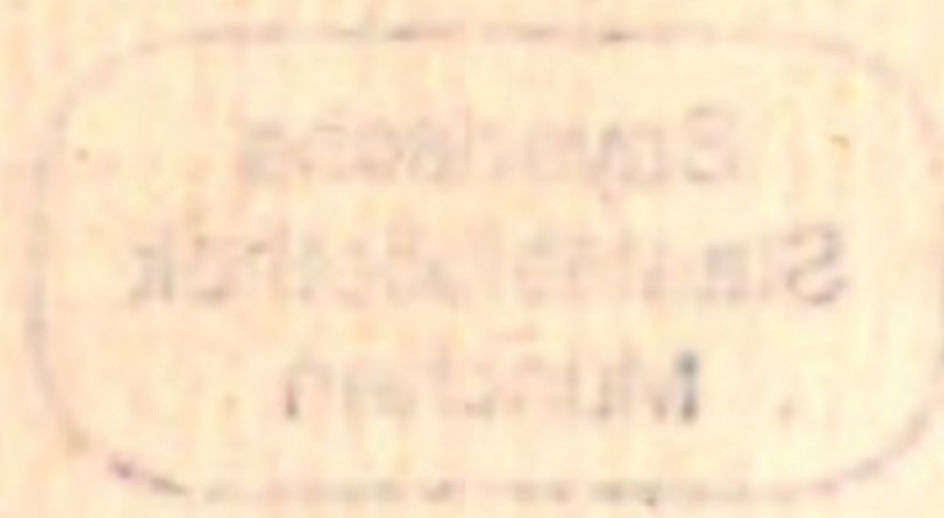


FLORENTIAE

EX TYPOGRAPHIA ALLEGRIANIANA

ANNO MDCCLXXXII.

REI BIBLIOGRAPHICAE PRAESIDIBUS PERMITTENTIBUS.



Vix exemplum adduci potest Reipublicae infelicitèr
administratae ad clavum sedentibus viris eruditis.

Bac. Verulamius in Lib. I. De augmentis Scientiarum.

PETRO LEOPOLDO

ARCHIDUCI AUSTRIAE

REGIO HUNGARIAE ET BOHEMIAE PRINCIPI

MAGNO ETRURIAE DUCI

Etc.

Etc.

Etc.

DEVOTUS NOMINI MAJESTATIQUE EJUS

PETRUS FERRONIUS

QUAE majoris futura momen-
ti hoc in Specimine conti-
nentur non paucis abhinc annis
mea inter adversaria jacuisse sa-
tis

tis pervulgatum dum beato Palladis otio latissime fruens Analyysi excolendae vacabam. Factum deinde est, ut vix quartum aetatis lustrum fuerim emensus cum REGIAE CELSITUDINIS TUAE Mathematici munere decoratum alio saepe distraxerint rei aquariae consilia, publicae oeconomes studia, Et ad moderandos Fluminum cursus, regendosque Etruriae TIBI subiectae fines diuturnae ac difficilimae peragrationes. Nec sane nuperrimo ausu analyticum hunc Commentarium edere statuissem nisi veteres illas schedas forte volutanti mihi occurrisset animadversio nondum incomparabilis beneficentiae
erga

v

erga me **TUAE** dono perquam
maximo, ne dicam miraculum;
palam vota vovisse. Utinam prae-
stantioribus in lucem editis mo-
numentum ære perennius Opti-
mo Principi dicatum superstruere
aliquando mihi fas esset! Verum
peractis tot tantisque Transalpi-
narum praesertim gentium labo-
ribus ad Geometriae Et Calcu-
li amplianda pomoeria periculosae
nunc plenum opus est aleae saga-
cissimis earum inventis aliquid ad-
dere. Jamdiu enim post Italorum
conatus doctrina excellentium, qui
Saeculo **XVII.** floruerunt, Geo-
metria ad summum adeo fastigium
evehcta, ut nostra aetate geome-

træ

trae titulo condecorari jure nequeat
 qui illorum tantummodo lucubra-
 tiones, tunc temporis ingeniosissi-
 mas ac etiam nunc cedro dignas,
 in succum Et sanguinem verterit.
 Faxint propitia Etruriae Numi-
 na, ut in eadem regione, quae ma-
 le feriatorum hominum contume-
 liis invulneratos ac nunquam mo-
 rituros Galilaeos, Torricellios, Vi-
 vianios aluit fovitque, sublimioris
 quoque Matheleos summus apex
 aevo hoc ipso vertente elucescat!
 Nihil equidem sub tanto Austria-
 co Principe vetat quomodocum-
 que admiranda in nobiliorum di-
 sciplinarum cultu sibi polliceri sta-
 tim ac ipse tam grandia molitus
 pro

pro omnigena *Physices Et Historiae naturalis eruditione paranda*,
cæterisque ornandis cognatis *Mathematicum Scientiis jam singulari*
Palatina Collectione Etruscae suae
gentis, exterorumque simul oculos
perculit, Et non inani spe litera-
rium orbem implevit forte futu-
rum, ut LEOPOLDUS alter
faustiori omine restituat Leopoldinam illam experimentalem Aca-
demiam, Mediceorum Siderum fa-
ma celebriorem. Interea, dum le-
viculam hanc Exercitationem non
gloriae aucupandae, sed quasi ae-
stuantis grati animi sensus pate-
faciendi causa Augusto TUO
Nomini inscriptam publici juris
feri

*fieri sinis, nihil aliud mihi supe-
rest, aeternumque manebit optan-
dum praeterquamquod in Scien-
tiarum decus, in Regum exem-
plum, Et in publica Etruriae com-
moda diutissime Imperium teneas,
universaeque ita consulas felici-
tati.*

Dabam Florentiae in Aedibus Patrii Athenaei
IV. Nonas Augusti An. Vul. Ær. MDCCLXXXI.

INDEX CAPITUM

AD COMMENTARII ARGUMENTUM
SINGULARITER EXHIBENDUM.

C A P U T I.

Animadversiones in Formulam Binomii Newtoniani.

C A P U T II.

Evolutio Serierum ad quantitates Exponentiales pertinentium.

C A P U T III.

Evolutio Serierum, quae numerorum Logarithmos exprimunt.

C A P U T IV.

Theoremata communia utrisque quantitatibus Exponentialibus, & Logarithmicis.

C A P U T V.

Evolutio Serierum Trigonometricarum.

C A P U T VI.

Quaedam de Seriebus Infinitis Quadraturam Hyperbolae, & Circuli complectentibus, aliisque argumenti occasione memoratis.

C A P U T VII.

Meditationes in Unitatis Radices Imaginarias, ubi de nonnullarum Aequationum resolutione, ac de Theorematis quoque Cotesiano, & Moivraeano.

C A P U T VIII.

Applicatio Calculi Infinitesimalis juxta vulgarem Potestatum methodum ad quantitates Logarithmicas, Exponentiales, & Trigonometricas.

C A P U T IX.

Illustratio celebris controversiae circa quantitatum negativarum Logarithmos.

C A P U T X.

Calculi Integralis paradoxon enodatum.



PROLEGOMENA HISTORICO - MATHEMATICA

I. **N** Eminem latere arbitror prima veluti lineamenta quantitatum Exponentialium duxisse acutissimum Leibnitium, & magni nominis geometram Joannem Bernoullium seniore. Usque ab anno elapsi saeculi nonagesimo quarto mentio facta invenitur aequationum Exponentialium $a^x = y$, $x^x = y$ in Litteris mutuo datis, quas servat *Commercii Philosophici & Mathematici* Volumen I. Laufannae ac Genevae editum anno MDCCXLV. Imo abunde constat multocius innotuisse Leibnitio & Bernoullio quantitates, quas primus Exponentiales, atque alter Percurrentes vocavit, cum in Epistolis ipsis & de solutione jam reperta quarundam aequationum Exponentialium, & de aequatione Exponentiali Curvam Logarithmicam *localiter* exhibente, & de omnigeno Percurrentium calculo sermo habeatur. Reapse Leibnitius in peculiari Schediasmate *de vera Circuli proportionem*, quod priori Volumine continetur *Diarii Lipsiensis* ad annum MDCLXXXII., aequatio-

nes gradus indefiniti analytico-transcendentes, veluti foret $x^x + x = 30$, designat, additque a nemine haecenus fuisse consideratas.

II. Non itaque repetenda Exponentialium consideratio, prouti celeberrimo Mathematicum Historiographo placuisse videtur, ab anno vulgaris aerae nonagesimo septimo supra millesimum sexcentissimum, quo nimirum publicam prodiit in lucem Opusculum Bernoullianum sub titulo *Principiorum Calculi Percurrentium in Commentariis Eruditorum Lipsiensibus* (*). Mirum autem semper erit semperque penes analyseos cultores incomparabilem Newtonum nullibi Exponentialia memorasse, quamvis ea non maximo distent intervallo a quantitatum Potestatibus constanti quolibet exponente praeditis, quarum theoricen mentis acie ferme divina summus ille Geometra tanta inventorum copia amplificavit. Perpetuum in Newtonianis scriptis silentium de quantitibus exponente variabili seu universos possibiles numeros percurrente affectis eo magis singulare videbitur dum hac in Exercitatione patebit ab ipsis formulis Potestatum cum exponente constanti hauriri illas quoque cum exponente variabili; ita ut ex unico fonte sine Infiniti mysteriorum subsidio fluere possint utrique quantitatum ad Potestates exaltandarum modi, quod adhuc erat in votis, & plurimi

(*) Unicum in illa brevi Diatriba correxit naevum Craigius respectu Tetragonismi Curvae Exponentialis $a^z = y^y$, quod liquet ex *Transactionibus* anni MDCXCVIII.

mi faciebat anno primo vertentis saeculi Joannes Bernoullius in brevi Syntagmate *de multisectione Anguli vel Arcus* Lipsiensibus *Commentariis* ejusdem anni comprehenso.

III. Et fanè cogitanti mihi, ac memoria repententi quantatum ad Potestates exaltationem saepe-numero usuvenit animadvertere, dum legebam potissimum, adolescens adhuc, Thomae Simpsonii *Fluxionum doctrinam* ubi de Exponentialibus agit (*), qua de causa non sit communis lex & origo evolutioni Potestatum a^m , z^m & a^x , z^x , seu quaenam vis aut difficultas obstat quominus univoca expressio inveniri posset a Neutonianis iisdem formulis eruta ad detegendum Exponentialium valorem. Tam arcto namque, amicoque foedere inter se convenire sentiebam Potestates a^m , z^m , & a^x , z^x , ut nedum statim fuerim auspicatus facillime enucleandam communem illarum derivationem, verum etiam mihi subito occurrerit summa confectariorum foecunditas ad intimam Exponentialium naturam perscrutandam, nodosque non paucos in Logometricis rebus solvendo aptissima. Nec mea me fefellit opinio: vix namque praeconcepti argumenti tractationi manum admove-ram, quando res voto cessit. Ipsamet enim Binomii formulam, quam Neutonus uberrimè aptavit Potestatibus universis exponente quocumque constante, integro, seu fracto, positivo, vel negativo affectis, manu-ducere pariter observavi ad obtinendas expressiones Po-
testa-

(*) In Sectione, ni fallor, VII. Londinensis Editionis anni MDCCL.

testatum illarum transcendentium, quae nuncupantur Exponentiales. Quomodocumque tamen hoc sit plusquam saeculari jam periodo absoluta, ex quo Neutonus percelebre Vallisii systema *Interpolationum* meditatus anno MDCLXIV. juxta *Commercii Epistolici* editi anno MDCCXII. de *analyfi promota* fidem in Binomii utillimam incidit formulam, nemo quod sciam dictae formulae sociavit alteram difficilioris indaginis Exponentialem, ideoque non algebraicam, sed Neutoniano more geometricè irrationalem.

IV. Longius, quam par esset, abiret diligentissima narratio tentaminum hætenus institutorum pro Exponentialium valore universaliter assequendo. Verum, ne inutilem penitus operam navasse videar, compendiarie oratione describere est animus quibus ortae principiis Series Exponentiales, qua methodo non satis directæ præcipui Geometrae illas deprompserint, quantaque subtilitate simpliciorē originem quasi renuentes argumentum Exponentialium a vulgatis Algebrae regulis necessario voluerint sejunctum. Hinc Exponentialium versatio in utroque Calculo finitorum & infinitorum magis fuit involuta; fundamenta algorithmi Potestatum algebraicarum nullo modo sufficere visa sunt explicationi Potestatum transcendentium enucleandae; & summis analyseos principiis absque necessitate multiplicatis perpetuam servarunt Geometrae Exponentialium separationem ab Elementari Calculo Potestatum, ac fortasse latentem aliquam hujusce divisionis causâ in Exponentialium metaphysicè obscuritatem.

V. Universae Germaniae decus Leibnitius methodum serierum, quas vocant assumptitias, arbitrarias, vel supposititias, Calculo nempe *Indeterminatorum* Cartesii subnixas, primus adhibuit, & tam anno MDCXCI., quam MDCXCIII. in *Commentariis Lipsiensibus* sub titulo *Quadraturae arithmeticae communis Sectionum conicarum centrum habentium*, ac *Supplementi Geometriae practicae* edidit hujusce inventi ope Progressionem aut Seriem numero terminorum infinitam, qua ex dato Logarithmo Numerus obtineretur. Eadem fuerat aggressus multos antè annos ratione non fortasse dissimili; siquidem in *Commentariolo*, quem praelo parabat usque ab anno MDCLXXV. *de quadratura Circuli arithmetica*, ut auctor ipse testatur in *Actis Lipsiensibus* anni MDCXCI, strata erat via regressus Seriei Logarithmi per Numerum ad aliam Numeri per Logarithmum. Vere jampridem, & saltem labente MDCLXIX., Seriem ipsam ex propria penu desumpserat ad grandia natus Newtonus, quemadmodum patet in aureo ejus *Opusculo Analysis per aequationes numero terminorum infinitas*, & adamussim ubi solvit Problema *inventionis basis ex area data*, radicum extractione oecumenica feliciter usus ope continuae substitutionis in approximationum methodis jamdudum receptae. Series Numerum promens ex cognito Logarithmo in memoratis elucubrationibus Leibnitii, ac Newtoni, quantitati Exponentiali respondet C^x , cujus basis C aequet Numerum qui pro Logarithmo hyperbolico vel naturali habeat unitatem, & Proto - Numerus Unitas sit;

XVI

fit ; ita ut, quamvis nulla formae Exponentialium mentio ibidem inita reperiatur, nihilo tamen minus Series ipsa veluti initium aut epocha jure omni erit existimanda primum inventae evolutionis Exponentialium . Juvat autem monere Series paulo supra descriptas non modo angustis limitibus contineri, cum nisi adjuvante substitutionum indirecta methodo impares sint evolvendis quantitatibus Exponentialibus a^x , b^x quacumque alia basi constante praeditis, & eo magis sine regressu ad Logarithmos applicari nequeant Potestatibus y^x , x^x duplici variabili affectis, sed quod maximi momenti est non recto tramite, quinimo summopere obliquo Calculi artificio diducantur ab antea cognitis Logarithmorum Seriebus .

VI. Logarithmicas Series nullus est qui non viderit Hyperbolae, ut ajunt, tetragonismum, & Infinitesimalis algorithmi regulas communi methodo praesupponere; quapropter, nisi primum Vallisiana inventa aut potiori jure Cavalerii & Fermatii labores Infinitum Algebrae subjecissent, necnon Gregorius a Sancto Vincentio palam fecisset anno MDCLVII. Conicum perinsigne Theorema de progressionem arithmetica arearum asymptoticarum Hyperbolae Apollonianae interea dum abscissae vel ordinatim applicatae geometricae progrediantur, & inde Nicolaus Hauffmannus, quem inopportuno latini idiomatis stylo Mercatorem appellant, usque ab anno MDCLXVIII. elegantem edidisset *Logarithmotechniam*, parique acumine Neutonus, atque Leibnitius Calculum differentialem

tialem, & integralem promovere curassent (*) in desideratis haftenus foret Series Exponentialia perhibens, cum praefati Geometrae non a finitorum Analysis sint eam affecuti, sed a quadratura Curvilinei spatii, quae praeter Infinitesimalem, aut equivalentem computandi modum omnes alios canones respuit. Eadem suppositione laborant ea, quae postea dederunt ingeniosissimi fratres Iacobus, & Ioannes Bernoullii; prior in *Parte quinta positionum de Seriebus infinitis* Basileae edita anno MDCCIV., ubi Infinitesimalis Calculi subsidio, & systematis Logarithmici cujuscumque Subtangente praecognita Numerum sive Antilogarithmum ex Logarithmo dato invenire docet, atque alter anno MDCXCIV., qui nuperrimo Infinitesimorum Calculo pariter adjuvante ferme idem confecit, ulteriusque, & saltem anno MDCXCVII., celebrem illam Seriem Exponentialia x^x respondentem in promptu habuit, prouti ipse affirmare videtur dum scriberet singulare Schediasma publici juris factum absque anni nota in suorum Operum Collectionis Volumine tertio. Ea, quae in posteriorum Geometrarum disquisitionibus, & praesertim ab Institutionum scriptoribus numero pene carentibus de exposito argumento sunt tradita, cum ab iisdem pendeant principiis, parique gressu procedant, nullo pertinent jure ad inventionis historiam.

(*) Priora Leibnitii jura in Algorithmi Differentialis inventionem sibi vindicandam non ab Actis Lipsiensibus anni MDCLXXXIV., ut vulgaris opinio est, sed ab ejus Epistola ad Oldenburgium Amstelodami anno MDCLXXVII. data sunt accersenda.

VII. Excipiendus attamen est summus inter Geometras eruditione, & facundia praestantissimus Edmundus Hallejus, qui tamdiu cognitam tritamque destituit semitam, & nuperi Infinitesimalis algorithmi, Quadraturae Hyperbolae, Arithmeticae Infinitorum Vallisii, atque Cavalerii Indivisibilium oblitus novo ausu Exponentialia persequens nec timuit Infiniti latebras, nec a molimine incoepto deflexit nisi praemio lato uberrimae novitatis. Ipse sidera, solemque intueri suetus imperturbata oculi, qua pollebat, dexteritate Infinitum sociavit Binomii formulae Neutonianae, statimque persensit enasci facile posse non tam simplices Logarithmicas Progressiones quam versa vice Series omnimodas Exponentiales. Qui consulat in *Philosophicis Londinensis Societatis Transactionibus* evulgatis anno MDCXCV. Hallejanam Dissertationem non dubius haerebit in iudicando felicem faustumque fuisse eventum tentaminis, cum reapse quaesita Series emerferit ab evolutione Potestatis Binomii $1 + L^m$ posito L Logarithmo, & indice m infinito. Praeclari ejusdem Astronomi postmodum insistens vestigiis Iacobus Bernoullius anno MDCCIV., quo typis mandavit supra laudatas *Positiones de Infinitis Seriebus*, in exaltatione Binomii ad Potestatem exponente n infinito affectam bonis avibus rei Exponentialis periculum fecit; verumtamen ad illam promovendam tanto deinceps elegantiae conatu, tantaque universalitatis copia accessit Leonardus Eulerus de sublimiori Mathefi optime meritis, & nusquam digna laude extollendus.

dus in Capite septimo *Introductionis in Analysin In-*
finitorum anno MDCCXLVIII., ut posthabitis ante-
 riorum Analystarum laboribus ipse sibi unus videat-
 ur aperuisse novam Exponentialia tractandi theo-
 ricen. Atque utinam Henrici Kuhnii perinsignis Geo-
 metrae anno MDCCLXIX. vita functi perpetuus in
 Eulerianam Introductionem Commentarius quatuor
 Voluminibus comprehensus, fatente Ioanne Bernoul-
 lio juniore potentissimi Borussiae Regis Astrono-
 mo (*), in publicum literarii orbis commodum ede-
 retur! Non modo, quae Tironum caprum fortas-
 se fugiunt in Opere super omnes eximio, emollien-
 da Euleri interpretem suscepisse facilis est conje-
 ctura, verum etiam immaturae divinationis nota ca-
 rebit praenunciatio, multa quoque ab eo amplificata,
 addita, ac Finitorum Algebrae restituta in Expo-
 nentialium, de quibus hic sermo, & in affini Lo-
 garithmorum doctrina.

VIII. Hisce praemissis observandum breviter
 censeo Series assumptitias pro Exponentialium evo-
 lutione saepius adhibitas vitio ambiguitatis fortu-
 naeque vicissitudine perquammaxime affici, nec so-
 lummodo vagas plerumque fieri, consentientibus &
 exemplo firmantibus Neutono, Taylora, Stirlingio,
 Gravesandio, Nicolao Bernoullio seniori, aliisque,
 quin etiam infelici successu aliquoties a veritate
 aberrare, dum non fuerit praecognita quaedam lex
 terminorum sive regnans forma Seriei. Non satis

(*) Ita asserit Auctor pag. 69. Sectionis VI. *Supplementi Collectionis &c.*

XX

igitur solido innititur fundamento methodus Leibniana, de qua superius differere occasio fuit; & quamvis caeteri Analyseos scriptores, differentiali ac integrali Calculo facem ferentibus, (*) inversa Serierum methodo usi a Logarithmicis pleniori demonstratione Exponentiales formulas deduxerint nihilo tamen minus videbatur optandum, ut geometricum Hyperbolae tetragonismum argumentatio pure algebraica non amplius in posterum saperet. Nulla vero de *Infinitorum Arithmetices* editae anno MDCLV., seu de *Mathematicarum Exercitationum* quas Cavalierius vulgavit anno MDCXLVII., applicatione ad Hyperbolici spatii dimensionem a Mercatore postea in *Logarithmotechnia* repertam specialis prolixiorque fit mentio, cum Potestatum numero infinitarum Summae ibidem traditae sola inductione aut analogia non raro regantur, quam methodum Graeci dicerent *ageometrian*, atque ideo grati tantummodo animi sensum permoveant veluti ingeniosa sed aliquantisper infirma praeludia plurimi existimanda, eo quod ad majora deinceps via sternebatur. Nec silentio praeterire in re mathematica decet subtilissimas etiam Halleji per Infinitum liberrimo cursu exspatiantis meditationes non levi difficultate laborare quando intelligere mens fuerit nitidam Exponentis ^m significationem; nam quandoque Infinitum absoluta plenitudine ab-

stra-

(*) Qui primus plagium indixit Leibnitio Calculi Infinitesimalis non Keilius juxta communem sententiam in Transactionibus anni MDCCVIII., sed Vallisus fuit in Praefatione Voluminis I. ejus Operum Oxoniae editi anno MDCXCV.

stractum, alia vice Infinitum quodam finito & determinato factore affectum, & denique Unitatem seu finitum Numerum ab Unitate diversum Protei instar eadem in lucubratione repraesentat, qua de re totum pene corrui argumentum super Infiniti fundamenta tanto robore tantaque animi defatigatione excitatum. Accedit quod infinities Infinita, & Infinita quacumque finita ratione invicem comparabilia Author ipse consideret ea repetens, quae longius differuerat in *Philosophicis Transactionibus* anni MDCXCII., & quae Geometriae sacramento firmanda assumpserunt post effata Vallisii Carraeus, Parentius, Grandius, Cramerus, nec non elegantissimus Fontenellius. Melioribus auspiciis, quod etiam supra monuimus, in Infiniti labyrintho se gessit Leonardus Eulerus, quum in praenotato Capite septimo *Introductionis* jam memoratae ambages ferme omnes vitaverit, Infinitum veluti numerorum assignabilium continuo crescentium limitem tantummodo praesupposuerit, & Hallejanum inventum adeo perpoliverit, ut proprio Marte novam Exponentialium derivationem reperisse censendus sit.

IX. Sunt qui in formularum Exponentialium pertractatione ita ratiocinantur quasi methodum universaliorum imitari peramaverint a Moivraeo in *Philosophicis Transactionibus* anni MDCXCVIII. pro Logarithmorum in Seriem evolutione adumbratam cum de generali Serierum Regressu ageret, seu de Radicum extractione ab Infinitinomio proposito. Consuli possunt prae caeteris Commentationibus,

nibus, ubi Moivraeana ista methodus renovatur, *Lectiones elementares Matheseos* Parisiensis postremae editionis MDCCLXXVIII. ab indefesso astrorum cultore Delacaillio jam concinnatae, & a Mariaeo auctae recensitaeque, necnon tertium *Opusculum Analyticum* inter Liburnensia nuperrime edita. Si tamen aequa lance pendamus relatae methodi summa elementa pronum erit evincere Series ibidem assumptitias vel indeterminatis affectas coefficientibus necessario adhibitum iri, & mancā mutilamque obtineri expressionem Series Exponentialis nisi Calculi infinite parvorum ope postea absolvatur. Usus itaque huc usque promotus Binomii Neutroniani in Exponentialibus eruendis nec tanta evidentia nec tanta felicitate donatur, quantam opinor suapte natura fortasse suscepturum ostendent sequentia; quod & in Logarithmis evenisse tam arcto foedere nexis cum quantitatibus Exponentialibus pari ratione fateri nullus dubitabo.

X. Ad argumenti, in quo versamur, illustrationem sit aliorum curae demandata libenter descriptio absolutissima historico-critica de ortu, primisque progressibus Logarithmorum Aera numerata ab anno MDCXIV., quo publici juris factus ab illustri Nepero *Mirificus Logarithmorum Canon*, ubi jamdudum computatis Lineis Trigonometricis seu Naturalibus assignantur respondentes Lineae Logarithmicae seu Artificiales, & non, ut vulgo creditur, communium Numerorum recensentur Logarithmi, quos nequidem in posthuma Operum Collectione anni

ni MDCXIX. edidit Joannis Neperi paulo ante vita functi filius Robertus, nec memorat Patris *Rabdologia* anno MDCXVII. typis mandata (*). Eo magis praetermittere libet obscuras in celeberrimae Scoti illius inventionis primatum dubitationes, quae Stifelium, & Byrgium Kepleri fide in *Tabulis Rudolphinis*, aut Longomontanum testante Oughtredo, atque Voodio in *Athenis Oxoniensibus* praeivisse concludendi vim facerent. Si qui primoribus tantum delibaverint labiis facilem Progressionum arithmeticarum una cum geometricis qualemcumque contemplationem inventoris nomine condecorandos fore decerneret literatorum Respublica, majori equidem jure Neperus ipse Logarithmorum genesim motuum consideratione complexus Fluxionum Calculum reperisse dicendus esset, & immortalis Keplerus ob principium notissimum in Parte secunda *Stereometriae doliorum* anno MDCXV. evulgatae veluti recentioris Maximorum & Minimorum methodi auctor ante Fermarium ipsum in Inventorum albo notandus. Non dubiam quamdam debilemque lucis emissionem, nec subodoratam tantummodo veritatem in Scientiarum orbe inventionis imperium constituere omnes norunt; digniorem

(*) Decessit Neperus senectute gravis post vitae annos emenses 67., & hominum mirabilibus adscribendum est ipsum, amicumque Briggium admodum senes Logarithmos numerasse tam immani labore, ut qui eorum methodum imitarentur fatente Speidellio in unico Numeri 2. Logarithmo assequendo opus Calculatorum octo simul per integrum anni lapsum adlaborantium consumerent; ac, si fides habeatur Pardiesio, viginti homines annorum viginti intervallo Galliae navarunt operam Logarithmicis Tabulis construendis, prae iis ab unico Briggio datis minus, minusque laboris facessentibus.

XXIV

rem hanc etenim sedem publico voto meretur si non omnibus numeris absoluta theorice, eo saltem promota, ut demonstrari evidenter, & applicari facile possit (*).

XI. Mirantur aliqui Neperum primo molimine Logarithmicos numeros versantem extemplo incidisse subtilitatis & praesensionis monstro in systema Logarithmorum hyperbolicorum (**) sine ulla initae supputationis ab Hyperbole derivatione, ac multo ante quam Gregorius a Sancto Vincentio de *Quadratura Circuli & Hyperbolae* agens peringeniosum illud detexisset Theorema, quo tantum duce inter se convenire antiquorum Synthesi facem ferente ostendi queat Logarithmorum Tabulas Neperianas, & Hyperbolae tetragonismum. Haec autem obversati portentis species statim evanescet dummodo monere fas sit simplicissimam prae omnibus esse, ac natura ipsa dictante nasciturae inventioni paratam methodum Neperianam pro Logarithmis arithmetice computandis primum adhibitam, cum inter innumerabilia systemata illud certe eligi debuisset praee omnibus facillimum, in quo Circuli Radius seu Unitas Trigonometrica 10000000000 toties in calculo Logarithmorum summanda vel subtrahenda commodiorem haberet Logarithmum, quale est nihilum a cyphra solita expositum, & Logarithmi nascentis

(*) Cartesius *Analystarum* primus Magnitudinum Potestates Indice Numerico appposito expressit. Ecquid? Exponentialem etiam Calculum invenisse affirmaremus?

(**) Consulantur numerus 139. & alter 351.

scentis positivi vel negativi, *abundantis* seu *defecti-*
vi, elementum foret in ratione aequalitatis ad quam-
 minimum decrementum sive incrementum Radii, non
 in qualibet alia minus nativa, & indeterminata ma-
 joris minorisve inaequalitatis ratione. Exemplo rem
 a Geometris hucusque nimis neglectam clariorem
 reddere satagam subsidiaria Logisticae aut Logarithmi-
 cae Curvae in Schemate I. descriptione utens, in
 qua delineanda Edmundus Guntherus Briggsii syn-
 chronus omnes praecessit, atque eximio geometri-
 cae synthesis penes Italos restitutori Laurentio Lo-
 renzino ansam dedit Logarithmicarum Linearum
 contemplationem ulterius porrigendi, uti Codices
 testantur autographi, vel saltem Authoris jussu con-
 scripti, in publica Bibliotheca Magliabechiana ab im-
 mortali feliciterque regnante PETRO LEOPOLDO
 novis aedificiis, librorumque muneribus aucta. Esto
 igitur in communi Logistica PEM Radius Unitatem
 Trigonometricam exhibens AE, a quo versus Y Ne-
 periano more sumantur Logarithmi *abundantes* Linea-
 rum Trigonometricarum GH, BF geometricae pro-
 gredientium, & radio minorum, dum e contra ver-
 sus Z computentur Linearum radio majorum NL,
 DC Logarithmi *defectivi*. Liqueat interserendo me-
 dias continuas sine limite proportionales Lineis EA,
 HG vel EA, LN tandem obtineri TS seu KR
 quamminime discrepantem ab Unitate EA. Posita
 itaque differentiola EQ, sive KX aequali AS, seu
 AR oritur systema primo intuitu facillimum Loga-
 rithmorum a Nepero institutum; at ex varia inae-
 qua-

qualitatis relatione inter EQ, AS, vel KX, AR alia omnia profluunt absque numero possibilia Logarithmorum systemata. Casu quodam evenit, ut quae Neperus pura arithmetica consideratione fretus investigaverit suos promens Logarithmos adamussim respondeant Arcis asymptoticis Hyperbolae Apollonianaee aequilaterae, quam affectionem elegantiae ususque foecundissimam serius norunt Geometrae.

XII. Supervacaneum hac occasione non erit caetera quoque systemata celebriora Logarithmorum obiter persequi, & speciali tentamine Tabularis originem atque differentias perscrutari, cujus aequae inventor praestantissimus idem Neperus. Altera, in qua nunc ociamur, Logarithmorum forma Unitati, quae est Protonumerus, tribuit pro Logarithmo cyphram seu nullitatem, dum ex adverso Unitatem assignat Denario, Denarium Radio 10000000000, positivosque dat Logarithmos numeris Unitate maioribus, & Unitate minoribus negativos. Labente anno MDCXVI. Henricus Briggs Neperum Edemburgi alloquens novam hanc didicit Logarithmorum institutionem, cui praecedente anno consimilem sponte sua conjectaverat; nam Usserio Chronologo scribens se concinnasse affirmabat Tabulam Trigonometricam, ubi Logarithmus Unitatis seu Radii erat nullitas, & Sinus-recti, qui foret Radii subdecuplus, ideoque proxime respondens Arcui vel Angulo $5^{\circ} 44' 21''$, Logarithmus numero 10000000000 aequalis assumebatur. Si ergo in eadem Logarithmica Curva Schematis I. imitari libeat Logarithmorum dispo-

dispositionem, quam dicunt Tabularem five Briggianam, opus erit pro Unitate determinandi inter ordinatim Asymptoto applicatas Lineam $\Delta\Phi$ ita constitutam, ut dum ipsa quamminime crescens evadit $\Theta\Gamma$ sit Logarithmus nascens $\Phi\Gamma$, seu ΔX ad incrementum inassignabile ΘX , in ratione minoris inaequalitatis proxime expressa a Fractione decimali 0.43429448 19032518276511289189166050822943970058036 66566114454 Unitati comparata, ineffabilis patientiae viro Abrahamo Sharpio eam ad sexaginta Locos Decimales sub finem emensi saeculi recensente, five quod idem valet ab Unitate & Numero 2.30 25850929940456840179914546843642076011014 88628772976033328 summopere in Arithmetica Logarithmorum percelebri. Eadem ratiocinatione ducente systematis Briggiani reciprocorum assequeremur invertentes memoratarum relationum terminos, adeo ut quae minoris inaequalitatis est Ratio in permutatam rationem Unitatis ad Numerum proximum 0.4342945, nimirum majoris inaequalitatis limitis elementi Logarithmi $\Lambda\Xi$ vel $\Psi\Omega$ ad limitem incrementi $O\Omega$ Unitatis $\Lambda\Psi$ transformetur. His insistendo principiis, cum infinities variari queant primorum incrementorum Numeri, & Logarithmi rationes, omnimoda difficultate carebit innumerabilia quoque in Logistica Curva delineare systemata, inter quae potissimum semper emicat Neperianum a simplicissima aequalitatis ratione depromptum. Qui amoenitatis analyticae causa praediligeret systema Logarithmorum tali caractere di-

XXVIII

stinctum, ut manente Nihilo pro Logarithmo Unitatis foret Unitas Logarithmus Binarii, oporteret elementorum nascentium rationem majoris inaequalitatis $\beta\omega$, vel $\mu\gamma$ ad $\nu\gamma$ ejusmodi lege moderari, ut numeri quamproximi 3.3219277 & 2.3025851, sive commodius 1. ac 0.6931472 eam exhiberent.

XIII. Nec praelaudatus Neperus, nec Eduardus Vrightius, qui Neperi ipsius recensione, & auspiciis faventibus Britannicae ad Indiarum Orientalium commercium augendum initae Societatis *Mirificum Canonem* Anglico idiomate ornavit (*), addidere Tabulas Logarithmicas Numerorum in serie naturali progredientium Logarithmis Trigonometricorum Sinuum, si forte excipiatur graphica quaedam methodus a Vrightio huic rei accommodata, vel subsidaria Briggio auctore Tabella, sed parum felici successu. Primi omnium Speidellius, atque Keplerus Logarithmos Numerorum, quos dixere rotundos, & naturales, ab Unitate ad Millenarium usque anno MDCXXIV. in Tabulas publicae utilitati inservientes poni curarunt. Sed unus in sexta *Novorum Logarithmorum* Anglici Operis editione, & alter in *Chiliade Logarithmorum* Lentiae vel potius Marpurgi tunc edita, atque in *Supplemento ad Chiliadem* anno MDCXXV. nec hilum innovarunt primum Neperi systema soli Trigonometricae praxi devotum, id quod
nulla

(*) Neperus ipse Versioni Anglicam Praefationem adjunxit, cumque fato cesserit Vrightius, Opus edidit anno MDCXVI. vel XVIII. Samuel filius Briggio adjuvante.

nulla ratione illustrasse hucusque videntur recentiores Geometrae. Et re quidem vera in Tabula Speidellii Logarithmus Millenarii est 6.90775, nempe triplus 2.30258 $\rightarrow$, qui est Neperianus Logarithmus Denarii, eodemque examine repetito constat

161180959 Logarithmum Unitatis seu $\frac{1}{10000000}$ ^{mae}

partis Radii assumptum ab Ioanne Keplero, de universa Mathesi ac praecipue Astronomia, & coelesti Physice optime merito, esse septuplo maiorem iuxta Logarithmorum Neperi theoricen 23025851 $\rightarrow$ Logarithmo Denarii. Nullam itaque expostulant peculiarem indaginem Logarithmi utriusque Tabulae superius perpensae, qui cum Neperianis adamussim conveniunt, & Areis Hyperbolae Apollonianae Asymptotis ad normam positae praeditae in sublimiori Geometria moris est aequales effingere. Qui vero Logarithmos Briggianos, seu Tabulares Hyperbolicis spatiis optaret pariter designatos Scalenam opus esset ita seligere Hyperbolen, ut Angulus asymptoticus proxime aequaret $25^{\circ} 44' 25'' 27'''$, vel etiam Hyperbolam primae Conjugatam cum Angulo asymptotorum prioris supplemento $154^{\circ} 15' 34'' 33'''$, enormiter aberrante ab Angulo $54^{\circ} 16'$, quem habet preli vitio, vel naevi lapsu inobservato omnigenae Matheseos Historiographus clarissimus Montuclas initio secundi Voluminis. Universaliter, si solum excipias Neperianum systema, caeteris omnibus Logarithmorum formis Hyperbolicam representationem admittentibus (*) gemina semper Hyper-

(*) Videatur numerus 351.

Hyperbola Problematis satisfaciet, nimirum earum alterutra, quae Hyperbolicum systema Vallisii sensu componunt, & aptius nominantur Hyperbolae inter se Conjugatae Geometrarum omnium consensu praeter Saurium intempestivo verbo vocantem Conjugatas Hyperbolas Ramos ejusdem Hyperbolae oppositos, quemadmodum fidem facit Volumen secundum *Completi Mathematici Cursus* editionis nuperimae.

XIV. Tanta equidem facilitate, qua varias Logarithmorum formas lustrare licuit, enucleatum iri non puto valde obscurum Logarithmorum genus a Spangenbergio recens traditum in Opusculo nusquam viso, frustraue totam per Etruriam quaesito, cui titulus est *Dissertatio de natura systematum Logarithmicorum in genere, & in specie de systemate Logarithmorum Solidorum*. Quamvis in gravissimo geometricae veritatis studio nullus conjecturae aut divinationi sit locus silentio premere nolim fortasse Solidos Logarithmos ab auctore appellari, quos Solida repraesentent instar Logarithmorum ab Areis Hyperbolicis designatorum, qui aliquo utut levi jure nuncuparentur Plani sive Superficiales. Unicum Schema II. hac hypothese subintellecta complecti equidem potest tria Logarithmorum eorundem symbola in triplicis naturae extensione considerata, videlicet Lineare in Curva Logistica ABC aequationis notissimae Exponentialis $y = C^x$, Planum in Hyperbola Apolloniana aequilatera DBE, aequationis $y = \frac{1}{x}$, & denique Solidum genus nunc conjectatum

tum in Linea FBG ordinis tertii $y^2 = \frac{1}{x}$, quam Neutonus appellat Hyperbolismum Parabolae alterius speciei, caeterique Analystae Hyperbolam cubicam, alii quadraticam aut secundi gradus seu generis. Non ambigitur enim Numeris HI, HK, HL, HM in progressionem quacumque crescentibus respondere in Logistica lineares Logarithmos o, QP, QN, QH, & in Hyperbola Apolloniana planos Logarithmos seu Quadrilinea o, BIKR, BILS, BIMT, sive Sectores o, HBR, HBS, HBT; veruntamen posset aliquantisper in dubium revocari memorata sine demonstrationis sanctione quadraticae Hyperbolae affectio. Palam autem fit Solidum ex circumvolutione genitum Hyperbolae FBG circa asymptoton HX habere elementum $S_{2R}^{\Pi y^2 dx}$, ideoque proportionale simpliciori expressioni $Sy^2 dx$, vel juxta Hyperbolae assignatam aequationem $S\frac{dx}{x}$, idest abscissae Logarithmo, quapropter Numeris ipsis HI, HK, HL, HM Logarithmi solidi aderunt Rotunda Frustra o, BΔZY, BΔΦΘ, BΔΣΨ, quod demonstrandum susceperam.

XV. Utinam Conicarum Sectionum, Curvarumque cognatarum contemplationi geometrica synthesis nunquam facta exul ab Italis vacare rursus amaret! Quantam elegantissimorum fructuum segetem in secreto positam colligeremus si antiqua ingeniorum sagacitate restituta Linearum affectionibus perquirendis operam sedulo impendere excitaremur, nec Calculi furor quandoque heu! nimis pro-

XXXII

prohiberet! Nonne indecorum Geometris synthesis vires otiosiores languere, dum indemonstrata hactenus sint pulcherrima de Apollonii Hyperbola, aliisque Conicis symptomata, quae non diuturno conamine, quinimo facillime ostendi posse quilibet experiretur? Jam annus vertebat MDCCLXIX cum anonymus Author in *Novis Lipsiensibus Commentariis* occasione tunc observati Cometae proposuit Schemate III. depictum

Theorema.

XVI. In universis Sectionibus Conicis ANM si per Focum F ducantur normalis FEG ad axem AB, duoque quilibet radii FN, FM, & recta FL angulum bisecans NFM, semper eveniet, quod PR perpendicularis FL ducta a puncto P concursus chordae protensae MN, ac rectae FG abscindat FZ, sive FT aequalem FE, vel Curvae datae Semi-Parametro.

XVII. Sit XCD Directrix Sectioni Conicae adscripta, cui ex puncto E aptetur EC parallela MPD, addaturque recta conjungens FD. Profluunt ex natura constantis rationis, quam vocant *determinantem*, proportionem $FM : MD :: FN : ND :: FE : EC = PD$, ideoque $FM : FN$, vel ob anguli bisectionem $ML : LN :: MD : ND$. Sunt igitur $MD' : LD' : ND' ::$, qua posita continua geometrica proportione ex Elementis constat rectum esse angulum LFD, videlicet fore DF aequidistantem PR, quapropter $FM : MD$
vel

vel $FE : PD :: FT : PD$, & $FE = FT$, quod idem valet de FZ ob ZFT aequicrura Trigonum.

XVIII. Alterum exemplum suppeditat theoreti-
ca propositio a Montucla sine demonstrationis nexu
promota & Hyperbolam Apollonianam adscribens
Curvarum Conchoidalium familiae. Si ostendere ve-
lis singularem, de qua fit sermo, affectionem finge
animo Schema IV. respiciens super rectam longitu-
dinis indefinitae AB fluere Triangulum rectilineum
cujuscunque speciei CDE hac lege constanti, ut
unum ex lateribus CE semper rectae praefatae AB
congruat in fluxu. Si fluente Triangulo CDE du-
cantur a puncto G veluti Polo Conchoidis per in-
variaturum punctum lateris H, H', H'' , Radii GHI ,
 $GH'I$, $GH''I''$, inquiraturque Locus Geometricus
punctorum I, I', I'' mutui concursus Radiorum sine
numero, & Laterum $CD, C'D', C''D''$ parallelismo
gaudentium facile Locus ille innotescet, dum a Po-
lo G digrediatur GL parallela CD . Revera in Tri-
gonis similibus GHL, IHC , seu $GH'L, I'H'C'$, vel
 $GH''L, I''H''C''$ erunt $IC : CH :: GL : LH, I'C' : C'H' :: GL : LH', I''C'' : C''H'' :: GL : LH''$; adeoque cum
aequantur inter se ex constructione $CH, C'H', C''H''$
palam fiet verificari proportionem Geometricas $IC : I'C' :: LH' : LH, I'C' : I''C'' :: LH'' : LH'$, sive posita
 $LF = CH$ haberi $IC : I'C' : I''C'' :: FC'' : FC' : FC$,
quae est proprietas Hyperbolae Conicae ad asym-
ptotos AF , & FM aequidistantem CD comparatae.
Obtinebitur itaque quando angulus ad C in Trian-
gulo generatore sit rectus aequilatera Hyperbola
Apolloniana.

XIX. Habet praeterea de elegantissimo Solido Hyperbolico Acuto manuscriptus Codex Palatinus, Florentiae in Regii Physices Musaei Bibliotheca nunc asservatus, Evangelistae Torricellii olim mei supra laudem Praedecessoris, nonnulla sparsim dicta, quae syntheticam lucem desiderarent. Tot inter perjucunda Theoremata pudet longius negligere demonstrationem illius ibidem adserti, & usque ab eximii Scriptoris Operum editione anno MDCXLIV. memorati, de Solidi pariter Acuti mensura geniti revolutione Scalena Hyperbolae circa Asymptoton. In duplici Schemate V. VI., quod ex mea *Linearum secundi Ordinis Synopsi* heic repetere juvat, fusiusque ex Opere dudum typis parato sub titulo *Evangelistae Torricellii Omnia, quae supersunt, nova haecenus inedita*, Solidum Cylindrico-Conicum AD CBEFA productum a rotatione circa Latus AB obliquanguli parallelogrammi ADCB evidens est aequare rectum Cylindrum DCEFD, & sic de similibus. Methodo igitur Geometris notissima, aut Archimedeae Exhaustionis circuitu compendiosiore facto, prius observandum opinor Elementum Solidi ab Area infinites magna YDCBH rotante circa asymptoton BH orti, videlicet Tubulum genitum revolutione Quadrilinei infinitesimi OLRP, sive Parallelogrammi VLRP fore ad Tubulum genitum rotatione alterius Parallelogrammi XQRP ut LR ad RQ, vel ex Hyperbolae natura ut AD ad AQ, vel denique ut Parallelogrammulum XQRP ad ZSRP. Ita ratiocinando de caeteris Elementis elicietur Solidum

lidum Acutum YDCBEFKH longitudinis infinitae esse ad Solidum Cylindrico - Conicum ADCBEFA, seu ad rectum Cylindrum FDCE, veluti Parallelogrammum ADCB ad Triangulum BDC, nimirum in dupla ratione; atque inde liquet Solidum Acutum YDAFKH Basi gaudens concavo - conica, vel convexoconica aequale esse subjecto Cylindro FDCE, Solidumque Acutum YDMFKH Basi circulari insistens aequare in Hyperbola primi Schematis amblygonia Summam Cylindri FDCE ac recti Coni ADMF, & differentiam in Hyperbola secundi Schematis oxygonia, quemadmodum Torricellius ante dimidium elapsi saeculi peringeniose nunciaverat.

XX. Ita meo quidem iudicio inutilem operam non collocarent etiam hoc aevo synthesis amatores in Conicarum Sectionum doctrina promovenda, quomodocunque exulta res sit primum a Graecis Scriptoribus, & deinde praesertim ab Italis post literarum in Orientalis Imperii lapsu Medicea liberalitate fulcitam instaurationem. Physice etenim universa, ac specialiter corporum per aethera errantium Philosophia, Curvarum unius curvaturae in Coni superficie descriptarum affectionibus non modo sustinetur, sed potius dicam alitur foveturque; nec satis extolli uspiam poterit Theorematum Conicorum commodum perinsigne etiam in sublimiori Calculo, difficillimisque Matheseos quaestionibus, cujus fortasse specimina quaedam Capitibus V. VI. VIII. & IX. sum allaturus.

XXI. Plerumque accidit, ut Synthesis, videli-

XXXVI

cet firmioris. mathematicae ratiocinationis, defectu in diversa abeant nonnulli Geometricarum disciplinarum studiosi. Sunt qui Logarithmos prouti rigido sensu Rationum exponentes, sive Rationum mensuras intelligant, quos inter nescio quo titulo memorentur Hallejus, Keilius, Smithius, Maclaurinius, totaque Anglicorum Geometrarum cohors, dum caeteri hanc Logarithmorum proprietatem veluti absurdam rejiciant, proscriptamque a Mathesis limine velint, praeceunte incomparabili Dalembertio tam in Articulo *Logarithmus* Encyclopedici Dictionarii editi anno MDCCLXVI., quam in sexto Voluminis primi Mathematicorum Opusculorum Schediasmate. Meum certe non est tantas componere lites; sed parvulo liceat Matheseos interpreti elementares geometricarum Rationum imagines in mentem revocare ad quaestionem penitus dirimendam, quam in Opere posthumo *Harmoniae Mensurarum* forte magis lubricam reddidit Rogerus Cotesius, ille nimirum eximius Geometra, qui Pembertonio, & Smithio fatentibus si diutius vixisset alter erat Neutonus. Rationum Exponentes absque dubio eas commensurant, adeo ut duplo Exponenti Ratio dupla conveniat, triplo tripla, simplici cum dimidio sesquialtera, duplo cum duabus tertiis partibus dupla-superbipartientertias, quartae parti subquadrupla, & sic de infinita Rationum geometricarum diversitate. Quo posito Logarithmus e contra duplae, triplae, sesquialterae, duplae-superbipartientistertiae, subquadruplae Rationis non est duplus, triplus, simplex

simplex cum dimidio, duplus cum duabus partibus
 tertiis, pars quarta Logarithmi rationis simplicis,
 idest non aequatur Rationum Exponenti; imo ex-
 cepta unica Rationum aequalium comparatione, ve-
 luti recte admonet Dalembertius, nullo modo pro-
 portionem servant Rationum Exponentibus Loga-
 rithmi. Non igitur Logarithmi eadem forma metiun-
 tur Rationes geometricas, qua circulares Arcus men-
 surant angulos, quomodo non raro sonant expressio-
 nes aequivoco, labilique significatu praeditae, de
 quibus hic mentio laconica occurrit. Peccat ergo
 Shervinii notissimum Tabularum Proemium editionis
 Londinensis MDCCLXXII., Clarkio recensente vulga-
 tum (*); nam semper, & ubique Arcus metientes pro-
 portionales sunt Angulis mensuratis, crescant ii seu
 decreascent, quod ex praemissis nullatenus verificatur
 de Logarithmis respectu Rationum. Verum cum fal-
 lax, & vitiosa fiat definitio Logarithmorum quasi
 forent vel *Rationum Exponentes*, vel *Rationum Men-
 surae*, facto tectoque veritatis jure substitui poterit
 observatio tutissima quod Logarithmi non Ratio-
 num multipliciter aut submultipliciter, sed *Ra-
 tionum compositionem* vel *resolutionem metiantur*, compo-
 sitionis seu resolutionis Rationum *Exponentibus* aequen-
 tur, *Mensuram* compositionis sive resolutionis exhi-
 beant, demonstrentque non quam multiplex sit, at
 quoties *composita* vel *resoluta*, duplicata, triplicata,
 sesqui

(*) Ibidem sine emendatione dicitur aliqui Logarithmos vocant Nu-
 meros Rationem exponentes.

sesquuplicata, subquintuplicata, uno verbo n^o plicata prima Ratio alteri comparanda. Haec ad medendam logomachiam, ad verborum malesonantium valorem determinandum, ad Graecae vocis *Logarithmi* nativam vim restituendam, & ad conciliandos fortasse non dispares sensus Scriptorum varie loquentium simul omnia conducunt.

XXII. Nec adjuncta Logarithmis verba *Magnorum*, & *Logisticorum*, quae passim occurrunt vel in Tabulis Astronomicis, vel in quorundam Authorum scriptis rei claritati non parum obesse inficiandum. Saepe usuvenit Mathesim non exactius callentibus subintelligere in cacophonia Logarithmorum *Logisticorum* singulare aliquid Logometrici Canonis systema, etsi reapse nil aliud ii sint nisi differentiae inter Tabularem Logarithmum constantem Numeri 3600, & Logarithmum aequae Tabularem cujuslibet minoris Numeri scrupulorum secundi Ordinis ad Calculi sexagesimalis commodum, & ad partium proportionalium computationem seu Tabularum penes Astronomos intercalationem majori facilitate obtinendam. Quare Logarithmos istos *Logisticos* a Tabularium intima systematis natura non segregatos confusionis vitandae gratia proprio differentiarum nomine inscribere noluerunt Geometrae, five potiori jure Logarithmos non dixere Briggianos non amplius ab Unitate, sed a Numero 3600 veluti computationis initio desumptos? Nemo ignorat summam in universo scientiarum orbe utilitatem asferre aptam verborum significationem,

nem, plerasque idearum obscuritates a minus castigato vocabulorum selectu originem ducere, atque supervacuarum definitionum mole non amplificari, sed contrahi ex adverso disciplinarum pomoeria; adeo ut in Grammaticorum parvi aestimata provincia ferme omnia Philosophi curae, solertiaeque sint commendanda. Sic qui pro Logarithmis arithmetica Complementa jam a Speidellio anno MDCXIX. excogitata adhibent Logarithmorum Tabularium, seu potius augent Tabulares Logarithmos ipsos Denario, Numerosque artificiales efformant, quos in Additamentis Operis Luini *de Progressionibus & Serieb* anno MDCCLXVI. evulgatis celebratissimo Boscovichio *Magnos* Logarithmos nominare placuit, nil in systemate Logarithmorum vulgarium substantialiter innovant. Ii namque tantummodo ad subtractionem commodius traducendam in additionem, & ad evitandam Logarithmorum, quos negativos vel defectivos dicunt, in Calculo molestiam emergere hac methodo sciunt Logarithmos decimali eadem progressionem, eodemque vulgari sive Briggiano computo manentibus tam leviter immutatos, ut cyphra vel nullitas non Unitatis, sed Fractionis 0.0000000001 sit Logarithmus.

XXIII. Posteaquam Briggio prae omnibus infabili labore contigit *Chiliadem primam* Logarithmorum anno MDCXVIII. Londini in publicam edere lucem, subindeque emenso sexennio Logometriam perficere tam in *Arithmetica Logarithmica* anno MDCXXIV. typis donata, quam in *Trigonometria*
Bri-

XXXX

Britannica, cujus admirandae Collectionis Tabulas Auctore fatis erepto (*) Henricus Gellibrandus vulgavit anno MDCXXXIII, defuetudine premi incoepere Neperi *Canon*, Batschii, Frobeniique, & Beniamini Ursinii Tabulae Neperiano more ab eo concinnatae usque ab anno MDCXXV. in usus amplissimos Trigonometricos. Novum aperuit vertente anno MDCXX. Briggianum iter in Trigonometria Edmundus Gunterus, qui *Scalas* construxit Logarithmicas, a Vingatio deinceps, Oughtredo, Milburnio, Partridgio, recensque ab acutissimo Lambertio perfectiores effectas, & *Canonem Sinuum, Tangentiumque Artificialium* tunc edidit, ac rursus anno MDCXXXIII. editionem sui *Canonis* iteravit una cum Briggii *prima Chiliade* in Opere *de Radio, & Sectoris*. Verum, quae Neperus, ac Briggs paucos ante annos, Adrianusque Ulaacus in *Trigonometria Artificiali* anno MDCXXVIII. publici juris fecerunt, extractionibus Radicum quadraticarum in tanta Numerorum copia & immenso laboris pondere repetitis Auctores praelaudati invenere; nec alia prostabat illa aetate contractior methodus ad extricandas tot calculorum ambages ab Hallejo potissimum, nec non in *Arithmetice* Malcolmi, in *Mathematicis* Sethi Vardi, & in Keilii *Euclide* fusius descriptas. Posteriora non memoro Petri Crugeri, Iona Moorii, Nathanielis Roei, Edmundi Vingati, Ioannis

(*) Quartum supra septuagesimum annum Briggs agebat tum cum fato cessit anno MDCXXX. XXXI., Londinensi stylo, editionem praefationemque Trigonometriae suae parans.

nis Neutoni, Shervinii, & Gardineri, ne de recentissimis loquar, spartamque jam occupatam usurpare videar nulla equidem utilitate, cum iisdem regatur mediarum geometricae proportionalium principiis, nec hilum in calculorum commodis augendis, atque Logarithmorum penu ditanda Auctores illi profecerint (*).

XXIV. Majora molitus Logarithmicae Scientiae primus sane novator subtilissimus Jacobus Gregorius, qui cum ausisset labente anno MDCLXVII. in Commentario *de vera Circuli & Hyperbolae quadratura* Spatia Hyperbolica ad Series analyticas convergentes revocare ope Theorematum circumscriptis, & inscriptis Polygonis pertinentium diversam quoque monstraverat semitam Logarithmorum computationi paratam (**). Accessere rei Logometricae felicitati eodem anni lapsu MDCLXVIII. Series elegantia nulli secunda, qua in *Transactionibus Philosophicis* magnus Geometra Vice-Comes Brounckerus Hyperbolae tetragonismum parallelogrammorum & triangulorum consideratione usus rationalibus numeris expressit, atque superius dicta Mercatoris Holsati *Logarithmotechnia* ab Jacobo Gregorio rigidae synthesis examine paulo post comprobata in *Geometricis Exercitationibus* primasque tenens in litteratorum Republica, cum nondum idem edidisset

f

set

(*) Compendium Tabulare sub nomine Shervinii editum anno MDCCVI. primus excogitaverat in *Trigonometria Britannica* Ioannes Neutonus.

(**) Triumviri clarissimi Brounckerus, Vallisus, & Collinsius iudicium tulere de Gregorii invento meae consonum descriptioni.

XXXXII

fet inventum Huddenius , quod anno potuisset
 MDCLXII. affirmante Leibnitio, vel potius anno
 MDCLVII. Brounckerus fatente Vallisio. Vigebat
 tunc inopinata humani ingenii sagacitas, qua tot
 horrens errorum tabem Natura sibi ipsi consulere
 visa, & in Saeculo decimo-septimo tantis inventis
 praeclaro scientiarum miracula promere. Hinc sta-
 tim alias ex *Logarithmotechnia* derivarunt Series pro
 Logarithmis naturalibus, vel hyperbolicis ad con-
 vergentiam magis tendentibus insignique commo-
 do praeditis ferme omnes singillatim Geometrae,
 sed praecipue Vallisius, Hallejus, Sharpiusque, cu-
 jus in calculis efficiendis stupendae celeritati debe-
 mus Logarithmos Tabulares ad sexaginta Locos
 praeter *Indicem* sive *Characteristicam* computatos, &
 respondentes in priori tentamine septem Numeris
 post Unitatem, excepto Quinario, quos vocant *Pri-
 mos*, deindeque Nota addita Logarithmos aeque
 Briggianos omnium Numerorum ab Unitate ad 100,
 & *Primorum* usque ad 1100, Logarithmosque hyper-
 bolicos Numerorum a Denario ad octavam Decadem
 pari solertia evolutos. Isti serierum Logarithmica-
 rum progressus partim ab Hyperbolae quadratura
 fluxere, seu methodo Mercatoris aliam in formam
 redacto, vel methodo *Differentiali* aut *Interpolatio-
 num* Neutoni, partimque a foecundiori Infinitesi-
 morum algorithmo sine Hyperbolae tetragonismi
 subsidio, quod feliciter obtinuit in *Harmonia Mensu-
 rarum* post prima *ratiuncularum* numero carentium
 ab Hallejo jacta semina nusquam satis laudandus
 Cote.

Cotesius. Universa ipse sensit Logarithmorum systemata pendere ab unica Serie, tantum affecta diverso Coefficiente, atque ab unica Logometrica lege, cujus vi in quolibet systematum sit Numeri Fluxio ad Logarithmi Fluxionem ut Numerus ad constantem, quae *Modulus* appellaretur. Posito hoc *Modulo*, qui in Hyperbolicis - Neperianis, sive Naturalibus Logarithmis est 1, in Briggianis vel Tabularibus 0,4342945 &c., in Briggianorum Reciprocis 2,3025851 &c., in systemate, cui Binarius pro Basi conveniat, 1,4426949 &c., eademque ratione obtinendus in omnium possibilium complexu, nulla amplius difficultas occurrit dum de Logarithmis agatur sine numero discrepantium systematum versa vice de uno in alterum traducendis.

XXV. Mirum autem quam opimam elargiatur messem recens acquisita Logometriam intuendi doctrina, cum idem sint *Modulus*, Coefficientis Seriei Logarithmicae, Exponens Rationis elementi Logarithmi ad minimum incrementum Numeri supra Unitatem, Sinus rectus Anguli Asymptotici Hyperbolae Aequilaterae Apollonianae vel multiformis Scalенаe, cujus Arcis peraequantur Logarithmi, & Subtangens Curvae Logisticae ad eos exhibendos accommodatae. Huic tantae molis argumento in sequentibus maximae utilitatis futuro, ne forte obscurum fiat, lucem afferent Schemata VII. & VIII., quorum primum satis ostendit ad abscissam AC, communiter Unitatem exprimentem, utriusque Hyperbolae Aequilaterae, & Scalенаe spectare aequales inter se atque AC,

XXXXIV

Rectas ordinatim Asymptoto applicatas CD, CV; qua de re si Areae DCTP, VCTQ, idest Logarithmi ejusdem Numeri AT in infinites minimas, sed pari numero Areolas subdividantur, erit harum quaelibet DOCI ad respectivam VLCI, nimirum & Areae integrae, in eadem Parallelogrammorum circumscriptorum ratione, videlicet ob aequalem basim CI, ut DC, vel VC Sinus-totus seu Radius in Aequilatera ad VB Sinum-rectum Anguli VCB, five Asymptotici MAT in Hyperbola Scalena. Patet quoque in Logistica VIII. Schematis, dum recta AB Unitas praesupponatur, fore Rationem Minimorum Elementorum AF, seu BC ad CD speciem ex antea dictis Systematis Logarithmici determinantem, quemadmodum EA Subtangens ad AB Unitatem, unde sequitur Subtangentem Systematis aequari Exponenti Rationis Systema ipsum determinantis, prouti ostensa suadent, ac Sinum-rectum Anguli Asymptotici eodem redire cum Exponente determinantis Rationis, vel cum Systematis *Modulo*. *Modulus* autem cum constans sit in Logarithmis omnium unius Systematis Numerorum, erit necessario Coefficientens Seriei Logarithmum per Numerum exprimendis, five Cotesiano more Logarithmus immutabilis & determinatae *Modularis Rationis*, quae Logometriae universalis Mensura omnium Geometrarum consensu aequalis Rationi transcendenti inter 2.718 281828459 &c. & 1, vel inter 1 & 0.367879441171 &c. refertur.

XXVI. Atque hic monere sufficiat Curvae Logari-

garithmicae Subtangentem superius duxisse adjumento communis, & jam usu obsoleti Trianguli *Characteristici* similis *Tangentiali*, quamvis mihi methodus placeat fortasse non inelegans, & Veterum demonstrationibus analoga Curvarum Tangentes, Asymptotos, Curvaturas, Puncta Multipla, & Singularia contemplandi absque incomparabilium vel nihilorum non satis castigata relatione a Fermatio primum, anno saltem MDCXXXVI., introducta, quinimmo absque Motuum inopportuna theorice eodem anno Curvarum Geometriae a Robervallio, & deinceps Torricellio, Fabrio, ac melioribus avibus a summo Neutono applicata, & absque Rationum Limitibus a Leibnitio, ipsomet incomparabilium comparandorum authore, usque ab anno MDCXCIV. feliciter consideratis. Hujus methodi specimen, ne ea quae in Capite VIII. de simplicioris Logisticae affectionibus erunt dicenda obscuritatem patiantur, ac ne novitas suspicionem pariat, in Parabola adjungam, atque Cycloide (*).

XXVII. Esto itaque Apolloniana Parabola Schematis IX., in qua a puncto B Tangens sit educenda. Linea primi Ordinis, seu recta Tangens quaesita in unico puncto B arcui Curvae finito circa B posito occurret, quapropter Aequatio primi Gradus Tangenti conveniens, & secundi Parabolae per finitum intervallum circa B adeo disponentur,
ut

(*) Hujus explicationis usum in num. 348. Lector non intempestivum sentiet.

XXXXVI

ut vel supra vel infra ipsa Tangens a Curva aliquantulum distet. Ope notissimi Canonis Analytici transformationis Axium Coordinatarum, seu ope Formulae Differentiarum finitarum non adhuc satis excultae Aequatio Curvae traducatur ad Axem abscissarum quemcunque per datum punctum B transeuntem, ad quem applicatim ordinatae spectent MON, M'O'N', M''O''N''. In Parabola facillime hoc efficietur etiam sine memoratis subsidiis sumendo CBD Axi primario parallelam, atque MON, M'O'N' ipsi normales; nuncupatis etenim EF = b, FB = a, & p Parametro habebitur aequatio ad Parabolam $pb + px = a^2 + 2ay + y^2$, vel simplicius $px = 2ay + y^2$, dum aequatio Localis ad Rectam erit $nx = my$, seu posito $n = p$ erit $px = my$, ideoque $2ay + y^2$ in Parabola = my in Recta-Linea, unde eruitur quod hypothesis facta $m = 2a$ sit semper ordinatim applicata positiva MO in Recta major MN in Parabola, ac ordinatim applicata negativa M'O', M''O'' &c. in Recta ex adverso minor M'N', M''N'' &c. in Parabola. Igitur Tangentis positionem indicabit Aequatio $px = 2ay$, nimirum Subtangens FG dupla erit FE, quemadmodum Conica estendunt. Specimen alterum sed in Transcendentibus Curvis praebeat Cyclois nec contracta nec protensa, at qualem prima vice depinxit prope annum MCCCCL. Cusanus, nec non MDXXX. Bovillus. Summatoria Aequatio Cycloidis in X.

Schemate delineatae est $y = \sqrt{2ax - x^2} + S \frac{adx}{\sqrt{2ax - x^2}}$ po-

fitis

fitis Axe $AE = 2a$, $AC = x$, & ad Axem Normali $CB = y$. Cum igitur translato in B abscissarum initio, ac designatis variabilibus CP, five BQ, CP', vel BQ' elemento alphabetico z, factis $QA, Q'A' = v$, abscissam AC puncto dato B respondentem litera b denotando, atque c ordinatim applicatam CB, evadat Aequatio Trochoidis

$$y - c = v = \frac{(2a - b)z + (ab - 2a^2)z}{\sqrt{2ab - b^2} \cdot 2(2ab - b^2)^{\frac{3}{2}}} \&c.$$

&c. dum Aequatio conveniens Rectae SBT fit ex familiaribus regulis $v = \frac{nz}{m}$, ideo si ponatur $\frac{n}{m} =$

$$\frac{2a - b}{\sqrt{2ab - b^2}} = \frac{EC}{CL} = \frac{CL}{AC}, \text{ nimirum si Tangens quae-}$$

fit SBT fiat Chordae AL parallela non ambigetur semper esse in parte positivorum ordinatim applicatas Rectae Tangentis majores, & in parte negativorum minores ordinatim applicatis Cycloidis. Iisdem insistentes vestigiis facillime consequemur exposita qualibet Curva situs Maximorum vel Minimorum, Nodos, Puncta Conjugata, Flexus utcunque Anguineos visibiles aut invisibiles, Regressus diversae speciei, Oscula, aliasve affectiones, quae, cum a parallelismo, respectu habito ad Coordinatarum Axes, aut ab impossibilitate, & multiplicitate Tangentium pendeant, vel a Parabola osculatrice Apolloniana, caeterisque altiorum graduum Parabolis pariter osculantibus, differentiarum finitarum ope statim innotescunt si in communi puncto B veluti Coordinatarum principio comparentur datae Curvae, & Rectae Lineae, aut Parabolae Aequationes Locales, absque eo quod

itine-

XXXXVIII

itinere jam patefacto diutius immorari sit opus.

XXVIII. Infinitesimalis Algorithmi auspiciis faustioribus Logometria universa in immensum herculeo ferme conatu promota. Novo robore freti in arenam descenderunt Geometrae, & Seriebus ingeniosissime convergentibus usi pro summa Integralis Calculi utilitate Logarithmos Hyperbolicos seu Naturales computare sunt aggressi, quorum parvulum specimen extat in *Fluxionum Doctrina* Simpsonii, & in Eulerianae *Introductionis* Capite VII., ac plenior habetur collectio in secunda Trigonometriae illustris Toaldi editione anni MDCCLXXIII., & indefessi Volframi Hollandi dono ad quadraginta octo decimales figuras seu locos in nuperrimo anni MDCCLXXVIII. Tabularum Berolini a Schulzio editarum primo Volumine. Cito etiam alii Antilogarithmos addidere, nimirum Logarithmorum propemodum dixeris Reciprocis, aut Numeros Logarithmis praecognitis respondententes. Alii in Briggiano vel Tabulari systemate *Antilogarithmicum Canonem* complere tentarunt, quos inter primi emicuerant Vallisioteste Harriotus, Varnerus, atque anno MDCCXIV. Longius Oxoniensis, & MDCCXLII. adsertis a Logarithmo 1 ad Logarithmum 100000 undecim Decimalium Locis Jacobus Dodsonius, Nauticae nec non Physicae rei peritissimus. Fuerunt nonnulli, qui Hyperbolicos consimiliter Linearum Trigonometricarum composuere Logarithmos, qui Versos angulorum Sinus recens in Sphaericis disciplinis utillime adhibitos respectivis

ſpectivis Logarithmis ornarunt (*), qui in Sharpii methodo diligentiffime profecerunt Logarithmos Briggianos in lucem edentes a Numero 999980 ad 1000020, nec non qui Tabulas ſubſidiarias pararunt ad converſionem Briggiani Logarithmi cujuſlibet in Hyperbolicum, & denique Gardineri Tabularum Londinenſis editionis MDCCXLII., atque Avenionenſis MDCCLXX. errores quamplurimos in Facultatis Aſtronomicae commodum publici juris fecerunt. Ita Mathematicorum votis, qui Gregorii, & Mercatoris methodo locupletatam Logometriam maxime optabant, ſatis recentiorum labores feciſſe non eſt qui non ſentiat, vixque ullum adeſſe vacuum in Logarithmorum Collectionibus nova cura ſolertiaque complendum, niſi forte Computo Sexageſimali ſubſtituendam cenſuerint in poſterum Aſtronomiae praefertim Calculatoriae vel Arithmeticae operam navantes facilioris Computi Decimalis in Trigonometria ab ipſo Briggio antiquitus ſelecti rationem, veluti pluribus in locis, nec immerito, Analyſtas hortatur junior Joannes Bernoullius.

XXIX. Verum, quamvis mirum in modum progrediente Scientia Infiniti pari inceſſerit paſſu Logarithmorum doctrina, atque utiliorum Canonum Summa, ad quorum historicam deſcriptionem, ſi amplior deſideretur praefertim de Tabularum chronologia

g

gia

(*) Sinuum - Verſorum ope Triangulorum Sphaericorum Analyſin nuperrimam ſuſtineri neminem fugit. Eos in Shervinii Tabulis quoque invenies.

L
gia sistenda, consulere oportet vel quartam acutissimi
Kaeftneri Schedam Goettingensi *Memorabilium Astro-*
nomicorum Collectione anni MDCCLXXIV. insertam,
vel Scheibellii Numerum septimum periodici Ope-
ris *Introductionis in Bibliographiam mathematicam*, vel
Parisiensem *Temporum Notitiam* anni MDCCLXXV.,
haudquaquam tamen ullus quod sciam omnem Lo-
gometricam Theoricen a nuda Analyfi Cartesiana Se-
riebus *assumptitiis*, & Infinitis atque Infinitesimis pe-
nitus rejectis haurire molitus, nec rem Goniometri-
cam sublimiorem sive Transcendentium omnium quan-
titarum a Circulo pendentium facultatem ejusdem
methodi simplicitate haecenus protulit superstructam.
Analogiam Hyperbolae, ac Circuli utut facillimam,
& sola synthesi quoque a Vallisio in Commentario
de Conicis Sectionibus anni MDCLV. affecutam pri-
mus nitere persensit Joannes Bernoullius in transitu
Logarithmorum realium ad imaginarios, cum cele-
berrimum Problema solveret in *Memorabilibus Regiae*
Scientiarum Academiae Parisiensis anni MDCCII. de
omnigenarum Fractionum rationalium Integratione.
Hanc ipsemet auctor longius protraxit Circuli atque
Hyperbolae mirabilem cognationem in singulari Li-
psiensis *Eruditorum Diarii* ad annum MDCCXII.
Disquisitione de *Angulorum vel Arcuum indefinitis*
Multisectionibus, tantaque luce mutua utriusque
Curvae affectionum, sive symptomatum correlatio-
nem perfudit, ut Logometriae argumentum quo-
quomodo illustrari nunc nequeat quin sponte velu-
ti sua & coronidis instar promptius perficiatur res
univer-

universa Trigonometrica ob latentem anteacto faeculo, & labentis initio detectam Angulorum ac Rationum quasi diceret Harmoniam, in qua deinceps novis inventis amplificanda adlaborarunt Cotesius, Smithius, Maclaurinius, & Valmesleyus.

XXX. Dubium fortasse suboriri posset, an Leibnitius prae ceteris suboluerit Logarithmorum imaginariorum significationem dum in Actis Lipsiensibus, quae referuntur ad annum MDCLXXXII., de Circuli agens tetragonismo affirmavit aequationes *Analytico-transcendentes*, sive Exponentiales pro Circulo a se jamdudum repertas? Nullam etenim datur assignare formam Aequationis Exponentialis omnia Circularis peripheriae puncta *localiter* exhibentis nisi Sectores aut Arcus Circuli imaginarii in computum veniant; quapropter saltem implicite elegans illa inter Circulum atque Hyperbolam analogia tunc temporis Leibnitio obversata summo jure videtur. Revera Aequationes $a^{2x\sqrt{-1}} - 2 \text{Cosin. } (x \text{ Log. } a)$ $a^{x\sqrt{-1}} + 1 = 0$, sive potius $C^{2x\sqrt{-1}} - 2\sqrt{-1}$. Sin. $x. C^{x\sqrt{-1}} - 1 = 0$, vel hisce simillimas, quas Capite V. praesupposita tantum Binomii Formula evolvam, cum Leibnitianis analytico-transcendentibus Aequationibus consentire, tametsi ignotae sint, necessario fatendum.

XXXI. Circularis Circumferentia quantitatum transcendentium equidem simplicissima numeris rationalibus adeo ingeniose coarctata subjectaque fuit,

ut vix dici possit Mathematicorum spem elusisse frustra omni aevo tentatam Circuli geometricam Quadraturam. Proportio namque ingenti conatu prolata anno MDCCXIX. ab optime de Cyclometria merito praestantissimo Lagnyo inter *Memorabilia Parisiensis Scientiarum Academiae* approximatione verae rationi tam consona aream Circuli perhibet, ut nedum praecedentes Petri Metii, Adriani Romani, Ludolphi Colonienfis, Abrahami Sharpii, ac Machinii stupendos labores, sed & imaginationis vim ita superet, ut veritatem ipsam mentiatur. Quidnam etenim accuratius in Matheseos praxi, vel in Peripheria Circuli numeris exprimenda postularent Geometrae rationem nacti Numerorum 1 atque 3.1415926535

89793238462643383279502884197169399375105
82097494459230781640628620899862803482534
21170679821480865132723066470938446 + ac-

7 —

qualem rationi Diametri ad memoratam Peripheriam, differentia existente minori parvulae & humano conceptu inassequendae fractionis decimalis valore ab Unitate cum additis dextrorsum cyphris centum viginti septem denominandae? Hic autem observandum velim appositos asteriscos denotare servato temporum ordine inventos Rationis illius limites ante Lagnyum a Francisco Vieta, Adriano Romano, Ludolpho, Sharpio, & Machinio, quemadmodum latius patebit si consulantur Collectio Operum Ma-

thema-

thematicorum Vietae anno MDCXLVI. recensente Francisco Schootenio publici juris facta, *Tabulae Mathematicae* a Sharpio editae, atque nuperrimarum Berolinensium Volumen II., *Synopsis Palmariorum Jonesii*, & *Historia tentaminum pro obtinendo Circuli Tetragonismo* a diligentissimo Montucla anno MDCCLIV. descripta. Silentio tamen vovendum non censeo, ne minus cauti in graves quandoque errores incidunt, Numeri paullo ante assignati 3.141 &c. figuras aut locos nonum supra vigesimum, atque trigesimum tertium puncto distinctos duplicem ostendere naevum Articuli, cui nomen *Quadratura* in Encyclopedico Dictionario, cum prioris vice habeatur 8, nec in altera Nota subsistat Geometrae Colonienfis ut ibi asseritur adproximatio, sed trium decimalium additione ulterius progrediatur; fallaciaque consimili deformatam scribendo pro Senario Quinarium in Proemio *Tabularum Shervinii* quintae Editionis figuram numericam ordine tertiam supra septuagesimam puncto pariter a caeteris Notis opportune sejunctam monendum non inutile duco. Proportione jam cognita Circularis Circumferentiae ad Diametrum facillime fluit ratio Circuli ad circumscriptum Quadratum, scilicet ea, quae Numerum 0.7853981633974483096156608458198757210492923498437764552437361480769541015715522496570087063355292669955370216283180766617734612 & Unitatem intercedit, inexactasque nimium esse post decimam Notam pronum erit inde deducere proportionem a Fergussonio datas in *Selectis Exercitationibus* 1: 0.88622692545276,

nec

nec non 1: 1.12837916709551 Diametri ad Lat-
tus Quadrati Circulum aequantis, ac vicissim Late-
ris Quadrati Circulo aequalis ad respondentem Dia-
metrum.

XXXII. Nascente Serierum Infinitarum theo-
ria (*) Arcus statim Circulares, Circulique Secto-
res, Segmenta &c. per Trigonometricas Lineas, Si-
nus, Cosinus, Versos - sinus, Tangentes, Cotangen-
tes, Secantes, Cosecantes, Chordas, terminis sine
numero plus minusve convergentibus expressa fuere,
Lineaeque ipsae Trigonometricae inversa Serierum
methodo Circuli Arcubus, eorumque Potestatibus
determinatae. Serierum, quas memoro, inven-
tio Trigonometriae sublimioris veluti epocham fi-
stit; verum, quamvis hac in arena certaverint si-
ne plagii suspicione Triumviri illustres Neutonus,
Iacobus Gregorius, atque Leibnitius, consentiunt
omnes praeivisse Neutonom, primumque anno
MDCLXVIII. Gregorium a se reperta edidisse in
Geometricis Exercitationibus, & saltem Serierum ele-
gantissimam Leibnitianae pro Circuli quadratura a-
damussim coincidentem ante annum MDCLXXIV.
jam Collinsio communicasse. Quidquid sit de pole-
mica Serierum circularium inventionis historia dubio
caret tunc temporis rei universae Trigonometricae
doctrinam, quae paucis exceptis elementaribus Trian-
gulorum Theorematis a Pitisci, Rhetici, Regio-
mon-

(*) Encyclopediae Editores Mercatori adscribunt in Articulo *Seriei*
inventionem Serierum Infinitarum. Fortasse publicationem intelligunt.

montani, Valentini Ottonis laboribus ad Neutronum usque nil aliud fuit nisi Tabularum ingenti Calculi vi acquisitarum congeries, Analyfi Algebraicae feliciter subiectam, ideoque ipsius pomocria supra descriptionem quamlibet progressu celerrimo amplificata. Nec minus Trigonometriae ulterius perficiendae contulerunt novae Infinitesimalis Algorithmi regulae, cum hoc duce instrumentoque potentiori licuerit Series majori convergentia donatas inventis addere, qua in segete defudarunt praesertim Stirlingius in *Tractatu de summatione & interpolatione Serierum*, Maclaurinius in *Fluxionum Theorice*, Thomas Simpsonius partim in *Fluxionum Doctrina*, partimque in *Mathematicarum Dissertationum Collectione*, & Leonardus Eulerus vertentibus annis MDCCXXXVI. VII. in *Petropolitanae Academiae Commentariis*.

XXXIII. Hinc oritur ratio, ob quam non immerito asseruerit Leibnitius Anno MDCXCII. in *Diario Eruditorum Lipsiensi* ope novarum methodorum Trigonometriam canonicam Algebrae penitus restitutam, & ferme a Tabularum antiqua necessitate liberatam: illi suppetere etenim viderat ex Infinitis Seriebus, de quibus fit sermo, tantam subsidiorum foecunditatem, ut singula absque Tabulis Trigonometrica Problemata resolvere Geometrae facile possent. Imo post fata Joannis Bernoullii, qui primus pereximio Schediasmate Lipsiensibus Actis anni MDCCXXII. inserto sub nomine *Theorematis novi pro dividendis multiplicandisque Angulis* ausit, rem hominibus improbam! Infinitum instar Finiti versare

in

in adipiscendos Sinuum, Cosinum, Tangentiumque valores praecognitis Arcubus, Leonardus Eulerus non tam Bernoullianas meditationes ditans exornansque, quam Trigonorum Rectilineorum, ac Sphaericorum Scientiam creato Calculo Sinuum, Cosinum, & aliarum Linearum a Circulo pendentium promovens veluti summus Trigonometricae Facultatis instaurator docuit iisdem Algebrae regulis, atque speciebus cohiberi posse quae antea videbantur respuere Cartesianam, & Infinitesimalem Analysin. Nova haec doctrina ob ejus supremam utilitatem in Coelestis Physices Neutoniana Theoria, videlicet ad errantium Corporum mutuas perturbaciones Tabularibus Aequationibus comprehendendas, Geometrarum ingenio quasi aemulantium citissime exculta insigni obtinuit incrementum ex quo, prouti ostendunt *Memorabilia Berolinensis Academiae* anni MDCCXLVI., Imaginarias quantitates omnigenas Realium instar in universali Magnitudinum Scientia adhiberi equidem posse maxima nobilitate ac facundia Dalembertius explicuit.

XXXIV. Tot tantisque suffultam praesidiis Circularem ac Sphaericam Geometriam quanto certatim conatu locupletaverint Synthesis atque Analyseos cultores quisque per se conjicere valet. Turpe nunc esse indemonstratas Sphaericorum Propositiones Aenigmatum instar longius detineri quilibet etiam non monitus intelliget. Sic, qui demonstrare suscepit elegans illud Sphaericorum Theorema a Matheoseos Historiographo ad calcem primi Voluminis sine demonstratione prolatum, gravisque usus

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in Gnomonice dum Lineas Horarias Italico seu Babylonico more describere animus sit, nimirum fore Lineas praedictas quocunque in Plano Horologii Sciotherici descriptas cujusdam Conicae Sectionis tangentes, votum facile impleret ratiocinatione subnexa, vel altera paullulum ab ea discrepante. Perspicitur in Schemate XI., quod omnes Circuli Sphaerae maximi, veluti OSRI, ABCD &c., quos inter rationalis Horizon ipse EQLD, tangentes Circumferentiam majoris Circuli circumpolaris EHGT, ideoque & oppositi sub Horizonte jacentis LYCR, tam istas, quam Circuli Aequatorii ZBSX, eique Parallelorum universorum VMK, V'M'K' proportionaliter secare. Exemplum praebeant eadem methodo caeteris aptandum Circumferentiae EQLD, OSRI, ideoque per Axem PF ducti intelligantur Declinationum Circuli PEVF, POUF, ductaeque Chordae EM, OΛ, MV, ΛU, EV, OU, quarum postremae protractae sint, ut in eodem puncto Δ cum protenso quoque Axe conveniant, prouti constat ex Elementis. Hoc praesupposito evidens est non modo Arcus EV, OU, ipsorumque Chordas inter se aequari, verum etiam ex natura situs Diametrorum ENL, ONR in Coni - Recti superficie circa Axem PNF, positionisque tangentialis Peripheriae EQLD in OSRI promotae semper aequari Arcus, Subtensasque EM, OΛ, & Angulos rectilineos ΔEM, ΔOΛ, five eorum Supplementa VEM, UOΛ; qua de causa in rectilineis Trigonis VEM, UOΛ reperietur fore Latus $VM = UA$, necnon Arcum $VUM = UMA$, sub-

LVIII

blatoque communi UM esse $MA = VU$, nimirum similem Arcui Circumpolaris EO . Patet itaque proportionalis sectio Peripheriae Circumpolaris praefatae, omniumque Parallelorum, ob quam affectionem nullus dubitabit, quod, si Circumpolaris $EHGT$ Peripheria dividatur in pares Arcus vigintiquatuor, perque omnia divisionis confectae puncta transeant quatuor supra viginti Circuli Maximi Tangentiales, ii munus gerant Circulorum Horariorum in Systemate computationis horarum a Babylonensibus, & Italicis Populis adhibito eas numerantibus a Solis ortu vel occasu Astronomico, seu potius ab ortu vel occasu taliter Crepusculari, ut aequali semper distet temporis intervallo ab ortus, aut occasus puncto sive momento Astronomico. Sed universa Plana Circulorum Tangentialium, quorum fit mentio, contingunt in Lateribus ENL , ONR , ANC Superficiem Conorum invicem oppositorum, qui apicem habent in Sphaerae Centro N , ac pro Basibus Circulos Circumpolares praedictos: Planum igitur Horologii Gnomonici quomodocumque secans Horarios Circulos Conicam Superficiem tangentes habebit pro communibus Planorum Sectionibus Lineas rectas Horarias Babylonenses - Italicas, quae Conum tangent, ideoque & Conicam Curvam ab intersectione Coni, Planique genitam, in quo Horologium depictum.

XXXV. Qui alacriori imaginandi facultate praediti Plano - Sphaerica statim concipiunt optimam segetem ex jam absoluto Theoremate colligere possent varios percurrentes casus pro diversa Sphaerae
Coele-

Coelestis obliquitate, contemplantesque in Recta Sphaera Horarios Circulos Babylonicos, vel Italicos cum usitatis Declinationum confundi, in Parallela cum Horizonte Rationali; necnon jucundam Sphaerae Frusti aequalibus Basium Circulis minoribus insistentis genesim derivarent supposito motu integri Circuli Maximi non rotatorio, ut penes Archimedem, sed turbinatorio, videlicet instar turbinantis rectae Lineae, Rectangulique motus, quo Superficies, ac Soliditas Frusti Cylindroidis Hyperbolicae Vrennio inventore per antea cognita gignitur. Majorem autem curam operae pretium esset impendere in adsequenda resolutionis pulcherrimi, utillimique Sphaerici Problematis veritate, quod recens exponitur in *Cometarum Specimine* Dionis Sejourii, cujus *Analyticarum Methodorum* artificio, & incomparabilis solertiae atque assiduitatis exemplo magnum debet incrementum tota res Astronomica, utpote novo perfectiore organo locupletata, & dignitati Geometriae restituta.

XXXVI. Secundam Problematis partem ad Longitudinem, Latitudinemque Astri cujuslibet C in XII. Schemate ex datis ejusdem Ascensione - Recta, & Declinatione cognoscendam Aequatio facilis continet $\text{Cofin. lat.} \times \text{Cofin. long.} - \text{Cofin. declin.} \times \text{Cofin. ascens.} = 0$, quae apertissime ostenditur veritati quammaxime adhaerens. In utrovis enim Triangulorum Rectangulorum Sphaericorum IAC respectu Aequatoris RIP, & IBC relate ad Eclipticam FIE liquet adesse Analogias $\text{Rad} : \text{Cofin. IA} :: \text{Cofin. CA} :$
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Cofin.

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Cofin. IC, atque Rad : Cofin. IB :: Cofin. BC : Cofin. IC, unde eruitur Cofin. IA $\times$ Cofin. CA = Cofin. IB $\times$ Cofin. BC ad sensum Auctoris.

XXXVII. Quo ad aliam vero difficiliorem partem demonstrandam altius quaedam repetenda sunt Sphaericorum Theoremata. Et primum in Triangulo CST habetur analogia Sin. ST $\times$ Sin. SC : Sin.

$$\left(\frac{CS + CT - ST}{2} \right) \times \text{Sin.} \left(\frac{ST + CT - SC}{2} \right) :: R^2 : R^2 -$$

$$\text{Cofin.}^2 \left(\frac{90^\circ + IA}{2} \right); \text{qua de re posito simplicitatis causa}$$

Radio $R = 1$, applicatisque Sinus summae Arcuum notissimo canone proportio in alteram mutabitur

$$\text{Sin. ST} \times \text{Sin. SC} : \left(\text{Sin.} \frac{CT}{2} \times \text{Cofin.} \left(\frac{SC - ST}{2} \right) + \right.$$

$$\left. \text{Cofin.} \frac{CT}{2} \times \text{Sin.} \left(\frac{SC - ST}{2} \right) \right) \left(\text{Sin.} \frac{CT}{2} \times \text{Cofin.} \right.$$

$$\left. \left(\frac{SC - ST}{2} \right) - \text{Cofin.} \frac{CT}{2} \times \text{Sin.} \left(\frac{SC - ST}{2} \right) \right) :: 1 : 1$$

$$- \text{Cofin.}^2 \left(\frac{90^\circ + IA}{2} \right), \text{vel potius in faciliorem Sin.}$$

$$\text{ST} \times \text{Sin. SC} : \text{Sin.}^2 \frac{CT}{2} \times \text{Cofin.}^2 \left(\frac{SC - ST}{2} \right) - \text{Cofin.}^2 \frac{CT}{2}$$

$$\times \text{Sin.}^2 \left(\frac{SC - ST}{2} \right) :: 1 : 1 - \text{Cofin.}^2 \left(\frac{90^\circ + IA}{2} \right), \text{nimi-}$$

$$\text{rum Sin. ST} \times \text{Sin. SC} : \left(\frac{1 - \text{Cofin. CT}}{2} \right) \left(\frac{1 + \text{Cofin. (SC - ST)}}{2} \right)$$

$$- \left(\frac{1 + \text{Cofin. CT}}{2} \right) \left(\frac{1 - \text{Cofin. (SC - ST)}}{2} \right) :: 1 : 1 - \text{Cofin.}^2$$

$$\left(\frac{90^\circ + IA}{2} \right), \text{aut Sin. ST} \times \text{Sin. SC} : \text{Cofin.} \left(\frac{SC - ST}{2} \right)$$

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— $\frac{\text{Cofin. CT}}{2} :: 1 : 1 - \text{Cofin.}^2 \left(\frac{90^\circ + \text{IA}}{2} \right)$. Ex hac analogia deducitur immediata Aequatio — $\frac{\text{Cofin. CT}}{2} = \text{Sin. ST} \times \text{Sin. SC} - \frac{\text{Cofin. (SC - ST)}}{2} - (\text{Sin. ST} \times \text{Sin. SC}) \text{Cofin.}^2 \left(\frac{90^\circ + \text{IA}}{2} \right)$, five $\frac{1 - \text{Cofin. CT}}{2} = \frac{1 - \text{Cofin. (SC - ST)}}{2} - (\text{Sin. ST} \times \text{Sin. SC}) \text{Cofin.}^2 \left(\frac{90^\circ + \text{IA}}{2} \right)$, feu $\text{Sin.}^2 \frac{\text{CT}}{2} = \text{Sin.}^2 \left(\frac{\text{SC} - \text{ST}}{2} \right) - (\text{Sin. ST} \times \text{Sin. SC}) \text{Cofin.}^2 \left(\frac{90^\circ + \text{IA}}{2} \right)$, quae in proportionem versa exhibet $\text{Sin.} \left(\frac{\text{SC} - \text{ST}}{2} \right) : \text{Sin.} \frac{\text{CT}}{2} :: 1 :$

$$\frac{\sqrt{\text{Sin.}^2 \left(\frac{\text{SC} - \text{ST}}{2} \right) - \text{Cofin.}^2 \left(\frac{90^\circ + \text{IA}}{2} \right) \text{Sin. ST} \times \text{Sin. SC}}}{\text{Sin.} \left(\frac{\text{SC} - \text{ST}}{2} \right)},$$

videlicet $\text{Sin.} \frac{\text{CT}}{2} = \text{Sin.} \left(\frac{\text{SC} - \text{ST}}{2} \right) (\text{Sin. B})$, qui Arcus B habeat pro Cofinu

$$\frac{\sqrt{\text{Cofin.}^2 \left(\frac{90^\circ + \text{IA}}{2} \right) \times \text{Sin. ST} \times \text{Sin. SC}}}{\text{Sin.} \left(\frac{\text{SC} - \text{ST}}{2} \right)}, \text{ veluti ada-}$$

missim, pleniusque consentit cum expressionibus a Sejourio prolatis ab Ascensione-Recta, Declinatione, ac Eclipticae Obliquitate tantum pendentibus, & in Astronomiae Sphaericae commodum, dum de re
Pla.

Planetaria & Cometographia agatur, recentiori praxi adjiciendis (*).

XXXVIII. Multa, quae Sphaericam respiciunt Astronomiam, peculiari disquisitione limanda, perficiendaque forent, cum saepe vel Formulae occurrant graviores demonstratione carentes, vel nimis adhuc perplexae majorem simplicitatem desiderent, vel diversis expositae modis vix eodem ducere videantur, vel denique in Coelestis Scientiae ditissima penu ignotae, neglectaeque delitescant. Jam in adolescentiae curriculo Astronomicae meditationi fuetus varias Aequationis Altitudinum Solis, quemadmodum vocant, *correspondentium* ad verum Meridiei punctum determinandum expressiones in unam redigere facile potui (**), nonnullaque evolvere Maxima, ac Minima in Sphaericis scitu sane non injucunda. Memini haec versantem eo temporis demonstrasse Synthesi facillima Infinitesimali juxta doctrinam Archimedaeam mirabile illud Theorema de Areae cujuscunque Trigoni Sphaerici dimensione, ignarum fundamenta demonstrationis inventae jecisse Leibnitium in *Lipsiensibus Commentariis* anni MDCXCII., & opus fere absolvisse in *Fluxionum Applicatione* Thomam Simpsonium. Nunquam autem fervida illa aetate, quod etiam nunc memoria repetam, oblectamentum coepi tanto imaginationis impetu ac voluptate favente conditum

(*) Leguntur praedictae Formulae Trigonometricae ad laudati Operis pag. 217. Editionis MDCCLXXV.

(**) Vid. Volum. III. pag. 337. *Collectionis in usum Astronomorum* ab Joanne Bernoullio editum Berolini anno MDCCLXXVI.

dítum, quam novam detegens in Sphaericorum doctrina correlationem Logísticas inter, atque Hyperbolas a Curvarum Loxodromicarum theoria derivatam. Libet hanc paucis explanare, ut eo magis constet de divitiis undequaque latentibus in geometrica Astronomiae Sphaericae, Scientiarumque affinium supellectile, excitenturque ad majora deinceps invenienda qui ab Jove summo meliorem nacti fortunam has disciplinas in deliciis habent, ipsarumque cultui potioribus vacare possunt auspiciis.

XXXIX. Loxodromiam in Nauticis rebus utilissimum Theorematum fontem quando stereographice projiciatur super Aequatoris Planum ibidem describere Logarithmicam Spiralem, dummodo oculus maneat in Polorum alterutro, Hallejus ostendit anno MDCXCVI.; idque patet si consulantur *Philosophicae Transactiones* ejusdem anni, & praecedentis in unum Volumen collectae. Vertente insuper anno MDXCIX. Eduardus Wrightius Mapparum, quas *reductas* nuncupant, primus equidem author crescentium Latitudinis Graduum insinuatione viam patefecerat, qua in nauticos usus facta recta Rhumborum lege Globi Terraquei superficies consideraretur instar rectae Cylindricae superficiei longitudinis infinitae, & super Aequatorem normaliter insistentis. His praesuppositis omnes norunt Loxodromicam in Sphaera Curvam tunc evadere Cochleam Archimedaeam circa Cylindrum contortam, cujus Spirae Cylindri latera veluti Meridianos Circulos sub aequalibus angulis secant, suntque Lineae brevissimae duo quaelibet inter superficiei Cylindricae

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LXIV

lindricae puncta jacentes. Accidit autem in nunc recensenda Loxodromiae artificialis transformatione Spiralem Cylindricam, cujus Projectio facilis Orthographica est Circulus, stereographice projectam, dum oculus ubicunque subsistat in Axe Cylindri, Spiralem Hyperbolicam loco Logarithmicae gignere, adeo ut in Superficie Circulis referta valeant Helices Logarithmicae, ac Hyperbolicae in altera rectis quasi Lineis contexta.

XL. Et re quidem vera sint Projectionis Stereographicae (*) Polus Y in Axe recti Cylindri indefiniti SOT constitutus, Baseosque Circulus AM NCB, superiusque, & inferius circumvoluta Helix Cylindrica Archimadaea GFEDBHKL in Schema-
te XIII. Selectis duo Helicis punctis P, R ducantur Cylindri Latera PC, RN, Radii OC, ON, & Axi Normales PQ, RT, nuncupenturque arcus BC, BN &c. x; OΔ, OZ &c. y; OB, OC, ON &c. a; OY, b; ac ratio constans BC : CP, seu BN : NR &c. = m : n, ex qua fluit simplex Aequatio CP, vel $OQ = \frac{nx}{m} = NR$, vel OT. Triangula autem similia YPQ, YΔO, aliaque YRT, YZO praebent analogiam $b + \frac{nx}{m} : a :: b : y$, quapropter Aequatio Localis Projectionis Stereographicae BΔZO erit $(b + \frac{nx}{m})y = ab$, competens Spirali Hyperbolicae.

XLI

(*) Schema inspiciatur attentius.

XLI. Fiat itaque Circulus Cylindricae Basi concentricus Radio $\frac{n}{m}$. OB, & Arcus a puncto ω vel ω' supputati exhibebunt $\frac{nx}{m}$, sive simplicius z , evadetque Aequatio $(b + z)y = ab = \frac{n}{m} \cdot a \times \frac{m}{n} \cdot b$, ac rursus ponendo Abscissarum initium in puncto Θ , vel Θ' , ita ut Arcus $\omega\Theta$, vel $\omega'\Theta' = YO$, seu b , ac $b + z = u$, denique obtinebitur simplicissima Aequatio $uy = \frac{n}{m} a \cdot \frac{m}{n} b$. Punctum concursus Circumferentiae-abscissarum, atque Helicis Φ , sive Φ' reperietur secundo Arcum $\Theta\Phi$, $\Theta'\Phi'$ aequalem $\frac{m}{n} b$; Ordinatum applicata longitudinis infinitae erit Radius protractus $O\Theta$, $O'\Theta'$; Subtangens constans aequabit $\frac{m}{n} b$; atque Asymptotos in Plano Helicis jacens summa facilitate elicietur ducendo rectam $OX = \frac{m}{n} b$ normaliter ad $O\Theta$, $O'\Theta'$, & $X\Sigma$ perpendicularem OX . Innumera puncta B, E, G, seu , D, F, supra Circulum Projectionis in eodem Cylindri Latere sita Radios projiciunt eadem in Recta dispositos, aequalesque $\frac{ab}{u}$, $\frac{ab}{\Pi + u}$, $\frac{ab}{2\Pi + u}$, $\frac{ab}{m\Pi + u}$, ubi Π Peripheriam integram Circularem significat Radio gaudentem $O\omega$, $O'\omega'$, vel $\frac{n}{m} a$. Loxodromiac, vel Cochleae Cylindricae angulo semi-

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recto

LXVI

recto distinctae Projectio habebit b Subtangentem ; b distantiam Asymptoti a Centro ; b Arcum consulto secatum pro inveniendi principio Abscissarum, quae tum numerantur in Circuli Baseos Cylindri Peripheria. Demum si a Polo Y elevetur $Y\Xi$ Normalis Axi Cylindri tali positione, ut occurrat Helici in Spirae puncto Ξ , haec Perpendicularis producta projiciet super Basim infinitae longitudinis Ordinatam centralem $O\Psi$, quae vicem gerit Axeos primarii Helicis Hyperbolicae ; nam ex parte Abscissarum negativarum ipsamet Curva paribus magnitudine Spirae circa Centrum O infinities revolutis, sed inverso situ contortis renovatur. Profecto hujus symptomaticis veritatem etiam genesis ista stereographica clare patefacit, eoquod indefinita Spiralis Cylindrica inferior ΞKL projiciat negativam Helicem Hyperbolicam, cujus geminum simul Ramum Joannes Bernoullius Reciprocam Helicem nuncupavit, dum Cylindrica apte vocaretur Directa exemplo Spiralis notissimae in Plano positae, & ab immortali Syracusano Geometra post acutissimum Cononem tot tantisque inventis ornatae.

XLII. Elegantiores has affectiones qui diligenter perpenderit multa in Geometriae penetralibus adhuc recondita, nec fortasse utilitatis expertia contineri non inficiabitur. Verum omnes carpere flores nec decet, nec libet, cum argumenti solummodo occasione ferente nonnullas obiter schedas evolverim in juvenilibus Studiis conscriptas, aliaque compendiosius attigerim, quae non satis perpolita premo amicae olim Palladis munere optato otium expectans.

EXPO-

EXPONENTIALIUM LOGARITHMORUM ET QUANTITATUM TRANSCENDENTIUM

QUAE A CIRCULO PENDENT THEORIA.

C A P U T I.

ANIMADVERSIONES IN FORMULAM BINOMII NEUTONIANI.

1. **C**UM universalis expressio a Newtono tradita pro exaltatione cuiuslibet Binomii, seu potius Polynomii ad Potestatem gradus indefiniti sit totius Commentarioli huiusce Analytici fundamentum incipere oportet ab ipsa Formula vindicanda: mancam etenim mutilamque non modo, verum etiam aliquoties fallacem depraedicant plurimi Institutionum Scriptores, immo et nonnulli summos inter Geometras iure merito adsciti. Istae, quas primum complector, vindiciae ad Formulam Newtonianam sartam tectam integre servandam parient saltem aliquam, utcunque exiguam, in Analyfi Finitorum novitatem, si forte omnis subnexae Exercitationis rueret utilitas.

2. In Algebrae Elementis passim ostenditur terminorum aequipollentium ope veritas Formulae Binomii Newtoniani donec de Potestatibus omni Exponente rationali, integro vel fracto, positivo seu negativo, praeditis agatur, adeo ut operae pretium non ducam iam toties repetita retexere. Plurimi vero interest demonstrationem ipsius Formulae subiungere casu, quo Exponentis Potestatis Binomii sit irrationalis sive transcendens, positivus vel negativus; ab hac etenim pender latissimus eiusdem Formulae usus in quantitatibus Exponentialibus evolvendis. Nec minoris existimo Formulam Newtonianam tueri a ceteris gravioribus naevis, qui eius universalitatem labefacerent, veritatemque minuerent, veluti quod incompleta unicum praebeat

valorem, et Functionis uniformis characterem mentiatur quando plures valores multiformium Functionum instar exhibere deberet; quod imaginarietatis vitium abscondat, realemque fallax se monstret tum etiam, cum imaginaria fiat, ideoque impossibilis; quod veram Radicem perfectae Potestatis aliunde praecognitam nullo modo restituat; saepiusque in falsam, seu divergentem Seriem numero terminorum infinitam, videlicet in Seriem Analyfi penitus inutilem transformetur.

3. De Potestatibus irrationali Exponente affectis Binomii Formulae subiiciendis pauculae prostant considerationes; de iis autem, quae transcendenti Exponente distinguantur, altum fere ubique silentium: vix etenim vires suas sunt experti Analystae circa istas Neutonianae expressioni cedendas. Quod attinet ad Potestatum irrationali Exponente praeditarum contemplationem Geometras omnes antecessere Vallisius, cum in *Arithmetica Infinitorum* non obscuram fecerit mentionem Potestatum $A^{\sqrt{3}}$, atque universalius Neutonus, cum in Epistola anno MDCLXXVI. ad Oldenburgium mis-

sa quantitates proposuerit formae $(x^{\sqrt{2}} + x^{\sqrt{7}})^{\sqrt{\frac{2}{3}}}$, quas in *Fluxionum Tractatu* non recte adserit Maclaurinius a Leibnitio nuncupatas Exponentiales. Hic etenim potius illas vocavit Interscendentes, ut ab Exponentialibus distinguerentur, quibus postremis deinde Christianus Hugenius nomen dedit Hypertranscendentium, seniorque Ioannes Bernoullius verbo quasi ludens alterum apposuit Hypotranscendentium: Leibnitianae nimirum voci excusa fuisse utraque nomina valde affinia, ac fortasse memoratae ambiguitatis origo. Post Neutoni Epistolam integer prope saecularis cyclus emensus antequam Dalembercius suspicione excitatus labilis nimium analogiae usque tunc adhibitae novam demonstrationem indigitaverit pro adstruenda Formulae Binomii Neutoniani fide etiam in unica hypothese Exponentium irrationalium; quod luculenter evincit Articulus vocabuli *Exponents* Dictionario Encyclopedico insertus. Adest ibidem, compendioque emicat Rationum Limitis theoricæ methodo a Neutono lata in *Principiis Philosophiae Naturalis*, a Varignonio contra Rollium scribente initio vertentis saeculi deinceps promota, atque a Dalembercio ipso in *Encyclopedia, et Miscellaneis* nitide admodum perpolitata ad Differentialis praesertim Calculi metaphysicæ illustrandam. Hac facem ferente nullum amplius inductionis argumentum, in rerum natura explicanda

canda saepe fallax, et rigidiorum canonum defectu superioris aetatis Geometris Keplero in *Stereometria*, Guldino in *Centrobarycis*, Vallisio in *Arithmetica Infinitorum*, aliisque tam familiare, semperque Mathesi indecorum, mereretur in medium prodire, cum ad Exponentialia Indice quolibet variabili non tantum irrationali, verum et transcendente gaudentia universim tractanda regulae quoque non desint inconcussae, quae geometricae veritati ab inductione haecenus latam iniuriam vindicare valeant.

4. Iisdem insistendo Dalembertii vestigiis Gregorius Fontana Cisalpinos inter Geometras summa tenens demonstrationem numero secundam concinnavit in Schediasmate sexto Voluminis V. ab Academia Physicorum Senensi anno MDCCLXXIV. editi pro tuenda evolutione Newtoniani Binomii indicem irrationalem habentis ex eo, quod irrationalis Exponens sit rationalium fractorum inassignabiliter discrepantium veluti Limes, obiterque primus ausit monere in Scholio ad calcem posito eadem Binomio accommodari posse transcendente etiam Indice praedito. Silentio praetereo priorem Cl. Authoris demonstrationem, ubi liquet veritatem exaltationis Binomii

$(a+x)^{\sqrt[n]{p}}$ ex Calculo Differentiali esse petitam instar methodi iamdudum adhibitae in variis Seriebus enucleandis, et in ipsa evolvenda Binomii Formula, hypothese facta cuiuscunque Indicis rationalis. Involvit etenim theoretica haec demonstratio Series assumptitias, et praesuppositam universalitatem etiam quoad Indices irrationales regulae differentiationis Potestatum, nimirum $d(a+x)^{\sqrt[n]{p}} = \sqrt[n]{p} \cdot (a+x)^{\sqrt[n]{p}-1} dx$, qui canon meo sensu, dum Exponens in irrationalitatem impingat, paulisper infirmatur, ac nisi analogia duce tuto et fideliter ostendi nequit per haecenus demonstrata (*).

5. Omnia, quae de Binomio inferius affirmantur, eadem de Polynomio, immo et Infinitinomio dicenda intellectum velim; nam Polynomium quodcunque $\pm x \pm b \pm y \pm z \pm d \&c.$ in Binomium verti $\pm x \pm u$ facili substitutione $\pm b \pm y \pm z \pm d \&c. = \pm u$ non est qui non videat, ideoque compendii solummodo gratia binomialis forma feligitur.

6. Binomia ad Potestates cuiuslibet Indicis exaltanda vel reapse Numeri sunt rationales, irrationales, transcendentes, vel recepto iam more sub

(*) Huius moniti quaeso Lectores meminerint, dum numerum 341. sunt perlecturi.

alphabeticis speciebus numerice interpretanda, et ad implicitam Unitatem referenda. Idem valet de Potestatum Indicibus, sive Exponentibus; quapro-

pter $(\pm y \pm z)^x$ rite explicatum mutabitur in $\left(\frac{\pm y \pm z}{c}\right)^{\frac{x}{b}}$, posita c homogenea Unitate Nominum Binomii, atque b homogenea Indicem x utcumque fractionum Unitate, quas Unitates subintellectas in sequentibus perpetuo negligam. Exaltatio etenim quantitatum cuiusque generis ad Potestates perfectas, sive imperfectas nihil est aliud quam multiplicatio, ac divisio latissimo sensu conceptae. Erunt itaque y, z, x Numeri, non semper communibus arithmetice cyphris expressi, sed quandoque Signis irrationalitatis, vel transcendentalibus notandi, dummodo Unitates c, b solita cyphra 1 maioris simplicitatis gratia expositae censeantur.

7. Dato quolibet Binomio $\pm a \pm b$, vel $\pm a \pm z$, seu $\pm x \pm y$ poterit semper in alterum redigi, cuius primus terminus sit Unitas positiva, nimirum $1 \pm d$, sive $1 \pm u$; atque licebit positivi valoris Binomium quodcumque considerare, cum de ipsius Potentiatione sit sermo. Et re quidem vera, si Aequatio numerica primi gradus, et resolutionis undequaque possibilis instituat $\pm a \pm b = 1 \pm d$, habebitur $\pm d = \pm a \pm b - 1$, atque eadem ratione $\pm u = \pm a \pm z - 1 = \pm x \pm y - 1$. Quando autem contingat esse $1 - d$, seu $1 - u$ in redacto Binomio Numerum negativum, tunc viceversa erit $d - 1$, vel $u - 1$ Numerus positivus, quo nuncupato f , seu v , divisioque in duo Nomina $1 \pm e$, sive $1 \pm t$ obtinebitur $(1 - d)^x = (-1)^x (1 \pm e)^x$, et $(1 - u)^x = (-1)^x (1 \pm t)^x$. Nec mirum in postremi casus evolutione suboriri Factorem $(-1)^x$; namque implicitus Factor $(+1)^x$, vel 1^x latet etiam in ceteris omnibus Binomiis valorem positivum habentibus, adeo ut $(1 \pm d)^x$, et $(1 \pm u)^x$ idem sint ac $1^x (1 \pm d)^x$, atque $1^x (1 \pm u)^x$, quod sedulo memorandum hic moneo. Numerorum inventio e , t nulla difficultate laborat, quum sit $\pm e = \pm a \pm b - 1 = d - 1$, necnon $\pm t = \pm a \pm z - 1 = \pm x \pm y - 1 = u - 1$, quemadmodum facillime patet.

8. Duos inter diversos finitae magnitudinis Numeros Rationales, positivi seu negativi valoris, utcumque proximos, concipi semper possunt sine fine existentes alii Numeri aequae Rationales. Sint enim datorum Rationalium maior m , minor n , differentia $m - n$; hac bisecta, rursusque biparti-

partita, & sic deinceps absque ullo limite, habebitur Rationalis $\frac{m-n}{2^r}$ quolibet assignabili, ac perceptibili Numero minor. Istam itaque differentiolam, quae nequaquam annihilari potest, sed Nihilum uti Limitem habet $\frac{m-n}{2^\infty}$, si vel addamus n reiterata vice, vel detrahimus ab m , palam fiet oriri numero infinitos, at valore finitos Numeros Rationales prius traditis m , & n interpositos. Series ergo Numerorum omnium possibilium Rationalium positivorum, siue negativorum non per saltum, veruntamen minima per intervalla, et inassignabiles differentias procedit, variationemque subit, ac veluti fluit Minimorum Elementorum Lege rite recteque servata.

9. In hypothesi Indicis x Rationalem, seu Transcendentem valorem habentis Potestas $(1 \pm z)^x$ concipi poterit inassignabiliter discrepans a Potestate $(1 \pm z)^y$, ubi Index y sit Numerus Rationalis, manente in utraque eodem valore variabilis z , quae nimirum in Exponentis variatione constans intelligatur. Nam Numerorum Rationalium Series positivorum, vel negativorum lento ita fluxu progreditur, ut vi Numeri praecedentis differentia inter maiorem, ac minorem evadat inassignabilis; quamobrem evidens erit tanto maiori ratione inassignabilem, ac imperceptibilem fore differentiam a proxime antecedente, vel sequente y Rationali Numero Irrationalis, seu Transcendentis x , qui Rationalium Progressionis minima implet vacua, siue inassignabilia spatia denuo subdividit. Posito ergo Indice $x = y \pm \omega$, quae Differentiola ω Numerum quocunque assignabili minorem designet, obtinebuntur $(1 \pm z)^y$ atque $(1 \pm z)^{y \pm \omega}$ Continuae Functiones y , quae quamminime crescit, vel decrescit per differentiolam ω ; et ideo constat ex natura Continua-

rum Functionum differre quoque $(1 \pm z)^{y \pm \omega}$ vel $(1 \pm z)^x$ ab $(1 \pm z)^y$ per Numerum inassignabiliter parvum. Hoc idem demonstrari pariter potest suppositis y, y' Numeris Rationalibus proxime minore, ac maiore quoad intermedium Irrationalem, vel Transcendentem x . Cum etenim in prioribus uni-

versaliter valeat expressio Neutroniani Binomii erit $(1 \pm z)^y = 1 \pm \frac{y}{1} z + \frac{y \cdot y - 1}{1 \cdot 2} z^2 \pm \frac{y \cdot y - 1 \cdot y - 2}{1 \cdot 2 \cdot 3} z^3$ &c., atque $(1 \pm z)^{y'} = 1 \pm \frac{y'}{1} z +$

y' .

$\frac{y^2 \cdot y' - 1}{1 \cdot 2} \cdot z^2 \pm \frac{y' \cdot y' - 1 \cdot y' - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c.$; quapropter $(1 \pm z)^{y'} = (1 \pm z)^y$
 $= \pm \frac{(y' - y)}{1} \cdot z + \frac{(y' - y)(y' + y - 1)}{1 \cdot 2} \cdot z^2 \pm$
 $\frac{(y' - y)(y'^2 + y'y + y^2 - 3y' - 3y + 2)}{1 \cdot 2 \cdot 3} \cdot z^3 \&c.$, nimirum ob multiplicatorem
 communem $y' - y$ sine Limite decrescentem fiet inassignabilis Numerus dif-
 ferentia $(1 \pm z)^{y'} - (1 \pm z)^y$, et eo magis $(1 \pm z)^{y'} - (1 \pm z)^x$, atque
 $(1 \pm z)^x - (1 \pm z)^y$.

10. Liquet coronidis instar $(1 \pm z)^\omega = 1$, seu vicissim $1 = (1 \pm z)^\omega$
 reddi posse Numerum quomodocunque minimum, et inassignabilem, dum sci-
 licet eadem repetatur hypothesis valoris quamminimi ω . Substituta enim pro
 1 fictitia Potestate $(1 \pm z)^0$ differentia, cuius fit sermo, evadet
 $(1 \pm z)^{0 \pm 0} = (1 \pm z)^0$, vel e contra.

11. Qui Functionum omnium Continuarum originem, atque indolem
 persequatur ex eo ipso, quod sint unius, vel plurium variabilium Functiones,
 nec variando formam immutent, ideoque operationibus algebraicis, seu tran-
 scendentibus semper iisdem singulari quolibet in casu componantur, nequa-
 quam dubitabit affirmare pene intuitivam theoreticam veritatem, Functiones
 nempe ita compositas saltum respuere, cum horum saltuum ratio nonnisi
 quaerenda foret in variabilibus, quae semper, semperque minima per incre-
 menta, siue decrements procedunt, aut procedere possunt. Functiones ipsae
 Arbitrariae, vel eandem non servantes formam, quae sublimiorem Integralis
 Algorithmi partem nuperrime locupletarunt, quamvis eiusdem formae, seu
 compositionis expertes, et verbo dicam in Analyfi exleges, raro tamen a pro-
 gressu per quamminima elementa recedent. Non etenim nomine tenus
 memoratae Arbitrariae Functiones $F, F', F'' \&c.$, nisi pro iis determinandis
 omnes penitus conditiones deficerent, discontinuitatem patiantur, verum
 salientes de forma in formam per minimos quoque gradus in Algorithmo
 Differentiarum Partialium instar ordinatim Applicatae Systematis cuiusque
 Curvarum se mutuo secantium progredi author est tam in *Tractatu Calculi*
Integralis sub anni lapsum MDCCLXV. edito, quam in Articulo quinto
 recentioris Schediasmatis *de determinatione Arbitrariarum Functionum Me-*
 morabilia

morabilia inter Parisiensis Scientiarum Academiae anno MDCCLXXI. evulgata sagacissimus Condorcetus. Adhuc equidem sub iudice lis erat ante recentissimos Condorceti labores, ac veluti pro aris et focis certabant primi ordinis Analystae partim affirmantes, partimque renuentes Continuitatem Functionum, quae Arbitrariae vocantur; sed tamen excepto rariori casu verae ipsarum discontinuitatis rigido sensu acceptae in praesentia videntur post dudum compositae quaestionis fatum legibus Continuitatis Geometricae omnino restituendae.

12. Irrationalium, Incommensurabilium, ac Transcendentium quoque magnitudinum differentia quamminima, et inassignabilis a Rationalibus, Commensurabilibus, atque Algebraicis homogeneis, de qua Num. 8. ac 9. specialiter agunt, fundamentum est universae Veterum Geometriae, et praesertim divini illius Archimedis, qui Exhaustionis, ut aiunt, methodum in Circulo, Sphaera, Conis, Cylindris, Conoidibus, Sphaeroidibus, Helice, Parabola feliciter expertus Curvilinea metiri quasi Limites Rectilineorum primus docuit, quidquid contra Parabolae Apollonianae tetragonismum Archimedeum immerito senserit in *Monito ad Geometras* Lipsiensibus Commentariis anni MDCLXXXVII. inserto Dethlerus Cluverius. Huic indirectae demonstrandi formae, quae falsam hypotheseos initio assumens differentiae assignabilis comparatas inter magnitudines syntheseos quodam miraculo tandem differentiam evincit inassignabilem, iterque sic parat ad magnitudinum difficiliorum mensuras, aliaque id genus cognoscenda, obiici aliquo fortassis iure posset incommensurabilia, et eo magis transcendentia nullibi nisi nomine existere, quum ex methodi ipsius natura a commensurabilibus, atque algebraicis ea distare queant inassignabili differentia. Ingeniosa equidem ratiocinatio; cui oppugnandae in re summopere periculosa, ubi tam lubricum est et fallere et falli, non inelegans opinor quaedam perfunctorie monere, ne consulto videar difficultatem elusisse, quam vix attingere una cum aliis Matheseos Scriptoribus celebriores Antiquorum Synthesis Interpretes, ac Instauratores in eximijs Operibus *Fluxionum*, ac *Sectionum Conicarum* Maclaurinius, atque Robertus Simpsonius. Exhaustionis, seu Limitum methodus, qua ruente corrueret tota fere Mathesis, ostendit solummodo Curvilinea a Rectilineis, Incommensurabilia a Commensurabilibus, Transcendentia ab Algebraicis differre posse per magnitudinem quacunque data minorem; verumtamen haec differentia non evanescit, nec ideo mutuum parit aequalitatem
nisi

nisi quando Rectilinea in Curvilinea, Commensurabilia in Incommensurabilia, Algebraica in Transcendentia desinunt; qua de re, cum semper aliqua maneat differentia, sine fine et Geometrarum nutu minuenda, haec signanter differentia sine limite subsistens non finit maximam Curvilineorum partem Rectilinea peraequare, et nusquam vult cum Commensurabilibus Incommensurabilia, cum Transcendentibus Algebraica confundi. Esto differentia Diagonalem inter, et Rectam commensurabilem Lateri Quadrati, Polygonum inter, et Circulum, Curvam inter finiti Generis Parabolici, atque Logisticam, in infinitum quamminima, videlicet semper alicuius valoris: haec ipsa synonyma expressio demonstrabit fallaciam completae aequalitatis, ac coincidentiae earum magnitudinum invicem relatarum. Repetitis enim utcumque subdivisionibus nunquam Curvae finiti Parabolici Generis ab Huddenio primum excultae Logisticam adamussim imitabuntur, nusquam Aequatio Algebraica systematis Rectarum componentium Polygona inscripta vel circumscripta in illam Circularis Peripheriae transmutabitur, nec Recta Lateri Quadrati Commensurabilis, differentia penitus evanescente, illius Diagonali aequari unquam poterit. Dum differentia quamminima vere, et geometrico rigore evanescat iam consummatus fuit transitus a Commensurabili magnitudine ad Incommensurabilem, a Rectilinea ad Curvilineam, ab Algebraica ad Transcendentem, quas magnitudines Natura ipsa extensionis ita inter se specie diversas constituit, ut nonnisi in Infiniti latebris, idest nunquam, coincidere possint. Caute igitur, sancteque interpretandum Exhaustionis principium, cum non ex eo, quod differentiam sine limite minuere doceat, Rationalia Irrationalibus, Rectilinea Curvilineis, Algebraica Transcendentibus conferens ideo nullam affirmet, misceatque nullo iure immiscenda; sed potius adiumento inassignabilis differentiae obliquo, indirecto, secundario feliciter utatur ad quaedam Rationalium, Rectilinearum, et Algebraicarum magnitudinum symptomata, affectiones, proprietates Irrationalibus, Curvilineis, Transcendentibus quoque magnitudinibus amplificandas, quando, ut saepe numero contingit, directa, et positiva methodus desit, supraque hominum vires ea symptomata demonstranda in Infiniti sint obscuritate reposita. Uno verbo dicam; certi sunt, absolutique fines in summis Matheseos principiis, quos recte noscere necessarium, circumire periculosum, praetergredi nefas. Sic in falsum vel invitus abiret qui in nova Indivisibilium Geometria aequalis, aut semper proportionalis, quemadmodum aiunt, spissitudinis elementorum

torum oblitus magnitudines anomala directrice invicem compararet; illeque nubem pro Iunone amplecteretur, qui Curvam sine ulla lege & scriptorio manus ductu delineatam Aequationi subiecturum promitteret, ipsamve humani cuiuscunque vultus iconem Curva Parabolici Generis, geometricam congruentiam quaerens, imitari mallet, veluti narrant de Huddenio inventore, ineptique iactavit se posse in nupera Scheda Lambergius. Longe pariter aberraret a veritate is, qui Guldini seu Pappi Centrobarycam Rotundorum Theoricen, ab Antonio Rocca eximio Cavalerii discipulo primum demonstratam, inconsultus aptare praesumeret Lineis aut Planis rotantibus circa Centrum vel Axem non in eadem directione, Planoque Rotantium positum; quod partim observasse memini Vincentium Riccatum in Schediasmate secundo I. Opusculorum Voluminis, ipseque rursus fallacem litteralem eius Principii significatum ostendi in Commentario iam typis parato *de Solidis Helicoidibus*, ac praecipue *de Cochlearium Superficierum Mensura*.

13. Hisce praemissis argumentum Binomii in universali Indicis significatione nullam difficultatem facesset. Sit igitur demonstrandum generaliter esse

$$(1 \pm z)^x = 1 \pm \frac{x}{1} \cdot z + \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2 \pm \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c. \text{ in infi-}$$

nitum. Negata aequalitate foret primum Aequationis membrum maius vel

$$\text{minus secundo. Si maius, habebitur } (1 \pm z)^x = 1 \pm \frac{x}{1} \cdot z + \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2$$

$$\pm \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c. + \Delta \text{ Functione variabilium } z, x. \text{ Iisdem manen-}$$

tibus valoribus z, x inveniatnr Numerus y rationalis ita proxime minor vel

maior x , ut $(1 \pm z)^x - (1 \pm z)^y = \omega < \Delta$, quod fieri posse ex Nume-

ro 9. iam liquet. Substitutione completa obtinebitur tandem Aequatio

$$\left(1 \pm \frac{x}{1} \cdot z + \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2 \pm \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c. \right) - \left(1 \pm \frac{y}{1} \cdot z \right.$$

$$\left. + \frac{y \cdot y - 1}{1 \cdot 2} \cdot z^2 \pm \frac{y \cdot y - 1 \cdot y - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c. \right) + \Delta = \omega; \text{ videlicet erit}$$

$\Delta < \omega$, et eo magis ex hypothese assumpta $\Delta < \Delta$, quae absurditas pari methodo eliceretur adversa hypothese facta minoris inaequalitatis.

14. Cui minus arrideat absolutae Exhaustionis Archimedaee ad utranque Analysin Cartesianam et Newtonianam applicatio, iste intimam Functionum naturam perpendens demonstrationem directam Theorematis facile sibi

efformabit. Symbolon assumo $\Phi(z, x)$ pro Functionibus similibus quibuscunque significandis magnitudinum z et x , Algebraicis vel Transcendentibus. Monendum autem duco z esse independentem ab x , sive Parametri utcunque variabilis munus adire, dum Functio $\Phi(z, x)$ Ordinatum-applicatam Lineae cuiuslibet Ordinis repraesentet, vel, quod idem est, z designare unam ex tribus Coordinatis ad Superficiem, cuius Aequatio Localis $y = \Phi(z, x)$. Experimentum prius faciam sublimioris Functionum Theorices in Potestatum Indicibus rationalibus x praedictarum elevatione, pro quibus viam obiter novam sterno ad id denuo comprobandum, quod ceteroquin constat ex Elementis.

Dum fieri posset fit $(1 \pm z)^x = 1 \pm \frac{x}{1} \cdot z + \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2 \pm \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c.$ $\pm \Phi(z, x)$, sive $(1 \pm z)^x - \Phi(z, x) = 1 \pm \frac{x}{1} \cdot z$

$+ \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2 \pm \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c.$ Tunc eadem ratione et analytica necessitate per Formulam ipsam Binomii, sive Polynomii foret etiam

$$\left(1 \pm \frac{x}{1} \cdot z + \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2 \pm \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c.\right)^{\frac{1}{x}} = 1 \pm z + \Phi\left(\pm \frac{x}{1} \cdot z + \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2 \pm \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c., \frac{1}{x}\right), \text{ ideoque } 1 \pm$$

$$\frac{x}{1} \cdot z + \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2 \pm \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c., \text{ vel } (1 \pm z)^x - \Phi(\pm z, x)$$

$$= \left(1 \pm z + \Phi\left(\pm \frac{x}{1} \cdot z + \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2 \pm \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c., \frac{1}{x}\right)\right)^x.$$

Ob Functionis similitudinem, cum tam $\Phi(\pm z, x)$, quam $\Phi\left(\pm \frac{x}{1} \cdot z + \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2 \pm \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c., \frac{1}{x}\right)$ ad eandem Lineam vel Superficiem seu Locum Geometricum prouti ordinatum Applicatae pertineant,

evidens fit identicas eas esse debere quando variables, a quibus pendent, separatim in ambabus aequentur. Fiat ergo $x = 1$, et z cuiusque ad libitum determinati valoris b . Functio $\Phi(\pm z, x)$ componitur ex $\pm b$ et 1

pari forma, qua Functio altera $\Phi\left(\pm \frac{x}{1} \cdot z + \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2 \pm \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c., \frac{1}{x}\right)$

+

$+ \&c., \frac{1}{x}$) componetur ex $\pm b$ et 1 propter $\frac{1}{x} = 1$, ac variabilem

$$\pm \frac{x}{1} \cdot z + \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2 \pm \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c. = \pm b \text{ evanescantiae}$$

caussa ceterorum omnium terminorum ob $x - 1 = 1 - 1 = 0$. Eveniet itaque casu $x = 1$ Aequatio $1 \pm b = \Phi(\pm b, 1) = 1 \pm b + \Phi(\pm b, 1)$, nimirum $-\Phi(\pm b, 1) = +\Phi(\pm b, 1)$; Aequatio impossibilis nisi hypothetice assumpta Functio semper sit 0 propter mutabilem quomodocunque b in dato Binomio.

15. Iisdem Functionum affectionibus insistentes in nihilum quoque verti $\Phi(z, x)$, si x foret alius rationalis, quinimmo etiam irrationalis Numerus vel transcendens, demonstrare possemus. At fortasse in postremis facilius res peragetur, quando animadvertamus prorsus servatis praecedentibus denominationibus Functionem $+\Phi(z, x)$ seu $+\Phi(b, x)$, supposita z constante valoris cuiuslibet b , additam Formulae Newtoniani Binomii, eius indolis fore, ut semper habeat finitum valorem, dummodo b et x finitorum ultra limites non excurrant, atque infinities evanescat duos inter quomodocunque proximos Numeros x , cum revera interiaceant sine fine Numeri rationales, in quorum singulis Formula Newtoniana omni rigore, et absque Functionis illius additione subsistit. Functio autem variabilis semper finita, quae in intervallo etiam minimo magnitudinis aequae finitae infinities evanescit, sive a 0 rursus transit ad 0, quiddam Quantum esse nequit, sed mera Nullitas erit; namque non datur Functioni finitae, quomodolibet parvae, infinities infinite inter extremos variabilis valores 0, et ∞ distinctis, seu non coincidentibus vicibus evanescere. Norunt profecto omnes in Analytica Scientia versati Functionem finiti valoris ac cuiusque compositionis et formae numero tantum infinitas, non infinities infinitas habere posse Radices reales, quibus singulis pro variabili substitutis Functio data in Nihilum abeat.

16. Iuvandae tamen imaginationis commodo studentes in Curvis Parabolici generis Anguinea, Trochoidali, vel Mixta, quas pingunt Schemata XIV.XV.XVI., Functionis $\Phi(b, x)$ progressum imitemur exemplo non modo eorum, quae dedere Deguafius in *Memorabilibus Academiae Parisiensis* sub annum MDCCXLI., et Bougainvillius in Capite sexto Introductionis, quam praefert *Calculi Integralis Tractatus*, verum etiam, saltem implicite, pluribus in Schedis Leonardus Eulerus, de Aequationum Radicibus po-

fitivis, negativis, imaginariis agentes. Nonne liquet, si Recta AB ut-
cunque parva metiatur duos quomodolibet parum distitos rationales valores
Numeri x , interque extremas sectiones sive concursus Axeos abscissarum et
Curvae A, et B in infinitum multiplicentur alii intermedii concursus B' ,
 B'' , B''' , B'''' &c. respondentes ceteris sine limite Numeris rationalibus, ap-
plicatim Ordinatam CI , CI' , CI'' , CI''' , CI'''' &c., videlicet, Functionem $\Phi(b, x)$,
semper semperque aucto concursuum sive sectionum numero minorem factam
tandem evanescere, ita ut Parabolicae Curvae perimeter ob innumerabilem in-
flexionum, regressuum &c. repetitionem etiam in minimo spatio contorta, et
ab Axe recedens per deviationes inassignabiliter parvas intactis Continuita-
tis legibus cum Axe ipso AB denique confundatur? Finge mediam Ordinatam
Infinites Longam Lineae Sinuum - Rectorum Circularium contrahi in
Rectam Longitudinis finitae, super quam transferri ideo debeant innu-
mera Sectionum mutuarum Puncta Ordinatim-applicatae, Curvaeque illius
Transcendentis. Hoc fieri nequit, dum recte mediteris, nisi Circulus ge-
nitor in punctum aut nihilum abeat, Curvae Sinuum in Rectam cadat
feu Ordinatarum Axem, omnesque Rami evanescant, ex eo quod, quan-
tumlibet parvus fuerit genitoris Circuli Radius, earum Sectionum pun-
cta super Rectam Infinitae Longitudinis disponderentur. Figurarum in aliis Li-
neis Parabolicis-Transcendentibus exempla similia volenti non deerunt.

17. Universalem Binomii Newtoniani evoluti veritatem praecedentium
ope firmatam, nullus sit, qui non meminerit fide veritatis eiusdem Binomii
Formulae in Potestatibus cuiuscunque Indicis rationalis penitus inniti; qua
de re, si forte accideret postremas has formulas in falsum ducere, et quae-
dam analytica artificia requirere pro Calculi vitio medendo, eadem cura,
idemque afferendum erit remedium, cum fundamentum labatur, Formulis quo-
que irrationali vel transcendente Indice praeditis. Quamplurimos huius
moniti usus etiam in re Logarithmica atque Exponentiali sequentia fac-
penumero patefacient. Meo autem iudicio praetermittendum non erit, quod
etiam numeri 7. facultate neglecta semper locum habeat Aequalitas $(z \pm v)^x$

$$= z^x \pm \frac{x}{1} \cdot z^{x-1} \cdot v \pm \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^{x-2} \cdot v^2 \pm \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^{x-3} \cdot v^3 \pm$$

$$\frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot z^{x-4} \cdot v^4 \pm \&c.; \text{ namque } (z \pm v)^x = z^x \left(1 \pm \frac{v}{z}\right)^x =$$

$$z^x \left(1 \pm \frac{v}{z}\right)^x$$

18. Interim a praeostensis uberrimum fructum colligamus amplificantes Analyseos Cartesianae pomoeria in Potestatum multiplicatione ac divisione exercenda, etiam dum eorum Index irrationalis sit, vel transcendens. Nonnisi labilis analogiae subsidio adhuc demonstrare licuit adhibitae saepius

confiniles . Haec omnia lucem, quae in desideratis erat, recipient generalli-
ori exemplo selecto multiplicationis, ac divisionis $(1 \pm y)^x \times (1 \pm y)^z$, et
 $\frac{(1 \pm y)^x}{(1 \pm y)^z}$. Evolvendo in priori casu Potestates Binomii obtinebitur $(1 \pm y)^x$
 $(1 \pm y)^z$

lis

Iis positis quisque experietur Coefficientes homologarum Potestatum y^1, y^2, y^3, y^4 &c. compendiosiores fieri dum loco singulorum scribantur aequales

$$\pm \frac{(x+z)}{1}, \pm \frac{(x+z)(x+z-1)}{1 \cdot 2}, \pm \frac{(x+z)(x+z-1)(x+z-2)}{1 \cdot 2 \cdot 3}, \pm$$

$$\frac{(x+z)(x+z-1)(x+z-2)(x+z-3)}{1 \cdot 2 \cdot 3 \cdot 4} \text{ \&c.}, \text{ atque hinc eruitur finalis}$$

Aequatio proposita $(1 \pm y)^x \cdot (1 \pm y)^z = (1 \pm y)^{x+z}$. Pari ratiocinandi modo, si Infinitinomialium multiplicationi divisio succedat, ostendi posset altera universalis Aequatio

$$\frac{(1 \pm y)^x}{(1 \pm y)^z} = (1 \pm y)^{x-z}, \text{ quod facilius etiam con-}$$

ficere datum est absque fastidiosa Infinitinomialium per alterum algebraica par-

$$\text{titione. Profecto Aequatio assumpta } \frac{(1 \pm y)^x}{(1 \pm y)^z} = (1 \pm y)^{x-z} \text{ vere identicam}$$

sefe praebet, quum multiplicando per denominatorem evadat, ut satis patet,

$$(1 \pm y)^x = (1 \pm y)^z (1 \pm y)^{x-z}, \text{ videlicet ex antea demonstratis } (1 \pm y)^x$$

$$= (1 \pm y)^{z+x-z} = (1 \pm y)^x. \text{ Fluit insuper ab eodem Theoremate aliud}$$

Potentiationis universalissimae $(1 \pm y)^x)^z = (1 \pm y)^{xz}$: ista etenim Potesta-

tis $(1 \pm y)^x$ ad novam Indicis z exaltatio nihil est aliud, quam repetita Mul-

tiplicatio $(1 \pm y)^x (1 \pm y)^x (1 \pm y)^x \dots (1 \pm y)^x (1 \pm y)^x$. Ve-

runtamen meliori forte ratione idem adsequemur observantes Potestatem $(1 \pm y)^x$

in Seriem explicatam perhibere Polynomium eiusmodi, quo rite tracta-

to, et uti supra redacto post ipsius instar Binomii evolutionem ad Po-

testatem z eveati, proveniet Formula identica respectu habito ad illam Pote-

statis pariter evolutae $(1 \pm y)^{xz}$ iuxta mentem Neutoni.

19. Uniformitatem Potestatum Binomii, quando aliunde Multiformes

noscentur, qui Formulae obiciunt Neutonianae, ii vel levius haec dictitant,

vel non Analysis vitio, sed Analystarum errori id adscribendum ignorant.

Hos inter numerare non ausim Viros equidem summos atque honoris caus-

sa nominandos Leonardum Eulerum et Dalembercium; nam Logarithmos,

qui Numeris adsint negativis, ingeniose versantes in *Memorabilibus Bero-*

linensis Academiae anni MDCCXLIX., et in VI. primi *Opusculorum* Volu-

minis

minis Schediasmate communem Geometrarum opinionem solummodo videntur repetiisse, et Logometricae illius controversiae stimulo rapti quaedam tunc ex propria penu severiora in Binomii rem illustrandam proferre nequaquam opportunum existimarunt. Sentiant Analystae quae velint: Expressionis a Neutono datae pseudo-uniformitas originem ducit ab incompleta fallacique methodo eam depromendi; nimio etenim analogiae amori ac blando laboriosae minus universalitatis commodo indulgentes, et legitimorum ideo canonum oblitos non immerito resolutionum induens insufficientiam, ne dicam errorem, vindex Algebra mulctat. Trium, quos in secunda percelebri Epistola ad Oldenburgium missa anno MDCLXXVI. Neutonus describit, modorum, Interpolationis nempe, Inductionis, necnon

Radicum extractionis respectu Binomii $(1 - x^2)^{\frac{1}{2}}$, aliorumve in Seriem evolvendorum, postremus unicus ille est, cui Analytico iure fidendum sit: ceteri inventionis campo aperiendo et Calculi commoditati adstruendae summopere utillimi cum primigeniis, rigidisque Algebrae regulis misceri nequeunt, ne nimia fortasse facilitate veritas obruatur. Quam cauto pede Inductioni seu Analogiae sit accedendum Matheseos universae passim docet Historia. Quaedam in praecedentibus occurrunt argumento, de quo agimus, magis affinia; nec deerit occasio in sequenti Capite VI. Interpolationum praesertim a Vallisio traditarum nonnullas, in quibus non ratiocinando, sed potius divinando aut periculose veritatem coniecit, aut aliquid humani passus est, indicandi (*).

20. Initium gravissimae investigationis ab exemplo ipso derivemus

$(1 - x^2)^{-\frac{1}{2}}$, cuius significationem, vero sensu biformem $\frac{1}{+\sqrt{1-x^2}}$ atque $\frac{1}{-\sqrt{1-x^2}}$, Eulerus censuit sub specie uniformis valoris a Neutoniana Serie

(*) Easdem, quas subnecto, observationes Methodis etiam universis Adproximationum inventis ab Halleio, Neutono, Raphsonio, Ioanne Bernoullio, Lagnyo, Thoma Simpsonio, aliisque ad Radices extrahendas aptari nemo est, qui non viderit. Perlegantur *Transactiones* Londinenses anni MDCXCIV., Bernoulliana *Lectio* LIII. *Calculi Integralis*, Simpsonianae *Dissertationes Mathematicae* ad pag. 102. &c.

$$1 + \frac{1}{2} \cdot x^2 + \frac{1 \cdot 3}{2 \cdot 4} \cdot x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot x^6 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot x^8 + \&c. \text{ proferri;}$$

quod sane ita esse nemo inficiabitur, si analogia tantum duce, et incon-

sulto Theorices unico fonte in Binomii Formula $(1 + z)^m = 1 + \frac{m}{1} \cdot z$

$$+ \frac{m \cdot m - 1}{1 \cdot 2} \cdot z^2 + \frac{m \cdot m - 1 \cdot m - 2}{2 \cdot 2 \cdot 3} \cdot z^3 + \frac{m \cdot m - 1 \cdot m - 2 \cdot m - 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot z^4 +$$

&c. pro z atque m peculiares valores $= x^2$, ac $= \frac{1}{2}$ substituantur.

Quanta hoc fieret iniuria, dum totam haurire Formulae vim, atque latissi-

am eius extensionem contemplari liberet, in universaliori prospectu Radi-

cis Quadraticae extractionis $(a^2 + x^2)^{\frac{1}{2}}$ facile legendum opinamur. Com-

munia etenim Algebraici Algorithmi principia perlustrantes reperiemus se-

cundam primi Nominis a^2 Radicem tam esse $+ a$, quam $- a$, ideoque

duplici affectam signo. Alterum Radicis terminum obtinebimus, ut moris

est, dividendo x^2 per $\pm 2a$; quapropter aequae gemino signo afficietur.

Ad tertium quod attinet clare patebit exhiberi a quotiente $\frac{-x^4}{4a^2}$ diviso per

idem $\pm 2a$, et sic de singulis in infinitum. Series itaque Newtoniana ab

intimis Calculi visceribus eruta, et $(a^2 + x^2)^{\frac{1}{2}}$ respondens, erit $\pm a \pm$

$\frac{1}{2} \cdot \frac{x^2}{a} \mp \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{x^4}{a^3} \pm \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \cdot \frac{x^6}{a^5} \mp \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \cdot \frac{5}{8} \cdot \frac{x^8}{a^7} \pm \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \cdot$

$\frac{5}{8} \cdot \frac{7}{10} \cdot \frac{x^{10}}{a^9} \mp \&c.$, nimirum biformis, ac, quemadmodum Algebra do-

cet, utraque aequalis valoris, sed signis oppositis positivo, et negativo di-

stincta. Permutato Binomio in $(a^2 - x^2)^{\frac{1}{2}}$, eodemque Calculi typo repe-

tito eveniet Radix quaesita quoque biformis, positiva, necnon negativa,

$\pm a \mp \frac{1}{2} \cdot \frac{x^2}{a} \mp \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{x^4}{a^3} \mp \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \cdot \frac{x^6}{a^5} \mp \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \cdot \frac{5}{8} \cdot \frac{x^8}{a^7}$

$\mp \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \cdot \frac{5}{8} \cdot \frac{7}{10} \cdot \frac{x^{10}}{a^9} \mp \&c.$, haecque Series cum praecedente per-

fecte consentit, dum Signa immutantur x^2 , et aliarum Potestatum Indicis

impariter-paris. Unitatem itaque dividendo per Seriem $\pm 1 \mp \frac{1}{2} \cdot x^2 \mp$

$$\frac{1}{8} \cdot x^4 \mp \frac{1}{16} \cdot x^6 \mp \frac{5}{128} \cdot x^8 \mp \frac{7}{256} \cdot x^{10} \mp \frac{21}{1024} \cdot x^{12} \&c. = \sqrt{1 - x^2}$$

prodibit notorum canonum partitionis Algebraicae subsidio $\pm 1 \pm \frac{1}{2} \cdot x^2 \pm$

$$\frac{1 \cdot 3}{2 \cdot 4} \cdot x^4 \pm \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot x^6 \pm \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot x^8 \pm \&c., \text{ nempe Formula biformem exhi-}$$

bens valorem, tam $\frac{+1}{\sqrt{1-x^2}}$, quam $\frac{-1}{\sqrt{1-x^2}}$, vel potius $\frac{1}{+\sqrt{1-x^2}}$, et $\frac{1}{-\sqrt{1-x^2}}$.

21. Ipsomet Analyticae ratiocinationis facili progressu constabit de pluribus Radicibus in Newtoniana Formula comprehensis etiam quum oporteat versari circa altiorum Ordinum Potestates, quas imperfectas vocare placuit

Geometris. Experiamur id in Cubica, seu tertia Radice Binomii $(a^3 \mp z^3)^{\frac{1}{3}}$.

Radix tertia, five Cubica Numeri a^3 triplex est, videlicet a , $\frac{-a + a\sqrt{-3}}{2}$,

$\frac{-a - a\sqrt{-3}}{2}$, unica realis, ac reliquae imaginariae. Nil vetat ternas has-

ce radices ob Calculi commodum speciebus a , b , c designare. Quum igitur plenitudinem expressionis optantes nullam trium Radicum a^3 praediligere debeamus in Radicis cubicae extractione, sed omnes eadem libella metiri, evidens erit primum Radicis terminum fore tam a , quam b , quam c ; ita ut unoquoque singulatim selecto ad Algorithmum prosequendum non unica, verum triformis Series numero terminorum infinita elicietur, realis nimirum $a \mp$

$$\frac{1}{3} \cdot \frac{z^3}{a^2} \mp \frac{1}{9} \cdot \frac{z^6}{a^5} \mp \frac{5}{81} \cdot \frac{z^9}{a^8} \mp \frac{10}{243} \cdot \frac{z^{12}}{a^{11}} \mp \frac{22}{729} \cdot \frac{z^{15}}{a^{14}} \mp \&c., \text{ ac}$$

$$\text{deinceps ambae imaginariae, vel impossibiles } b \mp \frac{1}{3} \cdot \frac{z^3}{b^2} \mp \frac{1}{9} \cdot \frac{z^6}{b^5} \mp$$

$$\frac{5}{81} \cdot \frac{z^9}{b^8} \mp \frac{10}{243} \cdot \frac{z^{12}}{b^{11}} \mp \frac{22}{729} \cdot \frac{z^{15}}{b^{14}} \mp \&c., \text{ necnon } c \mp \frac{1}{3} \cdot \frac{z^3}{c^2} \mp \frac{1}{9} \cdot$$

$$\frac{z^6}{c^5} \mp \frac{5}{81} \cdot \frac{z^9}{c^8} \mp \frac{10}{243} \cdot \frac{z^{12}}{c^{11}} \mp \frac{22}{729} \cdot \frac{z^{15}}{c^{14}} \mp \&c.: \text{ Universalis, valor}$$

Binomii exaltati ad Potestatem imperfectam Indicis fracti $\frac{m}{n}$, minimis termi-

nis expressi, veluti $(a \pm z)^{\frac{m}{n}}$, sit multiplex: aequivalet etenim Radici gradus n perfectae Potestatis ordinis m ; qua de re, quum ista evoluta Potestas vi Numeri 5. instar Binomii considerari queat, tot invenientur diversi hoc in

casu Formulae Neutonianae valores, quot admittat a^m varias Indicis n Radices, scilicet ex Algebrae Elementis valores numero n .

22. Multiformitas celeberrimae Formulae a Neutono prolatae eo magis constabit experimentum sumentibus in Binomiis, seu Polynomiis, quae ad Potestates Indicis suapte natura multiformis eleventur. Sit Magnitudo,

vel Quantitas Interscendens $(1 \pm z)^{\frac{n}{\sqrt{m}}}$, et in Seriem sine carentem evolu-

ta peraequet $1 \pm \frac{\frac{n}{\sqrt{m}}}{1} \cdot z + \frac{\frac{n}{\sqrt{m}} \cdot \frac{n}{\sqrt{m}} - 1}{1 \cdot 2} \cdot z^2 \pm \frac{\frac{n}{\sqrt{m}} \cdot \frac{n}{\sqrt{m}} - 1 \cdot \frac{n}{\sqrt{m}} - 2}{1 \cdot 2 \cdot 3} \cdot z^3$
 $+ \frac{\frac{n}{\sqrt{m}} \cdot \frac{n}{\sqrt{m}} - 1 \cdot \frac{n}{\sqrt{m}} - 2 \cdot \frac{n}{\sqrt{m}} - 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot z^4 \pm \&c.$ Facile intelligitur Seriem ipsam

ob n placitatem in omnibus, praeter primum, terminis valoris radicalis non spurii $\frac{n}{\sqrt{m}}$, necessario n formitatem induere. Tamen heic non sistit multi-

formitas asymmetricae Formulae, cuius nunc fit sermo; namque, quum $\frac{n}{\sqrt{m}}$ pro veri valoris Limite habeat Fractionem numeratore, ac denominatore

infinite magno distinctam, ideoque $1^{\frac{n}{\sqrt{m}}}$ fit reapse Infinitimultiplex, nemo nesciabitur fore quoque, ob diversos sine numero valores Potestatum Unitatis in quocunque termino subintellectarum, Infinitiformem ordinis $n \cdot \infty$ Formulam illam Neutonianam. Vera etenim Formula ita conscribi deberet

$1^{\frac{n}{\sqrt{m}}} \pm \frac{\frac{n}{\sqrt{m}}}{1} \cdot 1^{\frac{n}{\sqrt{m}}-1} \cdot z^1 + \frac{\frac{n}{\sqrt{m}} \cdot \frac{n}{\sqrt{m}} - 1}{1 \cdot 2} \cdot 1^{\frac{n}{\sqrt{m}}-2} \cdot z^2 \pm \frac{\frac{n}{\sqrt{m}} \cdot \frac{n}{\sqrt{m}} - 1 \cdot \frac{n}{\sqrt{m}} - 2}{1 \cdot 2 \cdot 3} \cdot 1^{\frac{n}{\sqrt{m}}-3} \cdot z^3$
 $\pm \&c.$, vel $1^{\frac{n}{\sqrt{m}}} \pm \frac{\frac{n}{\sqrt{m}}}{1} \cdot \frac{1^{\frac{n}{\sqrt{m}}}}{1} \cdot z^1 + \frac{\frac{n}{\sqrt{m}} \cdot \frac{n}{\sqrt{m}} - 1}{1 \cdot 2} \cdot \frac{1^{\frac{n}{\sqrt{m}}}}{1^2} \cdot z^2 \pm$
 $\frac{\frac{n}{\sqrt{m}} \cdot \frac{n}{\sqrt{m}} - 1 \cdot \frac{n}{\sqrt{m}} - 2}{1 \cdot 2 \cdot 3} \cdot \frac{1^{\frac{n}{\sqrt{m}}}}{1^3} \cdot z^3 \pm \&c.$, ubi Multiformitas valoris $1^{\frac{n}{\sqrt{m}}}$,

quae in Capite VII. melius elucescet, in causa est, cur praetermitti iure Analytico nequeant Unitatis Potestates; et hoc etiam inferius dicendis ex-

tenditur. Quod si in Binomii Potestate $(1 \pm z)^T$ Index appositus Radices significet Aequationis Algebraicae gradus p , partim reales, et imaginarias sub complexivo numero p , vel omnes eodem numero reales,

les, seu imaginarias, enascetur Formula $1 \pm \frac{T}{1} \cdot z + \frac{T \cdot T - 1}{1 \cdot 2} \cdot z^2 \pm \frac{T \cdot T - 1 \cdot T - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c.$, sane $\bar{p}$ formis quo ad diversos Coefficientium

valores; et excepto casu Radicum realium rationalium integrarum augebitur tantundem ista $\bar{p}$ formitas, quot Potestas 1^T diversis modis explicari possit. Sic duae, tres, quatuor, quinque, sex &c. non tantum reales, verum et integrae rationales radices, duae, tres, quatuor &c. ordinatim Applicatae reales, et integrae rationales cuidam communi Abscissae in Curva secundi, tertii &c. Ordinis respondentes, atque a T generatim expressae, Functionem gignent biformem, triformem, tetraformem, pentaformem, exaformem: neque novit Natura Calculi limitem in infinitum.

23. Transitu facto ab Asymmetris Exponentibus ad Transcendentes dabitur novas Infinitiformium Serierum species contemplari pariter sine ullo limite. Index Binomii Circularem exhibens Peripheriam Π , seu Numericam Magnitudinem instar Asymmetricarum rationaliter inexprimibilem, nisi Infinitis adiuvantibus numeratore, ac denominatore Fractionis, pariet Formulam Infinitiformem $1 \pm \frac{\Pi}{1} \cdot z + \frac{\Pi \cdot \Pi - 1}{1 \cdot 2} \cdot z^2 \pm \frac{\Pi \cdot \Pi - 1 \cdot \Pi - 2}{1 \cdot 2 \cdot 3} \cdot z^3 +$

$\frac{\Pi \cdot \Pi - 1 \cdot \Pi - 2 \cdot \Pi - 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot z^4 \pm \&c.$, quippe quum infinities varios contineant

valores Potestates imperfectae $1^{\frac{\Pi \cdot \infty}{\infty}}$, $1^{\frac{\Pi \cdot \infty}{\infty} - 1}$, $1^{\frac{\Pi \cdot \infty}{\infty} - 2}$, , $1^{\frac{\Pi \cdot \infty}{\infty} - n}$, &c. implicate omnibus in Seriei terminis comprehensae.

Idem quoque contingeret Transcendenti cuilibet Indici uniformis valoris, dummodo constaret de legitima praefati valoris non tantum realis, verum etiam imaginarii unitate, quod raro, vel nunquam accidet quando Signorum Summationis ope transcendentia Integralia *completa* erui debeant,

uti in Binomiis $(1 \pm z)^{Sx dx}$, $(1 \pm z)^{Sdx Sx dx}$, $(1 \pm z)^{Sdx Sdx Sx dx}$ &c., in quibus pro X intellectam velim Functionem variabilis x ita comparatam, ut expers sit Integrationis Elementum $X dx$. Sin tamen omissis aliis valoribus Transcendentium Indicum praenotatorum unus solummodo ex iis eligatur, veluti si in Serie, Aream Hyperbolae Apollonianae relatae ad Asym-

ptoton exhibente, unicum realem valorem adtendamus, ipsumque numeratore, ac denominatore Infinito affectum specie alphabetica H designemus, erit simpliciter Infinitiformis Formula Neutroniana $1 \pm \frac{H}{1} \cdot z + \frac{H \cdot H - 1}{1 \cdot 2} \cdot z^2 \pm$

$$\frac{H \cdot H - 1 \cdot H - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \frac{H \cdot H - 1 \cdot H - 2 \cdot H - 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot z^4 \pm \&c.: \text{E contra Infinities-}$$

Infinitiformis, seu gradus expressi ab $\infty \cdot \infty$, valor est Potestatis $(1 \pm z)^{A \cdot \text{Sin. } b}$

$$= 1 \pm \frac{(A \cdot \text{Sin. } b)}{1} \cdot z + \frac{(A \cdot \text{Sin. } b) \cdot (A \cdot \text{Sin. } b) - 1}{1 \cdot 2} \cdot z^2 \pm \frac{(A \cdot \text{Sin. } b) \cdot (A \cdot \text{Sin. } b) - 1 \cdot (A \cdot \text{Sin. } b) - 2}{1 \cdot 2 \cdot 3} \cdot z^3$$

+ &c.: namque posito e minimo omnium Circularium Arcuum Sinui b respondentium Series exposita infinities valorem mutabit substituendo pro $e = A \cdot \text{Sin. } b$ ceteros sine numero $\Pi + e$, $2\Pi + e$, $3\Pi + e$, $n\Pi + e$

$$\dots\dots\dots, \infty \Pi + e, \text{ necnon } \frac{\Pi}{2} - e, \frac{3\Pi}{2} - e, \frac{5\Pi}{2} - e, \frac{7\Pi}{2} - e, \dots\dots\dots,$$

$\frac{\lambda\Pi}{2} - e$ &c.; iterumque infinities ob valores sine numero discrepantes Potestatum Unitatis, quarum Indices Arcus praedictos, eosque Numero integro minutos singulatim complectuntur. Nec solum Infinities-Infinitiformis pari iure Binomii Series evaderet, quando illius Exponens Arcu quamcunque aliam Trigonometricam Lineam praecedente afficeretur, uti A. Cofin. b, A. Tang. b, A. Sin. Vers. b, A. Sec. b, A. Chord. b &c., veruntamen, latissimo sensu acceptos non artificiales trigonometricos, seu non utiliter logometricos, vel imaginarios aequae ac unicos reales valores complexim numerando, eidem Infinities-Infinitiformium Functionum classi adscri-

bentur evolutae Formulae Neutronianae Binomiorum $(1 \pm z)^{\text{Log. } d}$

$(1 \pm z)^{L \cdot \text{Sin. } c}$, $(1 \pm z)^{L \cdot \text{Tang. } c}$, ceterorumque sine fine comparandorum, de quorum Exponentium indole universa fusius in Capitibus II. III. V. et VII. erit differendum.

24. Hos multiformes Series a Neutono proditae valores facilius eundem obtinebimus, Calculorumque ambages maxima ex parte minuemus, dum in Binomii qualibet Potestate $(1 \pm z)^x$ evolvenda fingatur, ut Num. 7. superius dictum, Potestatem ipsam exhiberi a producto $1^x (1 \pm z)^x$, sive

$$1^x \left(1 \pm \frac{x}{1} \cdot z + \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2 \pm \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c. \right); \text{quo Ana-}$$

lyseos

lyseos artificio multiformitas Newtonianae Seriei, posito x Indice unius tantum valoris, in multiformitatem Potestatis 1^x reicietur, atque, posito x multiformi, partim pendebit multiformitas Formulae a Seriei Coefficientibus, partimque a varietate valorum 1^x . Ars omnis, quam circa versamur, iterum

$$\begin{aligned} & \text{elucescet, si memoremus, reapse } (1 \pm z)^x \text{ in Formulam verti } 1^x \pm \frac{x}{1} \cdot 1^{x-1} \\ & z^1 \pm \frac{x \cdot x-1}{1 \cdot 2} \cdot 1^{x-2} z^2 \pm \frac{x \cdot x-1 \cdot x-2}{1 \cdot 2 \cdot 3} 1^{x-3} z^3 \pm \&c., \text{ vel } 1^x \left(1 \pm \frac{x}{1} \cdot \right. \\ & \frac{z^1}{1} \pm \frac{x \cdot x-1}{1 \cdot 2} \cdot \frac{z^2}{1^2} \pm \frac{x \cdot x-1 \cdot x-2}{1 \cdot 2 \cdot 3} \cdot \frac{z^3}{1^3} \pm \&c.), \text{ five simplicius } 1^x (1 \pm \\ & \frac{x}{1} \cdot z^1 \pm \frac{x \cdot x-1}{1 \cdot 2} \cdot z^2 \pm \frac{x \cdot x-1 \cdot x-2}{1 \cdot 2 \cdot 3} z^3 \pm \&c.), \text{ quemadmodum supra.} \end{aligned}$$

Hinc uberrimus Potestatum imperfectarum 1^x in sublimiori Algebra usus; fecundissima Potestatum omnium imaginario Indice designatarum consideratio; quarum praesertim in Capite V. summa meditabimur principia; ac utillima Potestatum, et Radicum universarum 1^x evolutio, cui illustrandae in Capite VII. quamplurima occurrent fortasse non admodum inconcinna. Nunc tamen statuere oportet diversos 1^x valores, five analytico more Radices veluti cognitae praesupponi, quando occasio feret de Multiformitate Formulae Newtonianae nonnulla in posterum adfirmare. Nec dubium antilogiae oboriri unquam potest sumendo etiam vice $1^x (1 \pm z)^x$ alteram aequae veram analyticam expressionem $\frac{(1 \pm z)^x}{1^x}$, quum in Unitatis Radices pro-

fundius inspicienti facile persuasum sit ita esse compositas, ut eosdem reddant valores, vel potius se mutuo reproducant, si quoque Unitatem partiantur; quam affectionem, sane mirabilem, ac in ipso Capite VII. fusius per-

$$\begin{aligned} & \text{scrutatam, Elementa confirmant: namque } \frac{1}{\pm \sqrt{-1}} = \mp \sqrt{-1}, \frac{1}{\pm \sqrt{\pm \sqrt{-1}}} \\ & = \mp \sqrt{\pm \sqrt{-1}}, \frac{1}{\frac{1}{1 \pm \sqrt{-3}}} = \frac{1 \pm \sqrt{-3}}{2} \&c., \text{ ob Calculi invio-} \end{aligned}$$

labilem necessitatem.

25. Functiones diversorum sine numero valorum capaces, vel proprio nomine Infinitiformes communi Geometrarum consensu videntur inter non Algebraicas adscribi debere, quia vires Algebrae superant, nec finitarum magnitudinum Algorithmo liceat illas, quasi rebelles, fraenumque respuentes

spuentes, cohibere. Dum facer hic finis Algebraicas a Transcendentibus Functionibus dividens maneat, quaestio omnis dirimetur de apto vocabulo adiu-

dicando Potestatibus z^n , $z^{\sqrt[n]{m}}$, quas alii ut Algebraicas respiciunt, alii mendentur nomine Interfendentium: namque, ea duce Infinitiformitate utriusque Potentialis expressionis iuxta numeros 22. 23., controversia nominum ictu oculi evanescit, nullaque amplius dubitativa superest ratio istas summo iure Transcendentibus Formulis aut etiam, aut non adnumerandi.

26. Difficiliorem enodationem molimur detegere suscipientes Imaginarietatem Formulae Neutonianae tum, cum Potestas imperfecta, vel Radix Binomii dati fiat impossibilis, tametsi universi evolutae Seriei termini se praebeant Reales. Nullam vero in Analyfi contradictionem reperire fas erit, dummodo Potentiationis Algorithmum rite interpretari velimus. Duplex con-

ditio requiritur, ut imperfectae Potestates $(1-z)^{\frac{m}{n}}$ Imaginarietatis vitio subiiciantur: prima nimirum, quod Binomium $1-z$ fit quantitas negativa, videlicet $z > 1$, et altera, quod Indicis fracti denominator n sit par, ac ex adverso m numerator sit impar. Erit itaque $(1-z)^{\frac{m}{n}}$, sive ex praeostensis $1 - \frac{m}{1} \cdot z^1 + \frac{m \cdot m-1}{1 \cdot 2} \cdot z^2 - \frac{m \cdot m-1 \cdot m-2}{1 \cdot 2 \cdot 3} \cdot z^3 + \frac{m \cdot m-1 \cdot m-2 \cdot m-3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot z^4$ — &c. negativa magnitudo, pro quo Polynomio tute, ac simpliciter Binomium $1-x$ subrogari poterit, et hinc $(1-z)^{\frac{m}{n}}$ evadet $(1-x)^{\frac{1}{n}}$, sive $\sqrt[n]{1-x}$.

27. Ponatur primum Index $\frac{1}{n} = \frac{1}{2}$. Radicum Quadraticarum extractionis canones docent priorem huius Radicis $\sqrt{1-x}$ terminum esse ± 1 , residuumque

$-x$; quo diviso per ± 2 secundus Radicis terminus fiet $\mp \frac{x}{2}$, ac re-

siduum $-\frac{x^2}{4}$. Hoc iterum diviso per ± 2 $\mp x$ elicietur terminus tertius

$\mp \frac{x^2}{8}$, eritque novum residuum $-\frac{x^3}{8} - \frac{x^4}{64}$; quo rursus diviso per

$\pm 2 \mp x \mp \frac{x^2}{4}$ obtinebitur quartus Formulae terminus $\mp \frac{x^3}{16}$. Huic, et

codem

eodem progressu ceteris sine limite terminis $\mp \frac{5}{128} x^4, \mp \frac{7}{256} x^5$ &c. ad-
scribuntur residua $-\frac{5}{64} x^4 - \frac{x^5}{64} - \frac{x^6}{256}$, indeque $-\frac{7}{128} x^5 - \frac{7}{512} x^6$

$-\frac{5}{1024} x^7 - \frac{25}{16384} x^8, -\frac{21x^6}{512} - \frac{3x^7}{256} - \frac{81x^8}{16384} - \frac{35x^9}{16384} - \frac{49x^{10}}{65536}$, &c.:

at caute observandum praefata residua eorum ordine servato $-x, -$

$\frac{x^2}{4}, -\frac{x^3}{8} - \frac{x^4}{64}, -\frac{5}{64} x^4 - \frac{x^5}{64} - \frac{x^6}{256}, -\frac{7}{128} x^5 - \frac{7}{512} x^6 -$

$\frac{5}{1024} x^7 - \frac{25}{16384} x^8$, &c., ob x Unitate maiorem, adeoque semper crescen-

tem altiorum ipsius x Potestatum valorem, nunquam ad nullitatem verge-

re, nequidem progrediente in infinitum Newtoniana biformi Serie $\pm 1 \mp$
 $\frac{x^1}{2} \mp \frac{x^2}{8} \mp \frac{x^3}{16} \mp \frac{5}{128} x^4 \mp \frac{7}{256} x^5 \mp$ &c., nec idcirco negligi iure posse

Residuum postremum, seu de positiva agatur, seu de negativa Radice. Ad

hoc igitur, ut vera Radix emergat, non puris Evolutionis in Seriem Bino-

mii terminis, quomodocunque, ac etiam absque fine protractis, fidendum,

verum et Aequationem quandam correctionis addamus oportet ortam a Re-

siduo, nequaquam negligendo, at ceteris Seriei terminis cumulando; quae

correctio negligitur casu $1 - x$ positivi, videlicet x Unitate minoris, quia

crescente terminorum numero minuuntur Residua, tandemque in infinitum

minuta evanescunt. Quae cum ita sint, $\pm 1 \mp \frac{x^1}{2}$, vel $\pm 1 \mp \frac{x^1}{2} \mp \frac{x^2}{8}$,

seu $\pm 1 \mp \frac{x^1}{2} \mp \frac{x^2}{8} \mp \frac{x^3}{16}$, vel longius prosequendo $\pm 1 \mp \frac{x^1}{2} \mp \frac{x^2}{8} \mp$

$\frac{x^3}{16} \mp \frac{5}{128} x^4$, sive $\pm 1 \mp \frac{x^1}{2} \mp \frac{x^2}{8} \mp \frac{x^3}{16} \mp \frac{5}{128} x^4 \mp \frac{7}{256} x^5$, non ex-

hibebunt Radicem quaesitam Quadraticam Numeri negativi $1 - x$, sed

contra Analystae votum Radicem offerent Numeri semper, semperque po-

positivi, quum sit perfectum quadratum, et sine limite crescentis hoc ordine,

$1 - x + \frac{x^2}{4}, 1 - x + \frac{x^3}{8} + \frac{x^4}{64}, 1 - x + \frac{5}{64} x^4 + \frac{x^5}{64} + \frac{x^6}{256},$
 $1 - x + \frac{7}{128} x^5 + \frac{7}{512} x^6 + \frac{5}{1024} x^7 + \frac{25}{16384} x^8, 1 - x + \frac{21}{512} x^6 +$
 $\frac{3}{256} x^7 + \frac{81}{16384} x^8 + \frac{35}{16384} x^9 + \frac{49}{65536} x^{10}$ &c. in Infinitum. Non Geo-

metras

metras ergo Algebra fallit, attamen minus attenti Geometrae crassum in errorem impingerent, nisi Seriei terminis quolibet numero evolutis aliquid detraxerint compensationis causa, et proportionem habita ad Residua illa continuo excrefcentia.

28. Assumpto in exemplo coacervatio quorumcunque terminorum Newtonianae Seriei $\pm 1 \mp \frac{x^1}{2} \mp \frac{x^2}{8} \mp \frac{x^3}{16} \mp \frac{5}{128} x^4 \mp \frac{7}{256} x^5 \mp \&c.$ dicatur Δ ; erit $\Delta^2 = 1 - x + \Gamma$, quod signum Γ indicabit Numerum in progressu Seriei inassignabilem, et denique evanescentem dum x Unitatem non superet, verum, si excedat, significabit Numerum semper positivum, aucto terminorum numero in immensum auctum, & semper ita comparatum, ut $1 - x + \Gamma$ sit Numerus positivus. Sin velis non Radicem quadraticam Δ falsi Numeri positivi $1 - x + \Gamma$, sed veri propositi negativi $1 - x$, haec Radix promenda esset post evolutionem Formulae Binomialis a nova Aequatione $\Delta^2 - \Gamma = 1 - x$ scilicet $\pm \sqrt{1 - x} = \pm \sqrt{\Delta^2 - \Gamma}$; quapropter, cum ex ipsa constet Aequatione negativum iam esse Numerum $\Delta^2 - \Gamma$, utraque Radix ab Analyseos Oraculo rite recteque probato pro Imaginaria, vel Impossibili definietur. Algebra itaque obliquo, longiorique tramite ad Aequationem adamussim identicam $\Delta^2 - \Gamma = 1 - x$ Geometram ducit, prouti factis opportunis substitutionibus absque difficultate patebit.

29. Ratiocinatione consimili Theorice ipsam amplificabimus, dum loco Radicis Quadraticae Radices investigarentur quartae, sextae, octavae &c. Numerorum negativi valoris. Esto etenim Formula universalis vi Num. 26.

praecedentis $(1 - x)^{\frac{1}{2n}}$; haec in Radicale evolvetur facillimum $\sqrt[n]{(1 - x)^{\frac{1}{2}}}$.

Invento per Seriem Newtoni valore $(1 - x)^{\frac{1}{2}}$ in hypothesi x Unitate maioris proveniet biformis Functio Imaginaria, ideoque Imaginarietate turbabitur Numerus etiam $(1 - x)^{\frac{1}{2}}$ denuo exaltatus ad Potestatem imperfectam gradus $\frac{1}{n}$ ipsius Formulae Newtonianae subsidio. Si n denuo sit par, vel $2n$ Index initio assumptus ad Numerorum classem spectet Pariter-Parium, nova exsurgent Imaginaria in maius compendium aliis methodis contrahenda, quae non sunt huius loci.

30. Tanta in Algorithmo condendo subtilitati qui parcere malit, et inutilem

inutilem evitare studeat identicam Aequationem, cum generosiore praebere se munere Analysin deteget. Magnitudine namque Binomiali $(1-x)^{\frac{1}{2}}$ mutata in

productum $(-1)^{\frac{1}{2}} (x-1)^{\frac{1}{2}}$, Binomium $x-1$ ad normam Numeri 7. statim positivum evadit. Transitu tunc facto ad evolutionem Binomii habebitur $(x-1)^{\frac{1}{2}} = \pm x^{\frac{1}{2}} \mp \frac{1}{2x^{\frac{1}{2}}} \mp \frac{1}{8x^{\frac{3}{2}}} \mp \frac{1}{16x^{\frac{5}{2}}} \mp \frac{5}{128x^{\frac{7}{2}}} \mp \frac{7}{256x^{\frac{9}{2}}} \mp \&c.$,

qua in Serie non amplius occurrit imaginarietas, nec unquam fallacia abscondi potest, quia successiva Residua $-1, -\frac{1}{4x}, -\frac{1}{8x^2}, -\frac{1}{64x^3}, -$

$\frac{5}{64x^3}, \frac{1}{64x^4}, \frac{1}{256x^5}, \frac{7}{128x^4}, \frac{7}{512x^5}, \frac{5}{1024x^6}, \frac{25}{16384x^7}, \&c.$ ob

x Unitate maiorem continuo decrescunt, ac Formula in Infinitum protrahita abit in vere Arithmeticam Nullitatem. Hisce positis erit $(1-x)^{\frac{1}{2}}$

$= \pm \sqrt{-1} \left(x^{\frac{1}{2}} - \frac{1}{2x^{\frac{1}{2}}} - \frac{1}{8x^{\frac{3}{2}}} - \frac{1}{16x^{\frac{5}{2}}} - \frac{5}{128x^{\frac{7}{2}}} - \frac{7}{256x^{\frac{9}{2}}} - \&c. \right)$,

nimirum gemina effluet Radix Imaginaria, quemadmodum Regulae inconcusae Algebraicae in Elementis recensitae suadent. Immo, et admodum uni-

versaliter, si in Binomiali Potestate $(1-z)^{\frac{2m-1}{2n}}$ hypothesim repetendo Numeri $1-z$ negativi, videlicet z Unitate maioris, prius inveniatur abs-

que ulla difficultate Potestas Indicis $2m-1$, seu $1 - \left(\frac{2m-1}{1} \right) z^1 +$

$\left(\frac{2m-1}{1} \right) \left(\frac{2m-2}{2} \right) z^2 - \left(\frac{2m-1}{1} \right) \left(\frac{2m-2}{2} \right) \left(\frac{2m-3}{3} \right) z^3 + \left(\frac{2m-1}{1} \right)$

$\left(\frac{2m-2}{2} \right) \left(\frac{2m-3}{3} \right) \left(\frac{2m-4}{4} \right) z^4 - \&c.$, quae Potestas Polynomia erit

Numerus negativus, ideoque instar Binomii ex Numero 5. designari poterit

ope $1-y$, atque deinceps $(1-y)^{\frac{1}{2n}}$ transformetur in $(-1)^{\frac{1}{2n}} (y-1)^{\frac{1}{2n}}$,

$$\text{proveniet } (1-z)^{\frac{2m-1}{2n}} = \sqrt[n]{1-y} (y-1)^{\frac{1}{2n}} = \sqrt[n]{1-y} \times \pm \sqrt[n]{1-y} \\ \times \sqrt[n]{y-1}^{\frac{1}{2n}} = \sqrt[n]{1-y} \cdot \pm \sqrt[n]{1-y} \left(y^{\frac{1}{2n}} - \frac{1}{1(2n)^1 y^{\frac{2n-1}{2n}}} - \frac{2n-1}{1.2(2n)^2 y^{\frac{4n-1}{2n}}} - \right. \\ \left. \frac{8n^2-6n+1}{1.2.3(2n)^3 y^{\frac{6n-1}{2n}}} - \&c. \right), \text{ atque has expressiones ab Imaginarietate li-}$$

beras esse suspicari vel minimum cuiquam licebit.

31. Artificium ergo emicat unum idemque ad multiformitatem, et possibilitatem, seu impossibilitatem Formulae Neutonianae noscendam statuendamque, et illud consistit universaliter in repraesentatione Binomii $(1 \pm z)^x$ per $(1)^x (1 \pm z)^x$, sive per $(-1)^x (z-1)^x$, dum occasio feret esse z Unitate maius, Signoque affectum negativum. Penetralia itaque, ac mysteria Analyseos in Binomiis latentia simplici hoc flamine duce non tantum adire, verum insuper et referare continget. Nec diutius immorabor in re tam facili prosequenda addens, quod omnes iam viderint, universa, de quibus supra, eodem penitus modo adplicari Binomiis Indice negativo prae-

ditis, veluti $(1 \pm x)^{-\frac{1}{2}}$, $(1 \pm y)^{-\frac{m}{n}}$, $(1 \pm z)^{-x}$, quum reapse designent Potestates Indice gaudentes positivo $\frac{1}{1 \pm x^{\frac{1}{2}}}$, $\frac{1}{1 \pm y^{\frac{m}{n}}}$, $\frac{1}{1 \pm z^x}$, et ideo

divisione accedente eadem Multiformitatis, ac Imaginarietatis symptomata facta recta remaneant.

32. Polynomiorum, quae sint perfectae cuiuslibet Ordinis Potestates, evolutio in imperfectas Potestates aut Radices Ordinis inversi ope Series Neutonianae videtur demum nonnullis vel falsam praebere Magnitudinem, vel ipsam infinito terminorum numero illaqueatam intempestive recondere. De hac Formulae Neutoni non satis perspecta regressus veritate memini octo ante annos egisse in Epistola Physico-mathematica ad amicum Geometram, et simplicissimi tum casus Radicis Quadraticae ex Trinomio Quadrato $a^2 + 2ab + b^2$ educendae feliciter admodum Gordianum nodum solvisse. Verum nunc iuvat

Theorema

Theorema generaliter aggredi, Formulaeque Newtonianae tutamen pleno Marte fascipere. Quod ut clarius efficiatur post absolutissime demonstratam superius Newtonianam Seriem in Potestatis quolibet Exponente praeditae exaltatione, seu Binomii exaltandi, seu Multinomii cuiusvis sit sermo, Formulam universalem Potestatis Infinitinomii prae oculis habere oportebit, quam omnium primus Newtoni semitas exornans Moivraeus in *Philosophicis Transactionibus* anni MDCXCVII. ita expressit $(az^1 + bz^2 + cz^3 + dz^4 + ez^5 + fz^6 + \dots \&c. \dots \&c.)^m =$

$$\begin{aligned} & a^m z^0 + \frac{m}{1} a^{m-1} b z^1 + \frac{m(m-1)}{1 \cdot 2} a^{m-2} b^2 z^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} a^{m-3} b^3 z^3 + \frac{m(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3 \cdot 4} a^{m-4} b^4 z^4 \\ & + \frac{m}{1} a^{m-1} c z^1 + \frac{m(m-1)}{1 \cdot 2} a^{m-2} b c z^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} a^{m-3} b^2 c z^3 + \frac{m(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3 \cdot 4} a^{m-4} b^3 c z^4 \\ & + \frac{m}{1} a^{m-1} d z^1 + \frac{m(m-1)}{1 \cdot 2} a^{m-2} b d z^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} a^{m-3} c^2 z^3 + \frac{m(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3 \cdot 4} a^{m-4} c^3 z^4 \\ & + \frac{m}{1} a^{m-1} e z^1 + \frac{m(m-1)}{1 \cdot 2} a^{m-2} e^2 z^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} a^{m-3} e^3 z^3 + \frac{m(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3 \cdot 4} a^{m-4} e^4 z^4 \\ & + \dots \end{aligned}$$

&c. &c. &c. continuato similiter progressu, quousque permittant termini Polynomii elevandi, et Index m Potestatis quaesitae. Moivraeanam hanc Seriem Iacobus Bernoullius Logarithmorum, et Differentialis Calculi praecedente theorice inter Posthuma Volumine II. eius Operam a Gabriele Cramero recensitorum inserta comprobavit; ac perfecte congruere alteri, quam dedit Eulerus ad calcem Capitis IV. primi Voluminis *Introductionis in Analysin Infinitorum*, aliisque, quas ex Combinationum Scientia, aut ex directa Formula Newtoniani Binomii plerique eruerunt Geometrarum, perquam facillime patet.

33. Quibus in antecessum notatis extemplo ostendemus quod petitur generalissimam praebentes Indicis indefiniti Potestatem perfectam, cuius Radix invenienda Newtonianae Seriei solito more subiiciatur, videlicet

$$\left(a^n + \frac{n}{1} a^{n-1} z + \frac{n(n-1)}{1 \cdot 2} a^{n-2} z^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3} z^3 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} a^{n-4} z^4 + \dots \right)^{\frac{1}{n}}.$$

Hacc Radix necessitudine quadam abire debet in $a + z$, uti

liquet ex communibus Algebrae Cartesianae Elementis, nostraque interest opitulante Newtoni Formula comprobare. Adsequemur summatim demonstra-

$$\begin{aligned}
 a + \frac{1}{n}(a^n)^{\frac{1}{n}-1} \cdot \frac{n}{1} a^{n-1} z^1 + \frac{\frac{1}{n} \cdot \frac{1}{n} - 1}{1 \cdot 2} (a^n)^{\frac{1}{n}-2} \cdot \frac{n^2}{1} a^{2n-2} z^2 + \frac{\frac{1}{n} \cdot \frac{1}{n} - 1 \cdot \frac{1}{n} - 2}{1 \cdot 2 \cdot 3} \\
 + \frac{1}{n} (a^n)^{\frac{1}{n}-1} \cdot \frac{n \cdot n - 1}{1 \cdot 2} a^{n-2} z^2 + \frac{\frac{1}{n} \cdot \frac{1}{n} - 1}{n \cdot n} (a^n)^{\frac{1}{n}-2} \\
 + \frac{1}{n} (a^n)^{\frac{1}{n}-1}
 \end{aligned}$$

Ista autem in simpliciore formam coacta tribuet sequentem Aequalitatem

$$\begin{aligned}
 a + z^1 + \left(\frac{1-n}{1}\right) \frac{z^2}{a} + \left(\frac{2n^2-3n+1}{6}\right) \frac{z^3}{a^2} + \left(\frac{-6n^3+11n^2-6n+1}{24}\right) \frac{z^4}{a^3} &\&c. = a + z \\
 + \left(\frac{n-1}{1}\right) \frac{z^2}{a} + \left(\frac{-n^2+2n-1}{2}\right) \frac{z^3}{a^2} + \left(\frac{2n^3-5n^2+4n-1}{4}\right) \frac{z^4}{a^3} &\&c. \\
 + \left(\frac{n^2-3n+2}{6}\right) \frac{z^3}{a^2} + \left(\frac{-n^3+4n^2-5n+2}{6}\right) \frac{z^4}{a^3} &\&c. \\
 + \left(\frac{-n^3+3n^2-3n+1}{8}\right) \frac{z^4}{a^3} &\&c. \\
 + \left(\frac{n^3-6n^2+11n-6}{24}\right) \frac{z^4}{a^3} &\&c.,
 \end{aligned}$$

quemadmodum sine Serierum circuitu Analytica veritas postulat, et nunc Series ipsa decernit, quum praeter duos primos terminos ceteri omnes se mutuo evidenter elidant. Addere ulterius non piget eandem terminorum destructionem pari felicitate eventuram, si loco $+z$ substituissimus $-z$, quod permutatis alternatim signis unusquisque experietur, sed inutiliter novo inito

Calculo

strationem optatam. Vi etenim Formulae Infinitinomii, sive potius Binomii, cum Indice $\frac{1}{n}$ obtinebimus expressionem

$$\begin{aligned} & (a^n)^{\frac{1}{n}-3} \cdot \frac{n^3}{1} a^{3n-3} z^3 + \frac{\frac{1}{n}-1}{1} \cdot \frac{\frac{1}{n}-2}{2} \cdot \frac{\frac{1}{n}-3}{3} (a^n)^{\frac{1}{n}-4} \cdot \frac{n^4}{1} a^{4n-4} z^4 & \&c. \\ & \frac{n \cdot n-1}{1 \cdot 2} a^{2n-2} z^2 + \frac{\frac{1}{n}-1}{1} \cdot \frac{\frac{1}{n}-2}{2} (a^n)^{\frac{1}{n}-3} \cdot \frac{n^2}{1} a^{2n-2} \cdot \frac{n \cdot n-1}{1 \cdot 2} a^{n-2} z^4 & \&c. \\ & \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} a^{n-3} z^3 + \frac{\frac{1}{n}-1}{1} \cdot \frac{\frac{1}{n}-2}{2} (a^n)^{\frac{1}{n}-2} \cdot \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} a^{2n-4} z^4 & \&c. \\ & + \frac{\frac{1}{n}-1}{1} \cdot \frac{\frac{1}{n}-2}{2} (a^n)^{\frac{1}{n}-2} \left(\frac{n \cdot n-1}{1 \cdot 2} \right)^2 a^{2n-4} z^4 & \&c. \\ & + \frac{1}{n} (a^n)^{\frac{1}{n}-1} \cdot \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} a^{n-4} z^4 & \&c. \end{aligned}$$

Ista Calculo tentabit; namque in universali Speciei z valore negativos, ac positivos quoscunque contineri neminem latet.

34. Binomii $1 \pm x$ ad Potestatem cuiuslibet Indicis z elevati Series notissima more Newtoni evoluta convergit, dummodo x aequalis, vel minor sit Unitate; at ex adverso divergens fit, quoties x Unitatem superet, et eo magis divergit augescendo secundae Binomii partis valore numerico. Ratio convergentiae, seu divergentiae facillima hauritur ab ordine ipso terminorum Series Newtonianae

$$1 \pm \frac{z}{1} \cdot x^1 + \frac{z \cdot z-1}{1 \cdot 2} \cdot x^2 \pm \frac{z \cdot z-1 \cdot z-2}{1 \cdot 2 \cdot 3} \cdot x^3 + \frac{z \cdot z-1 \cdot z-2 \cdot z-3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot x^4 & \&c.,$$

cum luculentissime pateat, quod Potestatum x valores respectivi auctis Indicibus perpetuo minuantur casu $x < 1$, et contra crescant altera in hypothese $x > 1$. Casus quidam etiam distingui meretur, quo termini omnes, tametsi rarissimo, eundem servant valorem, ac prope diceres parallelismum. Veruntamen, quamvis a multis Analyseos Scriptoribus de hac Serierum affectione convergentes, parallelas, aut divergentes terminos respiciente sit actum, praesertim ab Iacobo Bernoullio in *Parte III. Positio-*

num de Seriebus Infinitis anno MDCXCVI. Basileae edita, a Varignonio deinde in *Memorabilibus Scientiarum Academiae Parisiensis* ad annum MDCCXV., et nuper fusiori oratione a principibus equidem in Mathesi viris Dalembertio, atque Eulero, nonnulla restant nihilominus amplificanda, vel saltem pauca, quae paradoxa videntur, ne mysterio fuescant Geometrae, aliquantisper lenienda. Longior autem labor esset opima hac in messe omnes veluti cereales aristas minutatim colligere; sed, dum aliis libenter universam Serierum provinciam, ubi scateat lacunis, aut adhuc obscuritatem patiatur, novo ordine, ut par est, pertractandam cedo, liceat ad rem nostram propius facientia, atque inferius dicendis utilia quaedam memoriae mandare.

35. Ac primum male opinantur qui de natura Seriei omen capiant a prioribus tantummodo terminis. Saepe in Binomiorum accidit Potestatibus divergentes prius occurrere terminos, et deinceps sequentes convergere in infinitum. Versa vice quandoque convergunt termini ipsi priores, et ceteri postea in immensum divergere conspiciuntur. Immo aliquando priores Serierum termini reales se praebere solent, et Imaginarietatis character, seu testifera Quanti impossibilis in posterioribus terminis oritur inexpectata. Sedulo itaque curandum, ne captiosa ingressus Seriei species una vel altera fallat; universitatis enim terminorum lex cognita unicus debet esse ratiocinandi fons pro vera Serierum classe, atque indole decernenda. Nec autem divergentium, et parallelarum Serierum familiam supervacaneam in orbe Analytico, ac multo minus falsam censeo, et ab Algebra segregandam. Series utcunque divergens, seu parallela, est vera Functio repraesentativa Functionis alterius, a qua regulis Algorithmi rite recteque observatis evolvitur, et hac ratione nil discrepat a convergentibus, eoquod Serierum alterutra loco Functionis, unde nascitur, ad Geometrae nutum substitui possit. Convergens, parallela, divergens Series numero terminorum infinita ita semper componitur, ut omnium terminorum Summa in infinitum coalescens pari pacto, eademque penitus methodo a datis Functionibus derivetur: hoc tamen interest discrimen, quod convergentium terminorum extrema Summa aequatur evolutae Functioni ob evanescentiam Residui, quum e contra divergentium, ac parallelorum extrema pariter Summa terminorum identica sit respectu datae Functionis, in computum tamen additis post terminos sine limite evolutos extremis infinitis seu finitis Residuis. Non Summa itaque, non Summae adpropinquatio a divergentibus sive parallelis Seriebus quaerenda veluti a

conver-

quia vel finitum $\pm \frac{1}{2}$, vel infinitum Numerum $\frac{\pm x^\infty}{1 \pm x}$ praebet, quum e contra casu x Unitate minoris, infinitesimum Numerum tribuat, vel Nihilum.

Posito itaque $x = 1$ fiet $\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1$
&c. $\frac{1}{2}$, scilicet Aequationem identicam fugata contradictione nancisce-

mur; ac similiter facto $x = -3$ in denominatore Fractionis $\frac{1}{1+x}$, five,
quod eodem redit, $x = 3$ in $\frac{1}{1-x}$, proveniet $\frac{1}{-2}$ seu $\frac{1}{2} = 1 + 3$
 $+ 9 + 27 + 81 + 243 + 729 + 2187 + 6561$ &c. $\frac{19683}{2}$, videlicet

altera identica Aequatio, cuius termini cumulatim homologis prioris Series
gignent Aequalitatem $\frac{1}{2} - \frac{1}{2} = 2 - 2 + 10 - 26 + 82 - 242 -$
 $730 + 2186 - 6562 \text{ \&c. } - \frac{19684}{2}$, seu $0 = 0$, Ecquid? Tantaene molis
erat obscuratam iamdudum simplicissimam Veritatem severiori iuri suo re-
stituere?

36. Usus Serierum parallelarum in Algebra amplissimus ad summae indaginis enodandas quaestiones, cuius spicilegium aliquod VI. Caput atque postremum inter sequentia exhibebunt, postulat, ut ipsarum originem adtentius perpendam. Parallelae Series numero terminorum infinitae sunt singularis facillimus modus Magnitudinis Exponentialis generalissimae $(1 + z)^x$: haec etenim Potestas rite evoluta dat Seriem $1 + \frac{x}{1} \cdot z^1 + \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2 + \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot z^4 + \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3 \cdot x - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot z^5 + \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3 \cdot x - 4 \cdot x - 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot z^6 + \&c.$, quae suppositis tam x , quam $-z = -1$ sponte traducitur in antea cognitam Progressionem $1 - 1 + 1 - 1 + 1 - 1 + 1 - \&c.$. Reperto Serierum harumce parallelarum universali prototypo monendum est in ipso latere affectionem memoratu digniorem, ac summopere infra dicendorum inlustrationi opportu-

nam, nimirum Seriem praefatam, dum $+z = -1$, semper semperque Nihilo aequalem esse, quicumque valor positivus indeterminato, et ad nutum variabili Indici x adsignetur. Constat evanescentia Seriei ab irrefragabili certe principio Aequalitatis $(1 - 1)^x = (0)^x = 0$: veruntamen haec Calculi vis, quae ingenium rapit non movet, clariorem confirmationem desiderat.

37. In confesso est Seriem $1 - \frac{x}{1} + \frac{x \cdot x - 1}{1 \cdot 2} - \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} + \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3 \cdot x - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3 \cdot x - 4 \cdot x - 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \&c. = (1 - 1)^x$ abrumpi, seu

finito componi numero terminorum, quoties x aequetur cuicumque Numero integro positivo. Ut super hanc priorem hypothesein quaedam nova adstruamus, transformetur Series ipsa in aequalem, vel potius identicam $1 - \frac{x}{1}$

$+ \left(\frac{x^2 - x}{2} \right) - \left(\frac{x^3 - 3x^2 + 2x}{6} \right) + \left(\frac{x^4 - 6x^3 + 11x^2 - 6x}{24} \right) - \left(\frac{x^5 - 10x^4 + 35x^3 - 50x^2 + 24x}{120} \right) + \left(\frac{x^6 - 15x^5 + 85x^4 - 225x^3 + 274x^2 - 120x}{720} \right) - \&c.$

quo facto ea rursus disponatur in ordinem Potestatum Numeri x Analyticum Triangulum aemulans

1	$-\frac{x}{1}$	$+\frac{x^2}{2}$	$-\frac{x^3}{6}$	$+\frac{x^4}{24}$	$-\frac{x^5}{120}$	$+\frac{x^6}{720}$	$\&c.$
	$-\frac{x}{2}$	$+\frac{x^2}{2}$	$-\frac{x^3}{4}$	$+\frac{x^4}{12}$	$-\frac{x^5}{48}$		$\&c.$
		$-\frac{x}{3}$	$+\frac{11x^2}{24}$	$-\frac{7x^3}{24}$	$+\frac{17x^4}{144}$		$\&c.$
			$-\frac{x}{4}$	$+\frac{5x^2}{12}$	$-\frac{5x^3}{16}$		$\&c.$
				$-\frac{x}{5}$	$+\frac{137x^2}{360}$		$\&c.$
					$-\frac{x}{6}$		$\&c.$

Demonstrare nunc opus erit casu $x = 1, x = 2, x = 3, x = 4, x = 5$
 &c. , $x = n$ Aequationes correspondentes $1 - x^1 = 0, 2 - 3x^1$
 $+ x^2 = 0, 6 - 11x^1 + 6x^2 - x^3 = 0, 24 - 50x + 35x^2 -$
 $10x^3 + x^4 = 0, 120 - 274x + 225x^2 - 85x^3 + 15x^4 - x^5 =$
 $0, \&c: \&c: , 1.2.3.4.5 n - +$
 $= \left(\frac{n+1.n+1.n.n-1.n-2}{1.2.2.3.4} \right) x^{n-3} \pm \left(\frac{n+1.n.n-1.3n+2}{1.2.3.4} \right) x^{n-2}$
 $= \left(\frac{n+1.n}{1.2} \right) x^{n-1} \pm x^n = 0$ tuto adferas esse, dummodo observetur quod Si-

gna superius apposita valeant pro quocunque Indice n pari, et inferius pro quo-
 libet impari, numerumque terminorum esse semper $n+1$ in unaquaque Ae-
 quatione. Nemo autem dubitabit univērsas hasce Aequationes necessario sub-
 sistere: prima etenim ipsarum Membra perfecte congruunt Numeratoribus su-
 pra descriptis $x^2 - x^1, x^3 - 3x^2 + 2x^1, x^4 - 6x^3 + 11x^2 - 6x,$
 $x^5 - 10x^4 + 35x^3 - 50x^2 + 24x, x^6 - 15x^5 + 85x^4 - 225x^3$
 $+ 274x^2 - 120x$ divisīs per x ; qua de re, cum isti Numeratores ita di-
 visi sint producta ex Factoribus $x - 1, x - 1.x - 2, x - 1.x - 2.$
 $x - 3, x - 1.x - 2.x - 3.x - 4, x - 1.x - 2.x - 3.x - 4.$
 $x - 5, \&c: , x - 1.x - 2.x - 3.x - 4.x - 5$
 $x - n + 2.x - n + 1.x - n,$ patebit singulas Aequationum, quarum
 fit mentio, Radices esse singulos ab Unitate incipientes usque ad n Nume-
 ros integros positivos. Interim elegans sane considerari meretur generalissi-

ma Exponentialis Aequatio $x^x - \left(\frac{x^2 + x}{1.2} \right) x^{x-1} + \left(\frac{3x^4 + 2x^3 - 3x^2 - 2x}{1.2.3.4} \right)$
 $x^{x-2} - \left(\frac{x^6 - x^5 - 3x^4 + x^3 + 2x^2}{1.2.2.3.4} \right) x^{x-3} +$
 $\left(\frac{15x^8 - 60x^7 - 10x^6 + 192x^5 - 25x^4 - 180x^3 + 20x^2 + 48x}{1.2.3.4.5.6.8} \right) x^{x-4}$
 $\pm 1.2.3.4.5 x - 2.x - 1.x = 0,$ vel potius, praemisso eodem
 monito quoad numerum terminorum $x+1$, et respectu Signi postremo termino
 adscripti, $x^x - \left(\frac{x+1.x}{1.2} \right) x^{x-1} + \left(\frac{x+1.x.x-1.3x+2}{1.2.3.4} \right) x^{x-2} -$
 $\left(\frac{x+1.x+1.x.x.x-1.x-2}{1.2.2.3.4} \right) x^{x-3} +$
 $\left(\frac{x+1.x.x-1.x-2.x-3(15x^3+15x^2-10x-8)}{1.2.3.4.5.6.8} \right) x^{x-4} \pm$

$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots x - 2 \cdot x - 1 \cdot x = 0$, quae facto x aequali cuilibet Numero integro positivo resolutionis semper est capax, habetque numero x universas reales, rationales, ac positivas Radices $1, 2, 3, 4, 5 \dots x - 2, x - 1, x$, five Divisores, aut Factores omnes, sui multiplicatione extremum terminum componentes. Lex Coefficientium adeo est comparata, quemadmodum constat, ut exhibeant Summam omnium Numerorum naturaliter progredientium ab 1 ad x inclusive, servatoque ordine Summas productorum ex binis, ternis, quaternis, quinis &c., uti postulat eiusdem praenotatae Aequationis Exponentialis natura, et Calculi Summae Potestatum Numerorum Naturalium subsidio perquam facillime adinvenitur.

38. Praetermissa uberiori tractatione, quam illa semper resolubilis Exponentialis Aequatio fortasse suppeditaret, periculum etiam ipsius methodi faciam dum x fuerit Numerus fractus, irrationalis, transcendens, positivus vel quilibet negativus, videlicet dum Series adlata evaserit numero terminorum Infinita. Sit primum Numerus Index x quicunque, sed positivi valoris. Secunda Analytici Trianguli Linea verticaliter computata, atque ut ceterae postea sequentes in infinitum protracta, offert Factum Numeri x in Harmonicam Seriem —

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots \right), \text{ nimirum,}$$

iam demonstrante Iacobo Bernoullio, — $x^1 \cdot \infty$, vel potius — ∞ . Quisque expectetur tertiam Lineam praebere medietatem Quadrati secundae, five $\frac{x^2 \cdot \infty^2}{1 \cdot 2}$;

quartam sextantem Cubi secundae, five — $\frac{x^3 \cdot \infty^3}{1 \cdot 2 \cdot 3}$; quintam autem vigesimam-

quartam partem Quadrato-quadrati secundae, five $\frac{x^4 \cdot \infty^4}{1 \cdot 2 \cdot 3 \cdot 4}$; et sic deinceps eadem

lege manente sine ullo limite; quae mira, atque hactenus inobservata progressio eo magis in Capite II. illustrationem recipiet. Orietur igitur iucunda satis Aequatio sine carens

$$1 - \frac{\infty \cdot x^1}{1} + \frac{\infty^2 \cdot x^2}{1 \cdot 2} - \frac{\infty^3 \cdot x^3}{1 \cdot 2 \cdot 3} + \frac{\infty^4 \cdot x^4}{1 \cdot 2 \cdot 3 \cdot 4} -$$

$$\frac{\infty^5 \cdot x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{\infty^6 \cdot x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \dots \&c. = 0, \text{ aut simplicius } 1 - \frac{\infty}{1} +$$

$$\frac{\infty^2}{1 \cdot 2} - \frac{\infty^3}{1 \cdot 2 \cdot 3} + \frac{\infty^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{\infty^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{\infty^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \dots \&c. = 0. \text{ Pri-}$$

num Aequationis Membrum exacte dividitur per Binomium $1 - x$, ac quantitatibus magnitudine incomparabilibus neglectis exhibet pro Quoto Aequationem ipsam expositam, quae Aequationis a partitione reproductio perpetua ostendit eam esse aequalem Potestati Binomii $(1 - x)^\infty = 0$. Idem liquet si Regula Huddeniana pro aequalibus datae Aequationis Radicibus investigandis adplicetur;

namque tunc nanciscemur $0 = \frac{\infty}{1} + \frac{\infty^2 \cdot x^1}{1} - \frac{\infty^3 \cdot x^2}{1 \cdot 2} + \frac{\infty^4 \cdot x^3}{1 \cdot 2 \cdot 3} -$

$\frac{\infty^5 \cdot x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{\infty^6 \cdot x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \&c. = 0$, seu redactam $1 = \frac{\infty \cdot x^1}{1} + \frac{\infty^2 \cdot x^2}{1 \cdot 2} -$

$\frac{\infty^3 \cdot x^3}{1 \cdot 2 \cdot 3} + \frac{\infty^4 \cdot x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{\infty^5 \cdot x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \&c. = 0$ superius posita adamussim identicam.

Quum itaque in unum redeant magnitudinis Infinitae, sive, ut melius dicam, universarum magnitudinum assignabilium Limitis respectu, tam ∞ , quam eius multiplex quodcunque ab x finito denominatum $x \cdot \infty$, erit universaliter,

quilibet valor x adsumatur, $1 = \frac{\infty \cdot x^1}{1} + \frac{\infty^2 \cdot x^2}{1 \cdot 2} - \frac{\infty^3 \cdot x^3}{1 \cdot 2 \cdot 3} + \&c. =$

$1 = \frac{\infty}{1} + \frac{\infty^2}{1 \cdot 2} - \frac{\infty^3}{1 \cdot 2 \cdot 3} + \&c. = (1 - 1)^\infty = (0)^\infty = 0$, dum-

modo Notam ∞ nunc adpositam non aliter interpretari possint Geometrae nisi

pro Harmonico Infinito $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c.$ designando; quam affectionem ab Infiniti veluti nucleoeductam

aggressus eram luculentissime demonstrare.

39. Nec inelegans puto paradoxon exinde genitum, scilicet ipsa in hypothesis, quod signum ∞ denotet Infinitum a Serie Numerorum naturalium reciproca Harmonice progrediente derivatum, cui vitandae confusionis causa Harmonici nomen imponere libet, fore $(1 - x)^\infty = 0$, posito pro x quocunque finito, Unitatem non elidente, sed ipsa minore, Numero positivo, etiam irrationali vel transcendente. Termini etenim exorti Infinitinomii $1 =$

$\frac{\infty^1 \cdot x^1}{1} + \frac{\infty^2 \cdot x^2}{1 \cdot 2} - \frac{\infty^3 \cdot x^3}{1 \cdot 2 \cdot 3} + \frac{\infty^4 \cdot x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{\infty^5 \cdot x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \&c.$ ita se mutuo de-

struunt, ut propemodum ob potentissimam Infiniti, seu Magnitudinis quacunque assignabili maioris efficaciam, tametsi mutationem patiatur variabilis Numerus x , maneat semper eadem proprietates perfectae elisionis Summae terminorum

$\left(1 - \frac{1}{x}\right)^{\infty} x^{\infty}$, quae expressio Infinitatem involvit dummodo x Bina-
 rium excedat. At in Binomii Potestate $(1 + x)^{\infty}$ Ordinis aequae Infinitatecupli,
 si Numerus $+ x$ positivus utcunque parvus ponatur, tum Infinitinomialium

$$1 + \frac{\infty^1 \cdot x^1}{1} + \frac{\infty^2 \cdot x^2}{1 \cdot 2} + \frac{\infty^3 \cdot x^3}{1 \cdot 2 \cdot 3} + \frac{\infty^4 \cdot x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{\infty^5 \cdot x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \dots$$

enascetur Series $1 + \frac{\infty^1 \cdot x^1}{1} + \frac{\infty^2 \cdot x^2}{1 \cdot 2} + \frac{\infty^3 \cdot x^3}{1 \cdot 2 \cdot 3} + \frac{\infty^4 \cdot x^4}{1 \cdot 2 \cdot 3 \cdot 4} \&c., \text{fi}$

nec iniuria, quum $(1 - 1)^{-x} = \frac{1}{0^x} = \frac{1}{0} = \infty$; eademque methodo

$$1 \pm \frac{8}{1} \cdot z^1 + \frac{\infty \cdot \infty - 1}{1 \cdot 2} \cdot z^2 \pm \frac{\infty \cdot \infty - 1 \cdot \infty - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c. \pm$$

$\frac{\infty \cdot \infty - 1 \cdot \infty - 2 \cdot \infty - 3 \cdot \infty - \dots - n \cdot \infty - n}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n \cdot n + 1}$ &c., universi Numeri negativi $-1, -2, -3$ &c., $-n+1, -n$ &c. Id et Divisionis regulae, et Huddenii canonis ope superius ostendimus, etsi fortasse aliquando scrupulum moverit nonnullis castigationis ingenii Geometris, praesertim quoad extremos Coefficientium Factores in genesi Exponentialis Formulae Eulerianae, ac celeberrimae expressionis iamdudum ab Iacobo Bernoullio traditae pro foenore computando Sorti datae per continuum temporis fluxum proportionaliter accedente. Praetermitto facillimum eventum etiam Indicis $x=0$; namque Series ipsa cum Algebrae Elementis consentiens dat $(0)^0 = (1-1)^0 = 1 - \frac{0^1 \cdot \infty^1}{1} + \frac{0^2 \cdot \infty^2}{1 \cdot 2} - \frac{0^3 \cdot \infty^3}{1 \cdot 2 \cdot 3} + \frac{0^4 \cdot \infty^4}{1 \cdot 2 \cdot 3 \cdot 4} - \dots$ &c., cui in Capite IV. fusiorem paravimus explicationem.

40. Caveant autem Analystae de Infinito saepius agentes a periculosa eius nominis gravesonantis maiestate, nec quiddam unquam reale in realium Limite excogitent. Ludere in Infinito plenum est opus fallaciae, quum ab incomparabilia comparantibus nil mirum si Magnitudinum monstra eliciantur. Qui Ordines varios, ac sine numero progredientes Infinitorum moliuntur meminerint quaeso Ordines aequae diversos Nihilorum simul adstrui debere. In Limite finitorum tractando nihil aliud adsumo nisi Calculi veluti Symbolum, quod sui semper consulens veritati integerrima Analysis eo casu suppeditat ad antilogias evitandas, riteque Algorithmum interpretandum. Sic prima et altera Aequatio $1 \mp \frac{\infty \cdot x^1}{1} + \frac{\infty^2 \cdot x^2}{1 \cdot 2} \mp \frac{\infty^3 \cdot x^3}{1 \cdot 2 \cdot 3} + \dots$ nullum involvunt mirabile dictu: sunt etenim $\infty - \infty = 0$, $\infty + \infty = \infty$, sive $\frac{m}{0} - \frac{m}{0} = 0$, et $\frac{m}{0} + \frac{m}{0} = \frac{2m}{0} = \infty$. Prior itaque Expressio, a qua Coronidis instar posterior educitur, significat specialem solummodo casum Aequationis indeterminatae $\infty = \infty'$, vel $\infty = \infty + n$, sive potius $\infty - \infty = n$, in quo n evanescat. Hoc, vero Infiniti Emblema ea ratione memoravi, deindeque rursus occurret, non quia ab Infinito sint in sequentibus eruendae Formulae pertinentes ad universam rem Logometricam, et Trigonometricam, verum ne novitas quorundam casuum Serierum obsit Analyticae veritati, foveatque in imaginario magnitudinum inassignabilium orbe mysteriorum cultores.

41. Tam opimas messis reliquias in Newtoniani Binomii campo a sagacissimis anteaetate aetatis colonis toties emenso colligi posse nemo fortasse suspicabatur; quod et spem facit divites adhuc fructus latere industrii collectoris novum laborem, curamque maximam expectantes. Vanum praeterea atque inutile foret praecipuis expositis circa Series divergentes alia subiungere, eoquod in superiori tractatione de Formularum quarundam imaginarietate facta recta servata etiam a Newtoniano Binomio abunde satis aperuerimus Analyseos penetralia videntes arcanum omne, Calculique praetensum errorem penitus dissipari, si Quantitates omnes residuae ceteris iam computatis, uti quisque debet, adnumerentur. Unum addamus oportet, nempe studium, atque artificium hac in re valde utillimum collocari dum divergentes Series in convergentes traducantur, ne Residua neglecta, et inassignabilia, propterea quod sint perpetuo excrefcentia, Formulas perturbent, ipsarumque denegent usum. Huiusce transformationis exempla nec in Algebra desunt, nec raro Capita, quae sequuntur, ostendent (*).



CA-

(*) Praecipue consulatur Pars II. Institutionum Calculi Differentialis Euleri.

C A P U T II.

EVOLUTIO SERIERUM AD QUANTITATES EXPONENTIALES
P E R T I N E N T I U M.

42. **A**ctis totius Commentarii veluti fundamentis nunc praecipuo argumento propius accedere sine obscuritate fas erit. Multa, quae aucta rerum copia mihi obversantur, vel respuam uti minoris momenti, vel primoribus tantum labiis delibabo, ne videar moli potius consulere, ac leviora sectari. Typum Calculorum Lectori Geometrae complendum relinquam, quum haec scripta sint non Analyseos ignaris. Omnia vero in ordinem inventionis non digeram: etenim saltu quodam saepius incognita divinantur, nec raro pervolat cogitatio quo nisi lento Syntheseos motu, longiorique perfectae demonstrationis circuitu ad Veritatem in sublimi positam osculandam non liceat conscendere.

43. Quantitates, aut Numeros omnes Exponentiales complectitur universalis Potestas v^x ; namque superiorum Ordinum Exponentialia v^{x^z} , v^{x^z} &c. evolventur facillime, praecognita simpliciorum, in quae resolvuntur, evolutione. Exponentialia v^x facto Numero $v = 1 \pm z$ evadent $(1 \pm z)^x$, qua in expressione continentur Numeri universi positivi Unitate maiores $1 + z$, ac Unitate minores $1 - z$, nec non Numeri omnes negativi $1 - z$ subintellecta Magnitudine z Unitatem positivam superante. Tam z , quam x Numeros fractos, vel integros, irracionales, seu racionales, transcendentes, vel algebraicos generalissima significatione designant, et in Indice x confusionis evitandae causa positivos, ac negativos valores pro nunc cumulamur. Senos itaque casus praebet considerandos Potestas $(1 \pm z)^x$: terni respiciunt Indicem x positivum, sed Numerum $1 \pm z$ vel Unitate positiva maiorem, vel ipsa minorem, vel absolute negativum; reliqui eodem ordine ad Indicem x negativum referuntur.

44. Esto primum positivus Index, aut Exponens x ; atque per ea, quae fusius in praecedenti Capite, ac praesertim num. 13. et seq. ostendimus, evolvetur

$$\begin{aligned}
 \text{vetur } (1+z)^x \text{ in } 1^x & \left(1 + \frac{x}{1} \cdot z^1 + \frac{x \cdot x - 1}{1 \cdot 2} \cdot z^2 + \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \right. \\
 & \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot z^4 + \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3 \cdot x - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot z^5 + \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3 \cdot x - 4 \cdot x - 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot z^6 \\
 & + \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3 \cdot x - 4 \cdot x - 5 \cdot x - 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \cdot z^7 + \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3 \cdot x - 4 \cdot x - 5 \cdot x - 6 \cdot x - 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \cdot z^8 \\
 & + \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3 \cdot x - 4 \cdot x - 5 \cdot x - 6 \cdot x - 7 \cdot x - 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} \cdot z^9 + \\
 & \left. \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3 \cdot x - 4 \cdot x - 5 \cdot x - 6 \cdot x - 7 \cdot x - 8 \cdot x - 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} \cdot z^{10} \&c. \right). \text{ Vi-}
 \end{aligned}$$

ce Potestatum z si disponatur haec Series in ordine Potestatum x , altera
orietur Aequatio

F

 $(1+z)^x$

$$(1+z)^x = (1)^x \times \left(1 + \frac{zx^1}{1} + \frac{z^2x^2}{1.2} + \frac{z^3x^3}{1.2.3} + \frac{z^4x^4}{1.2.3.4} + \frac{z^5x^5}{1.2.3.4.5} + \frac{z^6x^6}{1.2.3.4.5.6} + \frac{z^7x^7}{1.2.3.4.5.6.7} + \frac{z^8x^8}{1.2.3.4.5.6.7.8} + \frac{z^9x^9}{1.2.3.4.5.6.7.8.9} + \frac{z^{10}x^{10}}{1.2.3.4.5.6.7.8.9.10} + \dots \right)$$

quam ultra Potestates decimi gradus x , ac z , ubi sistit, et in horizontalibus et in verticalibus Analytici Trigoni Lineis longius protrahere nullius est laboris, et operis.

45. Interim animadvertimus Seriem istam a communiter adhibitis receptisque valde differre, cum quicunque adsumatur Index x , etiamsi integer positivus, abeat Series in infinitum tam horizontaliter, quam verticaliter, videlicet non tantum limite careat quoad Progressionem interminabilem Potestatum eiusdem Numeri x , verum etiam quoad ordinem Potestatum z vel Coefficientium. Nequidem Moivraeanum Infinitinonijum ad Potestatem cuiuslibet Indicis exaltatum hac duplici gaudet in Potestatum, et Coefficientium progressu cumulata veluti Infinitate; ipsique Exponentiali Seriei consimilem Formulam, nimirum utrovis sensu infinities procedentem, nusquam vidi, nisi forte exemplum quaerendum in singulari Serie, quam prodidit ingeniosissimus Astronomus Tobias Mayerus anno MDCCLVI.

coram

$$\begin{array}{r}
 \text{D} \\
 + \frac{z^6 x^6}{1.2.3.4.5.6} + \frac{z^7 x^7}{1.2.3.4.5.6.7} + \frac{z^8 x^8}{1.2.3.4.5.6.7.8} + \frac{z^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{z^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.) \\
 \hline
 \frac{z^7 x^6}{1.2.4.5.6} + \frac{z^8 x^7}{1.2.3.5.6.8} + \frac{z^9 x^8}{1.2.3.5.6.7.8} + \frac{z^{10} x^9}{1.2.3.4.6.7.8.10} \&c. \\
 \hline
 + \frac{23z^8 x^6}{1.3.4.5.6.8} + \frac{13z^9 x^7}{1.4.5.6.8.9} + \frac{29z^{10} x^8}{1.4.6.7.8.9.10} \&c. \\
 \hline
 \frac{z^9 x^6}{1.2.5.8} + \frac{z^{10} x^7}{1.2.4.6.8} \&c. \\
 \hline
 + \frac{3013z^{10} x^6}{1.2.4.5.6.8.9.10} \&c. \\
 \hline
 \&c.
 \end{array}$$

coram Goettingensi Societate differens *de motu Martis ab Iovis, Terrae-que attractione turbato* (*). Supervacaneum autem monitum opinor excipiendum esse unicum casum, quo z evanescat, propterea quod, eadem aeque manente hypothesi Indicis x etiam non integri positivi, nihilominus loco Infinitae Seriei Aequatio semper nascitur identica $(1)^x = (1)^x$, ceteroquin notis Analyseos canonibus consona.

46. Duplex itaque est modus Potestates Binomii seu Multinomii evol- vendi; primus a Neutono, aliisque Analystis hucusque in percelebri Formula comprehensus, quae Potestatum ordinem sequitur secundae partis Binomii; alter, quem nuper iuxta Indicis Potestates composuimus. Novus hic ordo

F 2

a prio-

(*) Nec in Operibus a Mayero editis, nec in Posthumis a Lichtenbergio, sed in Academiae Goettingensis Adversariis aut Collectionibus praecitata exstat Dissertatio ab Auctore perfecta die decima Aprilis.

a priori simplicissime derivatur: nam rite effectis multiplicationibus Formula Neutoniana respondens $(1+z)^x$ aequivalet expressioni $1 + \frac{x \cdot z^1}{1} +$

$$\left(\frac{x^2 - x}{1 \cdot 2} \right) z^2 + \left(\frac{x^3 - 3x^2 + 2x}{1 \cdot 2 \cdot 3} \right) z^3 + \left(\frac{x^4 - 6x^3 + 11x^2 - 6x}{1 \cdot 2 \cdot 3 \cdot 4} \right) z^4 +$$

$$\left(\frac{x^5 - 10x^4 + 35x^3 - 50x^2 + 24x}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \right) z^5 + \left(\frac{x^6 - 15x^5 + 85x^4 - 225x^3 + 274x^2 - 120x}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \right) z^6 +$$

$$\left(\frac{x^7 - 21x^6 + 175x^5 - 735x^4 + 1624x^3 - 1764x^2 + 720x}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \right) z^7 +$$

$$\left(\frac{x^8 - 28x^7 + 322x^6 - 1960x^5 + 6769x^4 - 13132x^3 + 13068x^2 - 5040x}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \right) z^8 +$$

$$\left(\frac{x^9 - 36x^8 + 546x^7 - 4536x^6 + 22449x^5 - 67284x^4 + 118124x^3 - 109584x^2 + 40320x}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} \right) z^9 +$$

$$\left(\frac{x^{10} - 45x^9 + 870x^8 - 9450x^7 + 63273x^6 - 269325x^5 + 723680x^4 - 1172700x^3 + 1026576x^2 - 362880x}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} \right) z^{10} +$$

+ &c.: cuius omnes termini adamussim consentiunt cum illis nuper evolutae Progressionis Exponentialis. Sed, quamvis utraque Formulae tam arcto foedere coniungantur, ut una ex altera prodeat mutato solum ordinis terminorum systemate, infra dicenda luculenter evincent Seriem in antecessum deductam num°. 44. esse universaliorē, ipsiusque propemodum species, peculiare sive casus Formulas omnes Neutoniano more solito explicitas.

47. Ad indolem Seriei generalis profundius introspiciendam haereat animo necesse est elegantissima proprietas rectam quamlibet Lineam in illo Trigono Analytico Hypothenusae suae parallelam, ductam per medium intervallorum, quae in prima sinistrorsum posita Columella verticali, et suprema horizontali seiungunt terminos Potestatum cognominum Numeri z , in censum adducere terminos omnes, qui supra Rectam sint, uti pertinentes ad eam Potestatem Binomii $1+z$ maximae Potestatis $\tilde{z}$ z vel x Indice denominatam. Ita Recta coniungens puncta A et B in Typo Seriei segregat terminos Cubi $(1+z)^3$, eademque ratione supra Rectam punctis C et D interpositam universi manent termini $(1+z)^7$. Liquet insuper, quod etiam a Serie genitricae effluit $1 + \frac{x \cdot z^1}{1} + \frac{x \cdot x - 1 \cdot z^2}{1 \cdot 2} + \&c.$, Numerum terminorum perfectis $1+z$ Potestatibus respondentium in Serie, cuius fit sermo, superare Unitate

tate Numerum Triangularem Latus habentem aequale Exponenti, five Indici Potestatis assumptae. Sic Quadrato-quadratum undecim vel $10 + 1$, nec non Potestas $(1 + z)^2$ sex supra quadraginta aut $45 + 1$, et generatim

$(1 + z)^n$, dum n integer ac positivus fuerit, $\frac{(n \cdot n + 1) + 2}{2}$ componitur ter-

minis. Constat denique, quod, si aliqua arte contigerit Coefficientes terminorum Seriei Exponentialis, quibus insunt ordinatim dispositae Potestates Indicis x , ab Infinitinomii forma compendiosius, quicumque sit Index x , liberare, et expressione monomia singulos cohibere, haec ipsamet Series universalissima ita comparata se quoque extendet repraesentandis Potestatum perfectarum valoribus, sed non parum diversimode ab usitata Formula Neutonica. Revera, quaevis methodus pro denotandis commodius sub unica specie Coefficientibus adhibeatur, manebit semper nihilominus inconcussa proprietas Seriem Exponentialem, explicando in Series eius Coefficientibus Monomii formam indutis, completi supra Lineas transversim ductas instar AB, CD inter termi-

nos continentes x^m , atque x^{m+1} , necnon z^m , ac z^{m+1} , terminos omnes Potestatis $1 + z$ Indice indefinito m integro positivo gaudentis, ideoque ceteros terminos infra Lineam numero carentes simul additos Nullitatem conficere. Haec Nullitas, tametsi obtineatur summatis partibus residuis sine limite protrahendis aliquorum Coefficientium, necnon singulis Coefficientibus ceteris nulla sui parte minutis Seriei Exponentialis, nil substantialiter mutat in valore Potestatis perfectae nova generalissima Serie exprimendo, quum idem valor Magnitudinis cuiuscunque servetur addito vel detracto ab ea Nihilo, qualibet hoc Nihilum multiformi analytica, seu transcendente quantitatibus se invicem elidentium expressione significare placeat, dummodo sit verum Nihilum. Quod maximi momenti opus ut firmitus aedificetur effingo iam redactam Seriem Exponentialem, et in unum terminum singillatim coalescente quovis ex Coefficientibus verti in $1 + Ex^1 + Fx^2 + Gx^3 + Hx^4 + Kx^5 + Lx^6 + Mx^7 + Nx^8 + Px^9 + Qx^{10} + \&c.$, ubi E Seriem Infinitam $+$

$$\frac{z}{1} - \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 3} - \frac{z^4}{1 \cdot 4} + \&c., \text{ F Seriem alteram } + \frac{z^2}{1 \cdot 2} - \frac{z^3}{1 \cdot 2} +$$

$$\frac{11 z^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{5 z^5}{1 \cdot 3 \cdot 4} + \&c., \text{ G proxime adpositam } + \frac{z^3}{1 \cdot 2 \cdot 3} - \frac{z^4}{1 \cdot 4} + \frac{7 z^5}{1 \cdot 2 \cdot 3 \cdot 4}$$

$\frac{5z^6}{1.4.4} + \&c.$, Speciesque sequentes similiter homologas Series adaequent.

Non pigeat diutius immorari in aequipollentia expressionum tanto discrimine compositarum, et addere in rem illustrandam exempla in Potestatibus perfectis primi, tertii, ac septimi gradus. Modus summopere a Neutroniano diversus ipsas Exponentialiter exhibendi tribuet has Formulas numero terminorum Infinitas, nimirum $1 + E + F + G + H + K + L + M + N + \&c.$ in Indice 1; $1 + 3E + 9F + 27G + 81H + 243K + 729L + 2187M + 6561N + \&c.$ in Indice 3; ac demum in Indice 7^{mae} Potestatis $1 + 7E + 49F + 343G + 2401H + 16807K + 117649L + 823543M + 5764801N + \&c.$ Istaе vero Formulae, utcunque sine carentes, reapse solummodo componunt vi Analytici Trianguli, ipsiusque genesis superius explicatae, $1 + z$, $1 + 3z + 3z^2 + z^3$, $1 + 7z + 21z^2 + 35z^3 + 35z^4 + 21z^5 + 7z^6 + z^7$, quemadmodum liquet substitutis in Trigono pro x respectivis valoribus 1, 3, 7. Hoc enim confecto, praeter terminos memoratos, qui Neutronianam expressionem constituunt, ceteri omnes per Coefficientes diffusi evanescent, et ea curiosa satis Lege, ut Nihilorum sine Limite cumulum offerant, si diagonaliter seu aequidistanter Lineis AB, CD computentur, prouti unusquisque facile experiundo reperiet. Inde deducitur Seriem Exponentialem, etsi infinites productam, intimius reddere eundem Potestatum perfectarum valorem, quem finitis numero terminis tribuit Formula a Neutono usitata, nec hilum ab ea discrepare nisi in terminorum addita coacervatione, qui adeo se gerant, moderenturque, ut dent Nihilorum, inefficacem quomodocunque, interminabilem progressionem. Supposito speciminis instar Indice $x = 3$, dum agatur de Cubis, si infra Rectam AB transversim reliquos terminos numeremus, nanciscemur profecto $\left(\frac{3}{4} + \frac{99}{24} - \frac{27}{4} + \frac{81}{24}\right)z^4 = 0$; $\left(-\frac{3}{5} - \frac{45}{12} + \frac{189}{24} - \frac{81}{12} + \frac{243}{120}\right)z^5 = 0$; $\left(-\frac{3}{6} + \frac{1233}{360} - \frac{135}{16} + \frac{1377}{144} - \frac{243}{48} + \frac{729}{720}\right)z^6 = 0$, eodemque systemate in infinitum.

48. Non amplius itaque in posterum mirabuntur Analystae Seriem Infinitam Exponentialium nedum quoad valorem numericum cum Formula speciei compositionisve toto coelo dissitae Neutroniani Binomii in casu Indicis x negativi universaliter aut positivi, sed fracti, irrationalis, vel transcendens consentire, quibus hypothesibus, ut omnes norunt, Neutoni quoque expressio
fine

sine limite progreditur, verum insuper, quod mirabilius videtur, implicite et infinito terminorum numero involutos absconditosque comprehendere singulos terminos Formulae ipsius Neutonianae quoties postrema haec abrumpatur, vel quum Index x fiat integer positivus m , finitoque terminorum numero perfecta Potestas Binomii $(1+z)^m$ sit praedita. Utrarumque

$$\begin{aligned} & \text{Formularum numeris omnibus absolutam aequipollentiam } 1 + \frac{m}{1} \cdot z^1 + \\ & \frac{m \cdot m-1}{1 \cdot 2} \cdot z^2 + \frac{m \cdot m-1 \cdot m-2}{1 \cdot 2 \cdot 3} \cdot z^3 + \frac{m \cdot m-1 \cdot m-2 \cdot m-3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot z^4 + \\ & \frac{m \cdot m-1 \cdot m-2 \cdot m-3 \cdot m-4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot z^5 + \&c.; \text{ atque } 1 + \frac{E}{1} m^1 + \frac{F}{1} m^2 + \frac{G}{1} m^3 \\ & + \frac{H}{1} m^4 + \frac{K}{1} m^5 + \&c. \text{ propemodum aenigmaticam profiteri potius a Cal-} \end{aligned}$$

culi inappellabili necessitate hucusque coacti, quam nitide in Infiniti latebris perlustrare, sibi que causa cognita suadere potuerunt Geometrae. Profecto, nisi communis origo, harmonia, ac reciprocum veluti cognationis vinculum, aut, dicam verius, tacita identitas ordine tantum elementorum integrantium permutato nunc detecta fuisset, summa adhuc, quam elicere datum est, evidentialiter lateret. Series ergo Exponentialis universis satisfaciens Binomii Potestatibus uti vere oecumenica Potestatum Formula emicat, quum Indici constanti, et variabili sine ullo laboris discrimine accommodetur, completaturque in infinito terminorum cumulo Summationem respiciente, atque, ut infra videbitur, transcendentaliter, celatam Formulam ipsam evolutam more Neutoni, veluti Cardanicae, iisque consimiles expressiones continent in imo recessu sub irrationali et difficili forma rationales Aequationis tertii gradus, et superiorum Radices, ac quandoque pro universalitate Analyticae resolutionis tuenda spuriam imaginarietatem commiscent, nonnisi subtili artificio vincendam Trigonometrico.

49. Uno verbo adferam Seriem Exponentialem non modo generalitate, veruntamen artis subtilitate, et legis, seu terminorum progressus elegantia Neutonianam Potestatum Formulam antecellere. Ea namque est Seriei $1 + Ex^1 + Fx^2 + Gx^3 + Hx^4 + Kx^5 + Lx^6 + Mx^7 + Nx^8 + Px^9 + Qx^{10} + \&c.$ perbella sane ac peculiaris affectio, quod singulari tantum valore primi Coefficientis Infinitinomii, seu $\approx E$ praecognito reliqui statim omnes $F, G, H, K, L, M, N, P, Q$ &c. eius ope, ac mira quadam facilitate determinantur. Qui enim vel

tantillum

tantillam examini subiiciat Seriem $+\frac{z^2}{1.2} - \frac{z^3}{1.2} + \frac{11z^4}{1.2.3.4} - \frac{5z^5}{1.3.4} +$
 $\frac{137z^6}{1.3.4.5.6} - \frac{7z^7}{1.4.5} + \frac{363z^8}{1.4.5.7.8} - \frac{761z^9}{1.5.7.8.9} + \frac{7129z^{10}}{1.5.7.8.9.10} - \&c. = F$ eam inve-
niet aequalem $\frac{E^2}{1.2}$, five medietati Quadrati Seriei quoque Infinitae $+\frac{z}{1} -$
 $\frac{z^2}{1.2} + \frac{z^3}{1.3} - \frac{z^4}{1.4} + \frac{z^5}{1.5} - \frac{z^6}{1.6} + \frac{z^7}{1.7} - \frac{z^8}{1.8} + \frac{z^9}{1.9} - \frac{z^{10}}{1.10} + \&c.$,
atque ultro detegat Seriem postsequentem $+\frac{z^3}{1.2.3} - \frac{z^4}{1.4} + \frac{7z^5}{1.2.3.4} -$
 $\frac{5z^6}{1.4.4} + \frac{29z^7}{1.3.5.6} - \frac{469z^8}{1.2.3.5.6.8} + \frac{29531z^9}{1.2.3.5.6.7.8.9} - \frac{1303z^{10}}{1.3.4.7.8.9} + \&c. =$
 $G = \frac{E^3}{1.2.3}$, vel sextae parti Cubi Seriei $+\frac{z}{1} - \frac{z^2}{1.2} + \frac{z^3}{1.3} - \frac{z^4}{1.4} +$
 $\frac{z^5}{1.5} - \&c.$, et pari processu indefinito alias Series $+\frac{z^4}{1.2.3.4} - \frac{z^5}{1.3.4} +$
 $\frac{17z^6}{1.2.3.4.6} - \frac{7z^7}{1.2.4.6} + \frac{967z^8}{1.2.3.4.5.6.8} - \frac{89z^9}{1.3.4.5.8} + \frac{4523z^{10}}{1.3.4.5.6.7.9} - \&c.$
 $= H = \frac{E^4}{1.2.3.4}$, nec non $+\frac{z^5}{1.2.3.4.5} - \frac{z^6}{1.2.4.6} + \frac{5z^7}{1.2.3.4.6} -$
 $\frac{7z^8}{1.3.6.8} + \frac{1069z^9}{1.2.4.5.6.8.9} - \frac{19z^{10}}{1.2.4.4.8} + \&c. = K = \frac{E^5}{1.2.3.4.5}; + \frac{z^6}{1.2.3.4.5.6} -$
 $\frac{z^7}{1.2.4.5.6} + \frac{23z^8}{1.3.4.5.6.8} - \frac{z^9}{1.2.5.8} + \frac{3013z^{10}}{1.2.4.5.6.8.9.10} - \&c. = L$
 $= \frac{E^6}{1.2.3.4.5.6}; + \frac{z^7}{1.2.3.4.5.6.7} - \frac{z^8}{1.2.3.5.6.8} + \frac{13z^9}{1.4.5.6.8.9} - \frac{z^{10}}{1.2.4.6.8} +$
 $\&c. = M = \frac{E^7}{1.2.3.4.5.6.7}; + \frac{z^8}{1.2.3.4.5.6.7.8} - \frac{z^9}{1.2.3.5.6.7.8} + \frac{29z^{10}}{1.4.6.7.8.9.10} -$
 $\&c. = N = \frac{E^8}{1.2.3.4.5.6.7.8}; + \frac{z^9}{1.2.3.4.5.6.7.8.9} - \frac{z^{10}}{1.2.3.4.6.7.8.10} + \&c.$
 $= P = \frac{E^9}{1.2.3.4.5.6.7.8.9}; + \frac{z^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c. = Q = \frac{E^{10}}{1.2.3.4.5.6.7.8.9.10}$
 $\&c. \&c.$

50. Hanc Coefficientium Progressionis Legem simplicissimam, ac satis perinucundam, quae Seriem Exponentialem $1 + Ex^1 + Fx^2 + Gx^3 + Hx^4 + Kx^5 + Lx^6 + Mx^7 + Nx^8 + Px^9 + Qx^{10} + \&c.$ commo-
diorem

diorem facit, redditque $1 + \frac{Ex^1}{1} + \frac{E^2x^2}{1.2} + \frac{E^3x^3}{1.2.3} + \frac{E^4x^4}{1.2.3.4} + \frac{E^5x^5}{1.2.3.4.5}$
 $+ \frac{E^6x^6}{1.2.3.4.5.6} + \frac{E^7x^7}{1.2.3.4.5.6.7} + \frac{E^8x^8}{1.2.3.4.5.6.7.8} + \frac{E^9x^9}{1.2.3.4.5.6.7.8.9} +$
 $\frac{E^{10}x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.$, quisquis proprio Marte evidenter sibi suadebit.

Quadratum etenim τE per notos Potestatum Binomii, sive Infinitinomii canones

$$\text{aequivalet } + \frac{z^2}{1} + \frac{z^3}{1} + \frac{z^4}{4} + \frac{z^5}{3} + \frac{z^6}{9} + \frac{z^7}{6} + \frac{z^8}{16} + \frac{z^9}{10} + \frac{z^{10}}{25} + \&c.,$$

$$+ \frac{2z^4}{3} + \frac{z^5}{2} + \frac{z^6}{4} + \frac{z^7}{5} + \frac{2z^8}{15} + \frac{z^9}{9} + \frac{z^{10}}{12}$$

$$+ \frac{2z^6}{5} + \frac{z^7}{3} + \frac{z^8}{6} + \frac{z^9}{7} + \frac{2z^{10}}{21}$$

$$+ \frac{2z^8}{7} + \frac{z^9}{4} + \frac{z^{10}}{8}$$

$$+ \frac{2z^{10}}{9}$$

cuius medietas post respectivas substitutiones Factorum in numericis Denominatoribus praebet $+ \frac{z^2}{1.2} + \frac{z^3}{1.2} + \frac{11z^4}{1.2.3.4} + \frac{5z^5}{1.3.4} + \frac{137z^6}{1.3.4.5.6}$

$$+ \frac{7z^7}{1.4.5} + \frac{363z^8}{1.4.5.7.8} + \frac{761z^9}{1.5.7.8.9} + \frac{7129z^{10}}{1.5.7.8.9.10} + \&c.,$$

nimirum, expressionem opportunis reductionibus additis $= F$. Similiter Cubus τE aequat

$$+ \frac{z^3}{1} + \frac{3z^4}{2} + \frac{3z^5}{4} + \frac{z^6}{8} + \frac{z^7}{4} + \frac{z^8}{6} + \frac{z^9}{27} + \frac{z^{10}}{12} + \&c.,$$

$$+ \frac{z^5}{1} + \frac{z^6}{1} + \frac{z^7}{3} + \frac{11z^8}{16} + \frac{z^9}{4} + \frac{3z^{10}}{32}$$

$$+ \frac{3z^6}{4} + \frac{3z^7}{4} + \frac{3z^8}{5} + \frac{3z^9}{16} + \frac{z^{10}}{2}$$

$$+ \frac{3z^7}{5} + \frac{z^8}{2} + \frac{11z^9}{20} + \frac{11z^{10}}{24}$$

$$+ \frac{z^9}{2} + \frac{3z^{10}}{7}$$

$$+ \frac{3z^9}{7} + \frac{3z^{10}}{8}$$

G

ideoque

ideoque eius subsextuplum tribuit $+\frac{z^3}{1.2.3} - \frac{z^4}{1.4} + \frac{7z^5}{1.2.3.4} - \frac{15z^6}{1.2.4.6}$

$+\frac{29z^7}{1.3.5.6} - \frac{469z^8}{1.2.3.5.6.8} + \frac{29531z^9}{1.2.3.5.6.7.8.9} - \frac{1303z^{10}}{1.3.4.6.7.8} + \&c.$, videlicet G.

Quadrato-quadratum τE quum sit ex Elementis Arithmeticae Universalis $+\frac{z^4}{1} - \frac{2z^5}{1} + \frac{17z^6}{6} - \frac{7z^7}{2} + \frac{967z^8}{240} - \frac{89z^9}{20} + \frac{9046z^{10}}{1890} + \&c.$, erit quo-

que eiusdem pars quarta supra vigesimam aequalis Seriei $+\frac{z^4}{1.2.3.4} -$

$\frac{z^5}{1.3.4} + \frac{17z^6}{1.2.3.4.6} - \frac{7z^7}{1.2.4.6} + \frac{967z^8}{1.2.3.4.5.6.8} - \frac{89z^9}{1.3.4.5.8} + \frac{9046z^{10}}{1.3.5.6.7.8.9} + \&c.$,

vel Coefficienti homologo H. Qua in affectione novis exemplis confirmanda ne longius, ac frustra tempus teram, saltu quodam quasi lusorio Potestatem gradus octavi, sive Quadrato-quadrati-quadratum perpendere liceat. Omnes norunt $E^8 =$

$+\frac{z^8}{1} - \frac{4z^9}{1} + \frac{29z^{10}}{3} - \&c.$; ac propterea pars aliquota τE^8 denominata

ab Unitatibus tercentum viginti supra quartam myriadem, vel brevius $\frac{E^8}{1.2.3.4.5.6.7.8}$

consentit cum $+\frac{z^8}{1.2.3.4.5.6.7.8} - \frac{z^9}{1.2.3.5.6.7.8} + \frac{29z^{10}}{1.2.4.5.6.7.8.9} - \&c.$, prouti

Coefficiens N adamussim se monstrat.

51. His rite peractis, si Seriem numero terminorum Infinitam, et elegantia, ac simplicitate nulli secundam $+\frac{z^1}{1} - \frac{z^2}{1.2} + \frac{z^3}{1.3} - \frac{z^4}{1.4} +$

$\frac{z^5}{1.5} - \frac{z^6}{1.6} + \frac{z^7}{1.7} - \frac{z^8}{1.8} + \frac{z^9}{1.9} - \frac{z^{10}}{1.10} + \&c.$, scilicet Functionem

istam variabilis z , signo Δ indicemus, assumentes Functionem memoratam veluti Notam Magnitudinis, quam in sequente Capite III., tametsi Transcendentem, intimius rimari, et inopinato compendio repraesentare fortuna dabit, progignitur universalis evolutio Numeri Exponentialis, aut Potestatis indefinitae, ac Indice quolibet praeditae $(1+z)^x = (1)^x \left(1 + \frac{\Delta x^1}{1}\right.$

$+\frac{\Delta^2 x^2}{1.2} + \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^4 x^4}{1.2.3.4} + \frac{\Delta^5 x^5}{1.2.3.4.5} + \frac{\Delta^6 x^6}{1.2.3.4.5.6} + \frac{\Delta^7 x^7}{1.2.3.4.5.6.7}$

$+\frac{\Delta^8 x^8}{1.2.3.4.5.6.7.8} + \frac{\Delta^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{\Delta^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.$), seu potius,

nonnul-

nonnullorum uti mos est Geometrarum, $\frac{(1+z)^x}{(1)^x} = 1 + \frac{\Delta x^1}{1} + \frac{A \Delta x^2}{2}$
 $+ \frac{B \Delta x^3}{3} + \frac{C \Delta x^4}{4} + \frac{D \Delta x^5}{5} + \frac{E \Delta x^6}{6} + \frac{F \Delta x^7}{7} + \frac{G \Delta x^8}{8} + \frac{H \Delta x^9}{9} +$
 $\frac{I \Delta x^{10}}{10} + \&c.$, posita nempe $A = \Delta x$, ceterisque speciebus, B, C, D, E,

F, G, H, I &c: terminorum valores proxime antecedentium designantibus. Nec scitu inutile cenfeo Seriem illam Exponentialem, quam tanta simplicitas fortunavit, in terminorum lege, ac progressu aliam imitari, cui ordinem tribuit prior Linea horizontalis explicata in numero 44., atque plenius in eam verti dummodo vice z , eiusque Potestatum substituuntur, a prima incipiendo, Potestates Seriei Limitem nescientis $\frac{z}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} -$

$\frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c.$ Huic postremae nomen Seriei Geometrico-Harmonicae ob Numeratorum Progressionem Geometricam impositum malim. Nemo autem priorem Lineam horizontalem evolutae Potestatis Binomii praefato in numero 44. coefficientium respectu compositam esse mirabitur ad instar Seriei notissimae ortae ab evolutione $\tau z (1+z)^\infty$ tam discrepantis a finita Potestate $(1+z)^x$, quando meminerit illam Lineam supremam complecti eos tantummodo terminos Potestatis totius in Seriem conversae, qui in Formulae Neutonianae Coefficientibus $\frac{x \cdot x-1 \cdot x-2 \cdot x-3 \cdot \dots \cdot x-m}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot m \cdot m+1}$

primum, aut altioris gradus unicum terminum Numeratoris evoluti, nempe $\frac{x^{m+1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot m \cdot m+1}$ respiciunt, quemadmodum etiam hypothesis statuit Indicis x Infiniti propter $-1, -2, -3, -4$ &c: prae ∞ in censum non computandos.

52. Qui Matheseos voluptate rapti per amoena libenter exspatiantur se delectari certe non desinent facillimam methodum nactis Exponentialia universa tractandi, quo, prope dixeris, manuducunt praecostensa symptomata. Etenim dato Binomio $1+z$ ad Potestatem Indicis x exaltando $(1+z)^x$, emerget Series petita hoc unico Canone facem ferente. „ *Evolvatur Binomium more Neutoniano, sed eam in Coefficientibus legem sequendo, quod prae*

Series Δ , ac ipsius Potestates ordinatim substituantur „. Profecto sic effluet

$$\begin{aligned} & 1 + \frac{x^1 \Delta^1}{1} + \frac{x^2 \Delta^2}{1.2} + \frac{x^3 \Delta^3}{1.2.3} + \frac{x^4 \Delta^4}{1.2.3.4} + \frac{x^5 \Delta^5}{1.2.3.4.5} + \frac{x^6 \Delta^6}{1.2.3.4.5.6} \\ & + \frac{x^7 \Delta^7}{1.2.3.4.5.6.7} + \frac{x^8 \Delta^8}{1.2.3.4.5.6.7.8} + \frac{x^9 \Delta^9}{1.2.3.4.5.6.7.8.9} + \frac{x^{10} \Delta^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: \end{aligned}$$

Seriei praecedenti optime congruens. Insuper autem memoriae mandetur Se-

riem illam Δ Geometrico-Harmonicam $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6}$

+ $\frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c$: talem esse secundi Nominis z Functio-

nem, ut, si Quantitas numerica Exponentialis haberetur v^x , et v maior Unita-

te, abiret in $\frac{v-1^1}{1} - \frac{v-1^2}{2} + \frac{v-1^3}{3} - \frac{v-1^4}{4} + \frac{v-1^5}{5} - \frac{v-1^6}{6} +$

$\frac{v-1^7}{7} - \frac{v-1^8}{8} + \frac{v-1^9}{9} - \frac{v-1^{10}}{10} + \&c$: , quod luculentissime patet

Aequationem intuenti $1 + z = v$, seu $z = v - 1$. Erit itaque $v^x =$

$(1 + (v - 1))^x = (1)^x \left(1 + \frac{\Phi x^1}{1} + \frac{\Phi^2 x^2}{1.2} + \frac{\Phi^3 x^3}{1.2.3} + \frac{\Phi^4 x^4}{1.2.3.4} + \right.$

$\frac{\Phi^5 x^5}{1.2.3.4.5} + \frac{\Phi^6 x^6}{1.2.3.4.5.6} + \frac{\Phi^7 x^7}{1.2.3.4.5.6.7} + \frac{\Phi^8 x^8}{1.2.3.4.5.6.7.8} + \frac{\Phi^9 x^9}{1.2.3.4.5.6.7.8.9} +$

$\frac{\Phi^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c$:), in qua Formula Functio Φ praenotatam Seriem,

alternis Signis Potestates Binomii $v - 1$ in Numeratoribus, et Numeros naturaliter progredientes in Denominatoribus complectentem, significat.

53. *Totas ergo vires impendere debemus in Functionis* $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3}$

$-\frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c$: natura, atque affectio-

nibus variis quaerendis, quum ab ea una obtineatur argumenti de Serierum Ex-

ponentialium evolutione complementum. Interea observandum Functionem illam,

dummodo z , ideoque Binomium $1 + z$ non fit Numerus positivus infinite ma-

gnus, valorem semper praebere finitum et positivum, necnon in hypothesi z ne-

gativi, veruntamen nec -1 , nec ultra, semper finitum et negativum. Quum ete-

nim tota Series $1 + \frac{\Delta x^1}{1} + \frac{\Delta^2 x^2}{1.2} + \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^4 x^4}{1.2.3.4} + \frac{\Delta^5 x^5}{1.2.3.4.5} + \frac{\Delta^6 x^6}{1.2.3.4.5.6}$

+

$$+ \frac{\Delta^7 x^7}{1.2.3.4.5.6.7} + \frac{\Delta^8 x^8}{1.2.3.4.5.6.7.8} + \frac{\Delta^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{\Delta^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.,$$

z atque x finitis manentibus positivis, finitum Numerum universaliter denotet $\frac{1+z^x}{1^x}$, nequit esse $\tau\delta$ Infinitum: secus enim infinities magna evaderet

Series ipsamet Exponentialis ob omnes terminos positivos. Insuper addo „*positivos* „ propterea quod neminem lateat casu $z = 1$, vel $1 + z = 2$,

$$\text{Seriem } \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} +$$

$$\&c: \text{ peraequare positivum certe valorem } \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \frac{1}{9.10}$$

$$+ \&c: , \text{ seu } \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \frac{1}{56} + \frac{1}{90} + \&c: \text{ in infinitum, videlicet dif-}$$

ferentiam illam duas inter Harmonicas Series, singillatim Infinitae magnitudi-

$$\text{nis, nempe } \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \&c: \text{ maiorem, ac denominatam ab}$$

$$\text{imparibus Numeris, et } \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \&c: \text{ minorem, ac}$$

denominatam a Numeris tam impariter, quam pariter paribus, esse positi-

vam finitam. Sin z minus Unitate fiat, quod eodem redit ac $\tau\delta$ $1 + z$ Bina-

rio minus, pari modo constabit de enunciati Theorematis veritate; nam $\frac{z^1}{1} >$

$$\frac{z^1 \cdot 1}{2}, \frac{z^3}{3} > \frac{z^3 \cdot 1}{4}, \frac{z^5}{5} > \frac{z^5 \cdot 1}{6}, \frac{z^7}{7} > \frac{z^7 \cdot 1}{8}, \frac{z^9}{9} > \frac{z^9 \cdot 1}{10} \&c: \text{ qua de re,}$$

$$\text{quum etiam ex hypothesi habeatur } 1 > z, \text{ obtinebimus eo magis } \frac{z^1}{1} > \frac{z^2}{2},$$

$$\frac{z^3}{3} > \frac{z^4}{4}, \frac{z^5}{5} > \frac{z^6}{6}, \frac{z^7}{7} > \frac{z^8}{8}, \frac{z^9}{9} > \frac{z^{10}}{10} \&c: , \text{ nimirum, Seriem } \frac{2z^1 - 1z^2}{1.2}$$

$$+ \frac{4z^3 - 3z^4}{3.4} + \frac{6z^5 - 5z^6}{5.6} + \frac{8z^7 - 7z^8}{7.8} + \frac{10z^9 - 9z^{10}}{9.10} + \&c: \text{ positivi, atque}$$

finiti valoris, tametsi sine carentem. Quo posito, cum nulli detur Functioni Quan-

titatis variabilis, quidquid in contrarium senserit aliquoties Leibnizius, transitus

a positivo ad negativum valorem nisi per Nihilum, aut Infinitum, et Functio,

de qua nunc agitur, evanescat solummodo casu $z = 0$, si de realibus, quas

hic contemplamur, magnitudinibus sermo instituatur, ut ex Seriei indol: li-

$$\text{quet, sit negativa casu } z \text{ Numeri negativi, quippe quae tunc vertitur in } -$$

$$\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} - \&c: , \text{ ac constanter finitum}$$

valorem

valorem exhibens fiat positiva casu z Unitate minoris, vel Unitati aequalis, positivum quoque valorem abscondere debet in infinitum protrahere, etiam si z superet Unitatem, scilicet $1 + z$ evadat maius Binario, quoquomodo Series ex convergente tum in divergentem mutetur. Inde profluit altera affectio, Seriem nimirum Signorum inversione ortam $-\frac{z^1}{1} + \frac{z^2}{2} - \frac{z^3}{3} + \frac{z^4}{4} - \frac{z^5}{5} + \frac{z^6}{6} - \frac{z^7}{7} + \frac{z^8}{8} - \frac{z^9}{9} + \frac{z^{10}}{10} - \&c.$ negativum obtinere valorem, quicumque Numerus positivus a specie z indicetur. Differentiam praeterea Serierum Geometrico-Harmonicarum $\frac{z^1}{1} + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \frac{z^9}{9} + \&c.$, atque $\frac{z^2}{2} + \frac{z^4}{4} + \frac{z^6}{6} + \frac{z^8}{8} + \frac{z^{10}}{10} + \&c.$ quum finitam reapse esse, vel finitae Quantitatis symbolum ostenderimus, quilibet positivus valor variabili z tribuatur, inferri poterit non tantum Progressiones istas Limite carentes, et finiti valoris quando $z < 1$ finita magnitudine inter se discrepare, verum etiam, quod subtilius videbitur, dum posito $z =$, et > 1 infinities magnae eadem singulae Series evadant. Nec apodictice demonstrari hic arbitror opus esse Series singulas praefatas $\frac{z^1}{1} + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \frac{z^9}{9} + \&c.$, atque $\frac{z^2}{2} + \frac{z^4}{4} + \frac{z^6}{6} + \frac{z^8}{8} + \frac{z^{10}}{10} + \&c.$ finitum tribuere valorem quum $z < 1$, et è contra quum $z > 1$ quocunque assignabili Numero maiorem: eoquod prima in hypothese per se pateat Progressiones Geometricas decrescentes $z^1 + z^3 + z^5 + z^7 + z^9 + \&c.$, $z^2 + z^4 + z^6 + z^8 + z^{10} + \&c.$, in quibus constans Ratio termini antecedentis ad consequentem sit $1 : z^2$, finitum complecti valorem, nempe $\frac{z}{1 - z^2}$, ac $\frac{z^2}{1 - z^2}$, maiorique iure $\frac{z^1}{1} + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \frac{z^9}{9} + \&c.$, $\frac{z^2}{2} + \frac{z^4}{4} + \frac{z^6}{6} + \frac{z^8}{8} + \frac{z^{10}}{10} + \&c.$; atque in altera hypothese Progressiones ipsas Geometricas, terminis earum quibuscunque in immensum crescentibus, excedere singulos homologos Serierum ambarum $\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \&c.$, $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \&c.$ infinito valore gaudentium, utpote in Harmonica Progressione.

54. Mirum autem non erit Seriem $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c.$

$-\frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c.$, ex duobus Infinitis se mutuo elidentibus veluti coacervatam, dum z Unitatem superet, afficere Formulam Exponentialem mendosa sub specie geminae Infinitatis, quae divergentium ratione terminorum Summationem respuat, saltem veritati propinquam, et nihilo tamen minus finitum valorem abscondat. Divergentia namque Seriei est ipsamet, quae diversa solummodo terminorum dispositione Formulam quoque Binomii Newtoniani in eodem casu z Unitate maioris perturbat, et a Residuis Potentiationis semper augescentibus originem ducit iuxta theoricen Capite praecedente satis explicitam. Attamen id commodi perquam maximi accidit in evolutione Formulae Exponentialis, quod, si remedium afferatur divergentiae unius Coefficientis secundi Seriei termini eliminandae idoneum, nil amplius difficultatis vel obstaculi superfit, quum ceteri Coefficientes universi primum illum ad perfectas Potestates elevatum quodammodo repetant. Quae utillima sane facilitas in Binomii Potestatibus more Newtoni evolutis penitus deficit, ubi terminorum divergentia tam intime, varieque diffunditur, atque immiscetur in universitate Progressionis, ut nec lex, nec methodus adsit illam extricandi, nisi prima Binomii pars maior secunda sumatur. Hanc artificii analytici formam ad medendam divergentiam Seriei Exponentialis imitari non possumus; proptereaquod vi numⁱ. 44. prior Binomii terminus inseparabiliter Unitas esse debeat, ne Exponentialitatem in non Exponentiales terminos evolvendam iterum redire non victam, seu reviviscere praepostero fato contingat. Varios subsequente in Capite Analystarum labores pro Serie Geometrico-Harmonica $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c.$ positiva, nec non eius inversa negativa undequaue convergente efficienda simplicioribus Algebrae Cartesianae principiis accommodaturi nunquam hoc interim mirari desinemus singularem ipsius Seriei characterem, quod, si z vel tantulum supra Unitatem augeatur, exemplo a convergente $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \&c.$, five, uti Hugenio scribere placuit, $\frac{2^2}{3^2-1} + \frac{2^2}{7^2-1} + \frac{2^2}{11^2-1} + \frac{2^2}{15^2-1} + \frac{2^2}{19^2-1} + \&c.$, in divergentem abeat, tametsi crescentes tunc Numeratores ab auctis semper in infinitum Denominatoribus temperentur. Et

re quidem vera nefas esset dubium haerere in hac divergentia fatenda, dum Numeratores geometricè, ideoque celerius, vividiusque augeantur, quam Denominatores arithmetice tantum crescentes. Monendum est potius priores solummodo Seriei terminos convergentiae prospectu quasi fallere donec z maneat infra Binarium, maiorique numero fore terminos convergentes, quanti minoris τ z superet Unitatem. Exemplo sint $z = 1 \frac{1}{10}$, et

postmodum $= 1 \frac{1}{3}$: eruentur alterutra in hypothesi Series, quae Limitem

$$\text{ignorant, } \frac{11}{10} - \frac{121}{100} + \frac{1331}{3000} - \frac{14641}{40000} + \frac{161051}{500000} - \frac{1771561}{6000000} + \frac{19487171}{70000000} \\ - \frac{214358881}{800000000} + \frac{2357947691}{9000000000} - \frac{25937424601}{100000000000} + \frac{28937424601}{1000000000000} - \\ \frac{3138428376721}{12000000000000} + \frac{34522712143931}{130000000000000} - \&c., \text{ atque } \frac{4}{3} - \frac{8}{9} + \frac{64}{81} - \\ \frac{64}{81} + \frac{1024}{1215} - \frac{2048}{2187} + \frac{16384}{15309} - \frac{8192}{6561} + \frac{262144}{177147} - \frac{524288}{295245} + \&c.$$

adeo comparatae, ut in priori detegi incipiat terminorum augmentum post undecimum, ac in posteriori tam citius a termino quinto. Nititur haec omnis theoria principio satis obvio, quod terminus consequens ad proxime antecedentem in qualibet harum Serierum Geometrico-Harmonicarum se habeat in Ratione $\frac{z}{m+1}$ ad $\frac{1}{m}$; et inibi m significat Numerum indefinitum naturalis Arithmeticae Progressionis. Exinde liquet fore terminos illius

Seriei convergentes donec $z < 1 + \frac{1}{m}$, sed necessario in divergentes postmodum verti, propterea quod m sine limite crescendo evadet tandem $z > 1 + \frac{1}{m}$; quae Aequationes $z < 1 + \frac{1}{m}$, $z > 1 + \frac{1}{m}$, perfecte conveniunt

cum aliis $\frac{m'}{m'-1} > z$, $\frac{m'}{m'-1} < z$, et simul ostendunt desinere tum periculofam convergentiam initialium terminorum, quum τ z Binario, aut $1 + z$ Ternario par fiat.

55. Ceterum, quidquid sit de singularibus hisce Numericis Progressionibus existente z Numero rationali, raturum firmumque universaliter habeatur

viciniam

viciniam non modo, sed punctum Limitis, dividitis in Serie $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c$: terminos antea convergentes a deinceps divergentibus, facillime aliquando, ac nitidissime significari a duobus terminorum Numeratoribus, ac Denominatoribus iuxta positus eiusdem respectivi valoris; qui parallelismus, dum eveniat, recurrit ad tot Seriei terminos, quot Numeri fractionarii τz simplicissimis cyphris expressi Numerator contineat Unitates. Revera nequit esse in rationalibus Numeris, qui Unitatem excedant fractione pro Numeratore Unitatem ipsam habente, τz prius $< 1 + \frac{1}{m}$, ac postea $> 1 + \frac{1}{m+2}$, nisi intermedia Aequalitas, eiusdem variabilis respectu, $z = 1 + \frac{1}{m+1}$ accesserit. Qua, et superioribus simul prae-

missis affectionibus nil mirum si, quo minor fuerit, etiam irrationalis, sive transcendens, excessus τz supra Unitatem, eo maior fiat convergentium Numerus terminorum, ac viceversa, quamvis Progressio vere pertineat ad divergentium Serierum genus; et id manebit inconcussum, utcunque semper semperque minuto excessu z supra Unitatem, quantumlibet, ac sine limite cumulentur initiales termini convergentes, atque minus cauti ii celent longissime incipientem divergentiam, et vitium, vel, melius dicam, inutilitatem Seriei.

56. Omni vero cura convergentiae Functionis Δ , vel Φ , scilicet Coefficientis secundi termini Progressionis Exponentialis, satis superque faciendi in praesentiarum relicta observationem meretur quammaximam Theorema hactenus indemonstratum „ Seriei $1 + \frac{\Delta x^1}{1} + \frac{\Delta^2 x^2}{1.2} + \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^4 x^4}{1.2.3.4} +$

$$\frac{\Delta^5 x^5}{1.2.3.4.5} + \frac{\Delta^6 x^6}{1.2.3.4.5.6} + \frac{\Delta^7 x^7}{1.2.3.4.5.6.7} + \frac{\Delta^8 x^8}{1.2.3.4.5.6.7.8} + \frac{\Delta^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{\Delta^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c$$

terminos ita universim esse compositos, ut, si convergens fiat Δ , tota quoque Series ipsa convergens necessario efficiatur, quomodo-

docunque augeatur magnitudo Indicis x „ Quilibet etenim terminus $\frac{\Delta^m x^m}{1.2.3.....m}$

ad proxime sequentem $\frac{\Delta^{m+1} x^{m+1}}{1.2.3.....m.m+1}$ se habet uti $1 : \frac{\Delta x}{m+1}$; quae Ra-

tio, quum Δx singulis in casibus definitum obtineat valorem, et ex adverso m in Serie Exponentiali absque limite, ac infinities crescere possit, tandem evadet minoris inaequalitatis, et idcirco post quosdam terminos divergentes, si adfuerint, ceteri omnes inseparabili necessitate convergent. En alterum prae communi formula Neutoniana commodum, quamplurimi faciendum in re pene omni algebraica, toties perpensae Progressionis Exponentialis. En etiam alterum Serierum vere convergentium exemplum, quae, tametsi divergentibus incipiant quandoque terminis, postea convergunt sine limite in infinitum, illarum ad instar, de quibus a Potentiatione, praecipue imperfecta, Binomii erutis Dalembertius differuit in trigesimoquarto Opusculorum quinti Voluminis Schediasmate (*).

57. Functio variabilis $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c: = \Delta$, seu y gignit, quemadmodum superius vidimus, Aequationem $\frac{(1+z)^x}{(1)^x} = 1 + \frac{y^1 x^1}{1} + \frac{y^2 x^2}{1.2} + \frac{y^3 x^3}{1.2.3} + \frac{y^4 x^4}{1.2.3.4} + \frac{y^5 x^5}{1.2.3.4.5} + \frac{y^6 x^6}{1.2.3.4.5.6} + \frac{y^7 x^7}{1.2.3.4.5.6.7} + \frac{y^8 x^8}{1.2.3.4.5.6.7.8} + \frac{y^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{y^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c:$, videlicet, si Index x fiat aequalis Unitati, $1+z = 1 + \frac{y^1}{1} + \frac{y^2}{1.2} + \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} + \frac{y^5}{1.2.3.4.5} + \frac{y^6}{1.2.3.4.5.6} + \frac{y^7}{1.2.3.4.5.6.7} + \frac{y^8}{1.2.3.4.5.6.7.8} + \frac{y^9}{1.2.3.4.5.6.7.8.9} + \frac{y^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c:$. Itaque sine subsidio usque nunc impetrato Coefficientium Indeterminatorum in Seriebus supposititiis, et absque Analyfi Aequationum per Series Infinitas, quas methodos vocant Geometrae Inversam Rationem, vel Regressum Serierum, mira facilitate nanciscimur Theorema per elegans „ a Serie numero terminorum infinita $y = \frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c:$ alteram sponte ori ri conversam $z = \frac{y^1}{1} + \frac{y^2}{1.2} + \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} + \frac{y^5}{1.2.3.4.5} + \frac{y^6}{1.2.3.4.5.6} + \frac{y^7}{1.2.3.4.5.6.7} + \frac{y^8}{1.2.3.4.5.6.7.8} + \frac{y^9}{1.2.3.4.5.6.7.8.9} + \frac{y^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c:$

(*) Consulat Lector inter alia sexcenta ipsius Serierum symptomatis specimina illud, quod in numero 54. etiam adtuli iuxta effatum in numero 35.

$$\begin{aligned}
& \frac{1}{1.2.3.4.5.6} + \frac{1}{1.2.3.4.5.6.7} + \frac{1}{1.2.3.4.5.6.7} + \frac{1}{1.2.3.4.5.6.7.8} + \frac{1}{1.2.3.4.5.6.7.8} \\
& \frac{1}{1.2.3.4.5.6.7.8.9} + \frac{1}{1.2.3.4.5.6.7.8.9} + \frac{1}{1.2.3.4.5.6.7.8.9.10} + \frac{1}{1.2.3.4.5.6.7.8.9.10} \\
& \&c: = \frac{1}{1.2} + \frac{1}{1.3} + \frac{1}{1.2.4} + \frac{1}{1.2.3.5} + \frac{1}{1.2.3.4.6} + \frac{1}{1.2.3.4.5.7} \\
& \quad \quad \quad \frac{1}{1.2.4.4.} \\
& \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} + \frac{1}{1.2.3.4.5.6} + \frac{1}{1.2.3.4.5.6.7} + \\
& \quad \quad \quad \frac{1}{1.2} + \frac{1}{1.3} + \frac{1}{1.2.4} + \frac{1}{1.2.3.5} + \frac{1}{1.2.3.4.6} +
\end{aligned}$$

in infinitum videlicet progressu supposito, obtinebimus $\frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3}$

$$\begin{aligned}
& + \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} + \frac{1}{1.2.3.4.5.6} + \frac{1}{1.2.3.4.5.6.7} + \frac{1}{1.2.3.4.5.6.7.8} + \\
& \frac{1}{1.2.3.4.5.6.7.8.9} + \&c: = \frac{1}{1.2} + \frac{4}{1.2.3} + \frac{9}{1.2.3.4} + \frac{16}{1.2.3.4.5} + \frac{25}{1.2.3.4.5.6} \\
& + \frac{36}{1.2.3.4.5.6.7} + \frac{49}{1.2.3.4.5.6.7.8} + \frac{64}{1.2.3.4.5.6.7.8.9} + \frac{81}{1.2.3.4.5.6.7.8.9.10} +
\end{aligned}$$

&c.; veluti adamussim se habet Series Bernoulliana superius exposita.

59. Radix Realis, quam circa versamur, et cuius mentio infra saepius occurret, ab Analystis iam computata ope Locorum Decimalium 1.71828182845904523536028 &c.; quemadmodum longius in Prolegomenis adnotavimus, socias habere posset innumerabiles alias Reales, seu Imaginarias Radices, quae analyticam Aequationem ipsam Infiniti Ordinis $1 -$

$$\frac{z}{1} + \frac{z^2}{2} - \frac{z^3}{3} + \frac{z^4}{4} - \frac{z^5}{5} + \frac{z^6}{6} - \frac{z^7}{7} + \frac{z^8}{8} - \frac{z^9}{9} + \frac{z^{10}}{10} - \&c: = 0 \text{ pari of-}$$

ficio functae resolverent. Universas huiusce Aequationis Radices Numero possibiliter infinitas, praeter Realem iam cognitam, firmum ratumque sit, si quae adsint, facile detegi Aequationis alterius praesidio Infiniti pariter Gradus

$$1 + z = 1 + \frac{\Delta^1}{1} + \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} + \frac{\Delta^4}{1.2.3.4} + \frac{\Delta^5}{1.2.3.4.5} + \frac{\Delta^6}{1.2.3.4.5.6} \\
+ \frac{\Delta^7}{1.2.}$$

$\frac{1}{1.2.4.4.5.6.6} + \frac{1}{1.2.3.4.5.6.7.9} + \frac{1}{1.2.3.4.5.6.7.8.10} + \&c.$ Idcirco, dum invicem addantur hae duae Series numero terminorum Infinitae, atque ordine subsequenti dispositae, quarum posterior vice $\frac{1}{1}$ subrogatur,

$$\frac{1}{1.2.3.4.5.6.7.8} + \frac{1}{1.2.3.4.5.6.7.8.9} + \frac{1}{1.2.3.4.5.6.7.8.9.10} + \&c.$$

$$\frac{1}{1.2.3.4.5.7} + \frac{1}{1.2.4.4.5.6.6} + \frac{1}{1.2.3.4.5.6.7.9} + \&c.,$$

in

$$+ \frac{\Delta^7}{1.2.3.4.5.6.7} + \frac{\Delta^8}{1.2.3.4.5.6.7.8} + \frac{\Delta^9}{1.2.3.4.5.6.7.8.9} + \frac{\Delta^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.,$$

$$\text{aut } z = \frac{\Delta^1}{1} + \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} + \frac{\Delta^4}{1.2.3.4} + \frac{\Delta^5}{1.2.3.4.5} +$$

$\&c.$, ponendo in Radicum generali expressione, post eiusdem resolutionem completam, valorem Magnitudinis, vel variabilis Functionis Δ

Unitati aequalem. Profecto Infinitinomialium $1 + \frac{\Delta^1}{1} + \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} +$

$$+ \frac{\Delta^4}{1.2.3.4} + \frac{\Delta^5}{1.2.3.4.5} + \frac{\Delta^6}{1.2.3.4.5.6} + \frac{\Delta^7}{1.2.3.4.5.6.7} + \frac{\Delta^8}{1.2.3.4.5.6.7.8} +$$

$$+ \frac{\Delta^9}{1.2.3.4.5.6.7.8.9} + \frac{\Delta^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.,$$

quum Regula Huddeniana duce gaudeat Radicibus omnibus inter se aequalibus, ac sit divisibile per $1 + \frac{\Delta}{\infty}$,

explicabitur simplicius Formulae ope $\left(1 + \frac{\Delta}{\infty}\right)^\infty$; quod ipsum constabit, si

meminerimus datum Infinitinomialium annihilari dummodo pro Δ Infinitum ne-

gativum substituatur (*). Erit itaque $1 + z = \left(1 + \frac{\Delta}{\infty}\right)^\infty (1)^\infty$, ni-

$$\text{mirum, } (1 + z)^{\frac{1}{\infty}} (1)^{\frac{1}{\infty}} = 1 + \frac{\Delta}{\infty}, \text{ vel } \infty \left(1^{\frac{1}{\infty}} (1 + z)^{\frac{1}{\infty}} - 1\right) = \Delta,$$

(*) Consulat Lector Numerum praecedentem 38.

Δ , five evolvendo Potestatem Binomii, $(1)^{\frac{1}{\infty}} \times \left(\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c: \right) = \Delta$. Sicuti ergo $\sqrt[1]{1}$ saltem unus valor est Unitas perspiciemus a secunda Aequatione $1 \left(\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c: \right) = \Delta$, uti dictum parcius in numero 57., directum ad primam transitum fieri; ideoque, quum ambae Aequationes penitus identicae sint, patebit luculentissime Radices universas postremae Aequationis easdem esse cum iis Aequationis conversae $1 + z = \left(1 + \frac{\Delta}{\infty} \right)^{\infty}$. Huic Aequationi determinatae primi Gradus, si

Functio Δ praecognita unicum habeat valorem, seu non multiformem admittat, unica Radix respondet $\left(1 + \frac{\Delta}{\infty} \right)^{\infty} = 1$, videlicet, posito $\Delta = 1$,

$$\begin{aligned} & \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} + \frac{1}{1.2.3.4.5.6} + \frac{1}{1.2.3.4.5.6.7} \\ & + \frac{1}{1.2.3.4.5.6.7.8} + \frac{1}{1.2.3.4.5.6.7.8.9} + \frac{1}{1.2.3.4.5.6.7.8.9.10} + \&c:; \text{ atque uni-} \\ & \text{versaliter, si } \Delta = m, \text{ unica Radix emerget } \frac{m^1}{1} + \frac{m^2}{1.2} + \frac{m^3}{1.2.3} + \frac{m^4}{1.2.3.4} \\ & + \frac{m^5}{1.2.3.4.5} + \frac{m^6}{1.2.3.4.5.6} + \frac{m^7}{1.2.3.4.5.6.7} + \frac{m^8}{1.2.3.4.5.6.7.8} + \frac{m^9}{1.2.3.4.5.6.7.8.9} \\ & + \frac{m^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c:. \end{aligned}$$

60. Nec mirum videatur Aequationi, utut numero terminorum, et altiori Potestatis gradu Infinitae, $m - \frac{z^1}{1} + \frac{z^2}{2} - \frac{z^3}{3} + \frac{z^4}{4} - \frac{z^5}{5} + \frac{z^6}{6} - \frac{z^7}{7} + \frac{z^8}{8} - \frac{z^9}{9} + \frac{z^{10}}{10} - \&c: = 0$ unicam tantum respondere Radicem, quum innumera Infinitinomia, et praecipue Series omnes Infinitae, in quas Polynomia evolvuntur, eadem finiti ac determinati Radicum numeri exempla praebeant. Ita Aequatio $m + z^1 - z^2 + z^3 - z^4 + z^5 - z^6 + z^7 - z^8 + z^9 - z^{10} + \&c: = 0$ punctualiter unica gaudet Radice

Radice $z = \frac{-m}{m+1}$, quae, dum hypothesis fiat $m = -1$, evadit $+\infty$;

idque congruit simplicioribus Analyseos regulis. Etenim $-1 + z^1 - z^2 +$

$z^3 - z^4 + z^5 - z^6 + z^7 - z^8 + z^9 - z^{10} + \&c. = \frac{-1}{1+z}$; atque $\frac{-1}{1+z}$

$= 0$, quod idem est ac $0 = \frac{1}{1+z}$, dat $z = \infty$. Radix autem singularis

$z = \infty$ reddit $\pm 1 \mp \infty^1 \pm \infty^2 \mp \infty^3 \pm \infty^4 \mp \infty^5 \pm \infty^6 \mp \infty^7 \pm \infty^8 \mp \infty^9$

$\pm \infty^{10} \&c.$ symbolum Nullitatis, non vere Nihilum, nisi addito nusquam

evanescente Residuo, ex eo quod Series in immensum divergat. Ecce igitur

finitorum Calculi subsidio non modo universalissime detecta Exponentialium

omnium magnitudinum evolutio, verum etiam coronidis instar partim dedu-

ctae, ac in Capite III. penitus intelligendae Formulae, quas omnium prior

Halleius proposuit, atque Eulerus locupletavit in Capite VII. *Introductio-*

nis in Analysin Infinitorum, universae iuxta Auctoris methodum rei Expo-

ponentiali, et Logometricae inservientes: nam quae ibidem scribuntur $(1 \pm L)^m$

$= 1 \pm q$, $1 \pm L = (1 \pm q)^{\frac{1}{m}}$, $(1 + kw)^i = 1 + x$, et $i\omega =$

$\frac{i}{k} (1 + x)^{\frac{1}{i}} - \frac{i}{k}$, hic designantur $(1 \pm \frac{\Delta}{\infty})^\infty = 1 \pm z$, necnon

$\pm \Delta = \infty \left(\frac{1}{1 \pm z} - 1 \right)$, atque adiuncta notione Logarithmici Modu-

li, de quo actum est in Capite subsequente, $(1 \pm \frac{\text{Log. } z}{M. \infty})^\infty = 1 + z$, et

$\pm \frac{\text{Log. } z}{M} = \infty \left(\frac{1}{1 \pm z} - 1 \right)$, vel $\pm \text{Log. } z = \infty \cdot M \left(\frac{1}{1 \pm z} - 1 \right)$.

Qui veritatum Algebraicarum reiteratam comprobationem in deliciis ha-

bent experiantur oportet nitidissimam substitutionem Radicis oecumeni-

cae inventae $\frac{\Delta^1}{1} + \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} + \frac{\Delta^4}{1.2.3.4} + \frac{\Delta^5}{1.2.3.4.5} + \&c.$ in

Aequatione $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \&c. = \Delta^1$, videbuntque iden-

ticam oriri reapse Aequalitatem

$$\begin{aligned}
& \frac{\Delta^1}{1} + \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} + \frac{\Delta^4}{1.2.3.4} + \frac{\Delta^5}{1.2.3.4.5} + \&c; = \Delta, \\
& - \frac{\Delta^2}{1.2} - \frac{\Delta^3}{1.2} - \frac{\Delta^4}{1.2.4} - \frac{\Delta^5}{1.3.4} \\
& + \frac{\Delta^3}{1.3} - \frac{\Delta^4}{1.2.3} - \frac{\Delta^5}{1.2.3.4} \\
& + \frac{\Delta^4}{1.2} + \frac{\Delta^5}{1.4} \\
& - \frac{\Delta^4}{1.4} - \frac{\Delta^5}{1.2.3} \\
& - \frac{\Delta^5}{1.2} \\
& + \frac{\Delta^5}{1.5}
\end{aligned}$$

eodemque successu confimilis, praeter primum, terminorum omnium homologorum elisio in infinitum pateret.

61. Ad pleniorē argumenti huiuscemodi intelligentiam proderit quoque novas aggredi resolutiones Aequationum magis compositarum $(1+z)^x$

$$\begin{aligned}
& = (1)^x \times \left(1 + \frac{\Delta^1 x^1}{1} + \frac{\Delta^2 x^2}{1.2} + \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^4 x^4}{1.2.3.4} + \frac{\Delta^5 x^5}{1.2.3.4.5} + \right. \\
& \frac{\Delta^6 x^6}{1.2.3.4.5.6} + \frac{\Delta^7 x^7}{1.2.3.4.5.6.7} + \frac{\Delta^8 x^8}{1.2.3.4.5.6.7.8} + \frac{\Delta^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{\Delta^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \\
& \left. + \&c; \right), \text{ quas eo laetius examini subiicio propterea quod iisdem superius}
\end{aligned}$$

memoratis rationibus reducantur ad $\frac{(1+z)^x}{(1)^x} = \left(1 + \frac{\Delta x}{\infty} \right)^\infty$. Quando $\Delta = 1$ iam observavimus fore $1+z = 2.71828182845904523536028 \&c;$, quo Numero posito C, erit $\frac{C^x}{1^x} = \left(1 + \frac{x}{\infty} \right)^\infty$; namque independenter etiam ab

omnibus praedemonstratis $\left(\frac{C}{1} \right)^x$ evanescere nequit, nisi fiat $x = -\infty$,

cuius tantummodo hypothesis fructus est $\frac{1}{C^\infty} = 0$: qua de re Radices omnes

aequales, ac numero carentes Aequationis $1 + \frac{x^1}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} +$

$$\frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^8}{1.2.3.4.5.6.7.8} +$$

$$\frac{x^9}{1.2.}$$

$\frac{x^2}{1.2.3.4.5.6.7.8.9} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: = 0$ erunt comprehensae in tam sim-

pliciori Aequalitate $x + \infty = 0$, vel $1 + \frac{x}{\infty} = 0$. Imo universalius, dum

$\Delta = m$, ac Binomium $1 + z$ aequale indeterminato pariter Numero N , emer-

get $\frac{N^x}{1^x} = 1 + \frac{m^1 x^1}{1} + \frac{m^2 x^2}{1.2} + \frac{m^3 x^3}{1.2.3} + \frac{m^4 x^4}{1.2.3.4} + \frac{m^5 x^5}{1.2.3.4.5} +$

$\frac{m^6 x^6}{1.2.3.4.5.6} + \frac{m^7 x^7}{1.2.3.4.5.6.7} + \frac{m^8 x^8}{1.2.3.4.5.6.7.8} + \frac{m^9 x^9}{1.2.3.4.5.6.7.8.9} +$

$\frac{m^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: = \left(1 + \frac{mx}{\infty}\right)^\infty$, cuius Formulae innumerae Radices in-

veniuntur praesidio Aequationis $1 + \frac{mx}{\infty} = 0$ tribuentis $x = -\frac{\infty}{m}$; atque hic

loci observandum a diverso valore supposito $\tau\delta$ Δ innumerasque variasque oriri N Bases Magnitudinum Exponentialium, quas inter eximia simplicitate supereminet

C. Optata igitur resolutio Aequationis Exponentialis $\frac{N^x}{1^x} = n$ pendebit ab al-

terius resolutione $(1)^x \times \left(\left(1 + \frac{mx}{\infty}\right)^\infty - n\right) = 0$, et casu $N = C$,

necnon $C^x = e$ ab ea simpliciore $(1)^x \times \left(1 + \frac{x}{\infty}^\infty - e\right) = 0$, de quibus

in Capite V. ac VII. locupletissima prostackit Analysis. Interim monemus $\tau\delta$

$(1)^x$ ex hypothesi facta $\frac{N^x}{1^x} = n$ nunc meram indicare Unitatem, atque

Radicum quaesitarum Aequationis $1 + \frac{x}{\infty}^\infty - e = 0$, necnon alterius

$1 + \frac{mx}{\infty}^\infty - n = 0$ unam esse in casu priori $\frac{(e-1)^1}{1} - \frac{(e-1)^2}{2} +$

$\frac{(e-1)^3}{3} - \frac{(e-1)^4}{4} + \frac{(e-1)^5}{5} - \frac{(e-1)^6}{6} + \frac{(e-1)^7}{7} - \frac{(e-1)^8}{8} +$

$\frac{(e-1)^9}{9} - \frac{(e-1)^{10}}{10} + \&c:$, atque in posteriori $\frac{1}{m} \left(\frac{n-1^1}{1} - \frac{n-1^2}{2} +$

$\frac{n-1^3}{3} - \frac{n-1^4}{4} + \frac{n-1^5}{5} - \frac{n-1^6}{6} + \frac{n-1^7}{7} - \frac{n-1^8}{8} + \frac{n-1^9}{9} -$

$\frac{n-1^{10}}{10} + \&c: \right)$. Quod facile ostendetur, dummodo memoria repetamus Ae-

quationem $\left(1 + \frac{\Delta}{\infty}\right)^\infty - (1 + z) = 0$ iuxta numeri 59. theoricen prae-

bere $\Delta = \frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10}$
 $+ \&c;$; ideoque homologas magnitudines comparando, substituendoque pri-
 mum x , ac secundo mx pro Δ , et vice $z + 1$ aequalem e , vel n , nimirum
 pro z aequipollentem $e - 1$, seu $n - 1$, praefati prodibunt valores inco-
 gnitae x , scilicet Aequationis geminae Radix Realis elicietur.

62. Elegantissimam hanc Serierum recensitarum cognationem, five po-
 tius mutuam regenerationem claudam insigni, atque ob ampliorem in sub-
 sequentibus usum animo intimius comprehendenda affectione singularis cuius-
 dam Magnitudinis Exponentialis. Usque a theorices, in qua adhuc su-
 mus, initio, et praecipue in numero 47., reperimus universaliter esse

$$1 + z = 1 + \frac{\Delta^1}{1} + \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} + \frac{\Delta^4}{1.2.3.4} + \frac{\Delta^5}{1.2.3.4.5} +$$

$$\frac{\Delta^6}{1.2.3.4.5.6} + \frac{\Delta^7}{1.2.3.4.5.6.7} + \frac{\Delta^8}{1.2.3.4.5.6.7.8} + \frac{\Delta^9}{1.2.3.4.5.6.7.8.9} +$$

$$\frac{\Delta^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.; \text{ ubi } \Delta^1 = \frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} -$$

$$\frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c.; \text{ nimirum rite perpenfa, uti}$$

factum fuit in num°. 47., Binomii Neutroniani Exponentialiter expressi For-
 mula ipsam Aequalitatem in Aequationem identicam verti $1 + z = 1 + z$;
 quod idem significat ac Series convergens numero terminorum Infinita

$$\frac{\Delta^1}{1} + \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} + \frac{\Delta^4}{1.2.3.4} + \frac{\Delta^5}{1.2.3.4.5} + \frac{\Delta^6}{1.2.3.4.5.6} + \frac{\Delta^7}{1.2.3.4.5.6.7}$$

$$+ \frac{\Delta^8}{1.2.3.4.5.6.7.8} + \frac{\Delta^9}{1.2.3.4.5.6.7.8.9} + \frac{\Delta^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c. = z, \text{ posi-}$$

ta Δ tali z Functione, ut congruat alteri Seriei numero terminorum Infini-

$$\text{tae } \frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10}$$

$+ \&c.$; propterea quod terminis omnibus se mutuo elidentibus unicus denique

$$\text{maneat } \frac{z^1}{1}. \text{ Quum igitur in num°. 61. viderimus esse } C^x = 1 + \frac{x^1}{1} + \frac{x^2}{1.2} +$$

$$\frac{x^3}{1.2.3}$$

$$\frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^7}{1.2.3.4.5.6.7} +$$

$$\frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^9}{1.2.3.4.5.6.7.8.9} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c., \text{ quia } C \text{ de-}$$

rivatur ab hypothefi $\Delta = 1$, nanciscemur exinde, dum vice x subrogemus

$$\frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \frac{x^9}{9} - \frac{x^{10}}{10} + \&c.$$

$$\frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} +$$

„ Aequationem memorabilem C

$$\frac{x^7}{7} - \frac{x^8}{8} + \frac{x^9}{9} - \frac{x^{10}}{10} + \&c. = 1 + x, \text{ vel potius } C = \frac{x-1^1}{1}$$

$$\frac{x-1^2}{2} + \frac{x-1^3}{3} - \frac{x-1^4}{4} + \frac{x-1^5}{5} - \frac{x-1^6}{6} + \frac{x-1^7}{7} - \frac{x-1^8}{8}$$

$$+ \frac{x-1^9}{9} - \frac{x-1^{10}}{10} + \&c.$$

$= x$, scilicet Potestas Infinitinomialis con-

scriptum pro Indice habens ita contrahetur, ut aequivaleat tantummodo primo eiusdem Indicis termino cum aucta Unitate. Sin loco $1+z$ quamlibet Potestatem $(1+z)^m$, aut generalissime $(1+z)^x$ feligere libeat, idem emergeret Theorema, sed maiori universalitate donatum, nempe deducere-

$$\text{tur } C \left(\frac{x-1^1}{1} - \frac{x-1^2}{2} + \frac{x-1^3}{3} - \frac{x-1^4}{4} + \frac{x-1^5}{5} - \frac{x-1^6}{6} + \right.$$

$$\left. \frac{x-1^7}{7} - \frac{x-1^8}{8} + \frac{x-1^9}{9} - \frac{x-1^{10}}{10} + \&c. \right) = x^y. \text{ Theorema ergo no-}$$

vum enascitur, cuius ope omnis Quantitas Exponentialis speciei x^y immutari facile possit in Exponentialem simpliciore speciei C^v , dummodo C semper habeat constantem praedefinitum valorem, ac v sit productum variabilis Indicis y per Functionem huiusmodi alterius variabilis x , qualis erat Δ respectu habito ad $1+z$. Notatu insuper dignum occurrit, quod in quaque variabilis z , vel variabilium z, x, y &c: Functione Φ expressa per hanc unicum symbolam, ac, si placeat, in unaquavis constante Magnitudi-

ne Φ semper contingat esse Potestatem Infinitinomii $\left(1 + \frac{\Phi^1}{1} + \frac{\Phi^2}{1.2} + \frac{\Phi^3}{1.2.3} \right.$

$$\left. + \frac{\Phi^4}{1.2.3.4} + \frac{\Phi^5}{1.2.3.4.5} + \frac{\Phi^6}{1.2.3.4.5.6} + \frac{\Phi^7}{1.2.3.4.5.6.7} + \frac{\Phi^8}{1.2.3.4.5.6.7.8} + \right.$$

$$\left. \frac{\Phi^9}{1.2.3.4.5.6.7.8.9} + \frac{\Phi^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c. \right)^m = 1 + \frac{m\Phi^1}{1} + \frac{m^2\Phi^2}{1.2} +$$

$$\frac{m^3\Phi^3}{1.2.3} + \frac{m^4\Phi^4}{1.2.3.4} + \frac{m^5\Phi^5}{1.2.3.4.5} + \frac{m^6\Phi^6}{1.2.3.4.5.6} + \frac{m^7\Phi^7}{1.2.3.4.5.6.7} + \frac{m^8\Phi^8}{1.2.3.4.5.6.7.8}$$

$$+ \frac{m^9\Phi^9}{1.2.3.4.5.6.7.8.9} + \frac{m^{10}\Phi^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.; \text{ et universalius } \left(1 + \frac{\Phi^1}{1} + \right.$$

$$\left. \frac{\Phi^2}{1.2} + \frac{\Phi^3}{1.2.3} + \frac{\Phi^4}{1.2.3.4} + \frac{\Phi^5}{1.2.3.4.5} + \frac{\Phi^6}{1.2.3.4.5.6} + \frac{\Phi^7}{1.2.3.4.5.6.7} + \right.$$

$$\left. \frac{\Phi^8}{1.2.3.4.5.6.7.8} + \frac{\Phi^9}{1.2.3.4.5.6.7.8.9} + \frac{\Phi^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c. \right)^x = 1 +$$

$$\frac{x\Phi^1}{1} + \frac{x^2\Phi^2}{1.2} + \frac{x^3\Phi^3}{1.2.3} + \frac{x^4\Phi^4}{1.2.3.4} + \frac{x^5\Phi^5}{1.2.3.4.5} + \frac{x^6\Phi^6}{1.2.3.4.5.6} +$$

$$\frac{x^7\Phi^7}{1.2.3.4.5.6.7} + \frac{x^8\Phi^8}{1.2.3.4.5.6.7.8} + \frac{x^9\Phi^9}{1.2.3.4.5.6.7.8.9} + \frac{x^{10}\Phi^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.: \text{ Est}$$

$$\text{etenim } \left(1 + \frac{\Phi^1}{1} + \frac{\Phi^2}{1.2} + \frac{\Phi^3}{1.2.3} + \frac{\Phi^4}{1.2.3.4} + \frac{\Phi^5}{1.2.3.4.5} + \frac{\Phi^6}{1.2.3.4.5.6} \right.$$

$$\left. + \frac{\Phi^7}{1.2.3.4.5.6.7} + \frac{\Phi^8}{1.2.3.4.5.6.7.8} + \frac{\Phi^9}{1.2.3.4.5.6.7.8.9} + \frac{\Phi^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c. \right)^x$$

$$= (C^\Phi)^x = C^{x\Phi} = 1 + \frac{x\Phi^1}{1} + \frac{x^2\Phi^2}{1.2} + \frac{x^3\Phi^3}{1.2.3} + \frac{x^4\Phi^4}{1.2.3.4} +$$

$$\frac{x^5\Phi^5}{1.2.3.4.5} + \frac{x^6\Phi^6}{1.2.3.4.5.6} + \frac{x^7\Phi^7}{1.2.3.4.5.6.7} + \frac{x^8\Phi^8}{1.2.3.4.5.6.7.8} + \frac{x^9\Phi^9}{1.2.3.4.5.6.7.8.9} +$$

$$\frac{x^{10}\Phi^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.: \text{ Idipsum praeterea comprobatur ab Aequalitate su-}$$

$$\text{perius cognita } 1 + \frac{\Phi^1}{1} + \frac{\Phi^2}{1.2} + \frac{\Phi^3}{1.2.3} + \frac{\Phi^4}{1.2.3.4} + \frac{\Phi^5}{1.2.3.4.5} +$$

$$\frac{\Phi^6}{1.2.3.}$$

$$\frac{\Phi^6}{1.2.3.4.5.6} + \&c: = \left(1 + \frac{\Phi}{\infty}\right)^{\infty}, \text{ ex qua statim fluit } \left(1 + \frac{\Phi^1}{1} + \frac{\Phi^2}{1.2} + \frac{\Phi^3}{1.2.3} + \frac{\Phi^4}{1.2.3.4} + \frac{\Phi^5}{1.2.3.4.5} + \frac{\Phi^6}{1.2.3.4.5.6} + \&c:\right)^x =$$

$$\left(\frac{\Phi^2}{1.2} + \frac{\Phi^3}{1.2.3} + \frac{\Phi^4}{1.2.3.4} + \frac{\Phi^5}{1.2.3.4.5} + \frac{\Phi^6}{1.2.3.4.5.6} + \&c:\right)^x =$$

$$\left(1 + \frac{\Phi}{\infty}\right)^x = \left(1 + \frac{\Phi}{\infty}\right)^{x\infty} = \left(1 + \frac{x\Phi}{\infty}\right)^{\infty}, \text{ quemadmodum luculen-}$$

$$\text{ter actualis evolutio confirmat, ideoque } = 1 + \frac{x\Phi^1}{1} + \frac{x^2\Phi^2}{1.2} + \frac{x^3\Phi^3}{1.2.3} +$$

$$\frac{x^4\Phi^4}{1.2.3.4} + \frac{x^5\Phi^5}{1.2.3.4.5} + \frac{x^6\Phi^6}{1.2.3.4.5.6} + \&c: \text{ sine ullo Limite.}$$

63. Vix necessarium opinor ut moneam Formulam Exponentialem usque nunc pertractatam, in quam nempe evoluitur Potestas Binomii universalissima $(1+z)^x$, vel y^x , seu $(1+(y-1))^x$, omnes possibilium Exponentialium ac Potestatum casus complecti, inter quos simplicissimus ille est, cuius tantum plerique Geometrarum sermonem facere solent, a^x , sive $(1+(a-1))^x$, quum prima Formula in posteriorem abeat, ceteris omnibus iisdem manentibus, quando τz y, aut z variabilis vice constans a, vel $a-1$ substituatur. Ne toties, ac perperam idem monitum renovemus silentio in posterum praeteribimus incedentes ad Exponentialium Theoriae complementum Formulas minus universales, utpote coronidis instar a generalioribus sponte effluentes. Calculi quoque inutilem pompam censeo vitandam in peculiariter evolvendis, densa exemplorum far-

ragine quasi ruente, Magnitudinibus Exponentialibus z^x , z^y &c: in infi-

nitum, quarum species forent simpliciores a^x , z^b , c^x , z^a , z^x &c:;

sunt etenim universae istae Formulae iisdem subiectae regulis, quibus Exponentialia dudum evoluta. Expressio quippe z^{xy} dupliciter Exponentialis ea-

dem erit cum altera $(1+(z-1))^y$, vel $(1+(z-1))^{(1+x-1)^y}$,

sive ex antecedentibus, ut per se liquet, aequabit $(1+(z-1))^{1+\frac{\Delta^1 y^1}{1}}$

+

$$+ \frac{\Delta^2 y^2}{1.2} + \frac{\Delta^3 y^3}{1.2.3} + \frac{\Delta^4 y^4}{1.2.3.4} + \frac{\Delta^5 y^5}{1.2.3.4.5} + \&c:$$

, nimirum verte-

tur in Formulam simpliciter Exponentialem Indice Infinitinomio praedi-

tam, ubi Δ solitae Functionis τ_x x Infinitinomiae $\frac{x-1}{1} - \frac{x-1^2}{2} + \frac{x-1^3}{3}$ $-\frac{x-1^4}{4} + \frac{x-1^5}{5} - \frac{x-1^6}{6} + \&c$: Typum significat. Dum igitur In-dici Infinitinomiali maioris compendii causa subrogetur Ψ , Functio scilicetiampridem cognita variabilium x, y , obtinebimus $z^x = 1 + \frac{E^1 \Psi^1}{1} +$ $\frac{E^2 \Psi^2}{1.2} + \frac{E^3 \Psi^3}{1.2.3} + \frac{E^4 \Psi^4}{1.2.3.4} + \frac{E^5 \Psi^5}{1.2.3.4.5} + \frac{E^6 \Psi^6}{1.2.3.4.5.6} + \&c$; supposita $E =$ $\frac{z-1}{1} - \frac{z-1^2}{2} + \frac{z-1^3}{3} - \frac{z-1^4}{4} + \frac{z-1^5}{5} - \frac{z-1^6}{6} + \&c$: Pa-

ri ratione omnes eliminabuntur expressiones Exponentiales in Formula tripli-

citer Exponentiali $v^{\frac{y}{x}}$. Haec etenim summa facilitate redigitur in

$$(1 + (v - 1)) \left(1 + \frac{E^1 \Psi^1}{1} + \frac{E^2 \Psi^2}{1.2} + \frac{E^3 \Psi^3}{1.2.3} + \frac{E^4 \Psi^4}{1.2.3.4} + \frac{E^5 \Psi^5}{1.2.3.4.5} + \frac{E^6 \Psi^6}{1.2.3.4.5.6} + \&c \right)$$

, videlicet, dum Infinitinomium, quod Indicem expri-

mit, Sigla Λ , et Functio $\frac{v-1}{1} - \frac{v-1^2}{2} + \frac{v-1^3}{3} - \frac{v-1^4}{4} + \frac{v-1^5}{5}$ $-\frac{v-1^6}{6} + \&c$: altera K denotentur, fiet $v^{\frac{y}{x}} = 1 + \frac{K^1 \Lambda^1}{1} + \frac{K^2 \Lambda^2}{1.2}$ $+ \frac{K^3 \Lambda^3}{1.2.3} + \frac{K^4 \Lambda^4}{1.2.3.4} + \frac{K^5 \Lambda^5}{1.2.3.4.5} + \frac{K^6 \Lambda^6}{1.2.3.4.5.6} + \&c$: ipsamet semper re-deunte forma Infinitinomialis Functionis. Spurias non ultra persequor $z^{\frac{y}{x}}$
Formulas bis Exponentiales; quum maiorem simplicioribus $z^{\frac{y}{x}}$ mendose diffi-

cultatem minentur, verum substantialiter nihil discrepent a simpliciorum

y
z evo-

z^y evolutione, quoniam substituendo x^a pro y resolvuntur in $1 + \frac{\Delta^1 x^a}{1} +$

$$\frac{\Delta^2 x^{2a}}{1.2} + \frac{\Delta^3 x^{3a}}{1.2.3} + \frac{\Delta^4 x^{4a}}{1.2.3.4} + \frac{\Delta^5 x^{5a}}{1.2.3.4.5} + \frac{\Delta^6 x^{6a}}{1.2.3.4.5.6} + \&c.; \text{ ac } \Delta \text{ de}$$

$$\text{more Functionem } \frac{\overline{z-1}^1}{1} - \frac{\overline{z-1}^2}{2} + \frac{\overline{z-1}^3}{3} - \frac{\overline{z-1}^4}{4} + \frac{\overline{z-1}^5}{5} -$$

$$\frac{\overline{z-1}^6}{6} + \&c: \text{ repraesentat. Similia similibus applicentur; et una vice ani-}$$

madvertisse non taedeat prae universis Exponentialibus distinctam sedem me-

rer x^x , x^{x^x} &c.; quae unicum variabilem comprehendunt exhibent Se-

$$\text{ries numero terminorum Infinitas } 1 + \frac{\Delta^1 x^1}{1} + \frac{\Delta^2 x^2}{1.2} + \frac{\Delta^3 x^3}{1.2.3} +$$

$$\frac{\Delta^4 x^4}{1.2.3.4} + \frac{\Delta^5 x^5}{1.2.3.4.5} + \frac{\Delta^6 x^6}{1.2.3.4.5.6} + \&c.; \text{ atque } x$$

$$+ \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^4 x^4}{1.2.3.4} + \frac{\Delta^5 x^5}{1.2.3.4.5} + \frac{\Delta^6 x^6}{1.2.3.4.5.6} + \&c: \text{, five } 1 + \frac{\Delta^1 H^1}{1} +$$

$$\frac{\Delta^2 H^2}{1.2} + \frac{\Delta^3 H^3}{1.2.3} + \frac{\Delta^4 H^4}{1.2.3.4} + \frac{\Delta^5 H^5}{1.2.3.4.5} + \frac{\Delta^6 H^6}{1.2.3.4.5.6} + \&c.; \text{ manente } \Delta$$

$$= \frac{\overline{x-1}^1}{1} - \frac{\overline{x-1}^2}{2} + \frac{\overline{x-1}^3}{3} - \frac{\overline{x-1}^4}{4} + \frac{\overline{x-1}^5}{5} - \frac{\overline{x-1}^6}{6} + \&c.; \text{ et}$$

H repraesentativa admissim Functione Indicis Infinitomii Potestatis x superius descriptae.

64. Quomodocunque prima fronte Exponentialia haecenus considerata $(1+z)^x$, y &c: praeseferant simplices solummodo variables Magnitudines z , x , seu y , v , nihilo tamen minus quilibet numero variables, ac variabilium Functiones subintelligi ibidem posse neminem latet, quod satis erit heic

monuisse. Sunt qui altioris Ordinis Exponentialia z^x facilius evolvere opinentur ea convertentes in Productum ex numero Infinitis Seriebus Limite carentibus, quae originem ducerent a seiunctis Exponentialibus per z^x

$$\text{multiplicatis } z^{\frac{\Delta^1 v^1}{1}}, z^{\frac{\Delta^2 v^2}{1.2}}, z^{\frac{\Delta^3 v^3}{1.2.3}}, z^{\frac{\Delta^4 v^4}{1.2.3.4}}, z^{\frac{\Delta^5 v^5}{1.2.3.4.5}}, z^{\frac{\Delta^6 v^6}{1.2.3.4.5.6}} \&c.; \text{ quasi}$$

haec

haec infinita multiplicatio nec aequipolleret consuetae expressioni dupliciter

$$1 + \frac{\Delta^1 v^1}{1} + \frac{\Delta^2 v^2}{1.2} + \frac{\Delta^3 v^3}{1.2.3} + \frac{\Delta^4 v^4}{1.2.3.4} + \frac{\Delta^5 v^5}{1.2.3.4.5} +$$

Exponentiali z

$$\frac{\Delta^6 v^6}{1.2.3.4.5.6} + \&c:$$

, aut aliquantisper Analysis nova methodo locupletaret. Quinimo tantum abest, ut a praefata Exponentialium Serierum numero Infinitarum infinities repetita multiplicatione vel hilum res Algebraica promoveatur,

$$(1-z)^x = (1)^x \times \left(1 - \frac{z^1 x^1}{1} + \frac{z^2 x^2}{1.2} - \frac{z^3 x^3}{1.2.3} + \frac{z^4 x^4}{1.2.3.4} - \frac{z^5 x^5}{1.2.3.4.5} \right.$$

$-\frac{z^2 x^1}{1.2}$	$+\frac{z^3 x^2}{1.2}$	$-\frac{z^4 x^3}{1.4}$	$+\frac{z^5 x^4}{1.3.4}$	$-\frac{z^6 x^5}{1.2.4.6}$
$-\frac{z^3 x^1}{1.3}$	$+\frac{11z^4 x^2}{1.2.3.4}$	$-\frac{7z^5 x^3}{1.2.3.4}$	$+\frac{17z^6 x^4}{1.2.3.4.6}$	$-\frac{5z^7 x^5}{1.2.3.4.6}$
$-\frac{z^4 x^1}{1.4}$	$+\frac{5z^5 x^2}{1.3.4}$	$-\frac{5z^6 x^3}{1.4.4}$	$+\frac{7z^7 x^4}{1.2.4.6}$	$-\frac{7z^8 x^5}{1.3.6.8}$
$-\frac{z^5 x^1}{1.5}$	$+\frac{137z^6 x^2}{1.3.4.5.6}$	$-\frac{29z^7 x^3}{1.3.5.6}$	$+\frac{967z^8 x^4}{1.2.3.4.5.6.8}$	$-\frac{1069z^9 x^5}{1.2.4.5.6.8.9}$
$-\frac{z^6 x^1}{1.6}$	$+\frac{7z^7 x^2}{1.4.5}$	$-\frac{469z^8 x^3}{1.2.3.5.6.8}$	$+\frac{89z^9 x^4}{1.3.4.5.8}$	$-\frac{19z^{10} x^5}{1.2.4.4.8}$
$-\frac{z^7 x^1}{1.7}$	$+\frac{363z^8 x^2}{1.4.5.7.8}$	$-\frac{29531z^9 x^3}{1.2.3.5.6.7.8.9}$	$+\frac{4523z^{10} x^4}{1.3.4.5.6.7.9}$	$\&c.$
$-\frac{z^8 x^1}{1.8}$	$+\frac{761z^9 x^2}{1.5.7.8.9}$	$-\frac{1303z^{10} x^3}{1.3.4.6.7.8}$	$\&c.$	
$-\frac{z^9 x^1}{1.9}$	$+\frac{7129z^{10} x^2}{1.5.7.8.9.10}$	$\&c.$		
$-\frac{z^{10} x^1}{1.10}$	$\&c.$			
$\&c.$				

cuius differentia a priore in hoc tantum consistit, quod Series numero terminorum Infinitae verticaliter dispositae habeant terminos universos eodem signo, non ut prius alterno notatos, et vicissim aliae Series horizontaliter iacentes gaudeant terminis omnibus alterno, non eodem veluti antea signo

tur, vel maior claritas praesto sit, ut potius de istius methodi inefficacia proferre liceat iudicium, quum haec eo sistat, quo altera methodus longius progredi non dedignatur.

65. Secunda, quam supra narravimus, hypothesis $(1 - z)^x$ post iam explicita nullam afferet difficultatem, at nova adaperiet Serierum Infinitarum argumenta tractanda. Is, qui ad instar numeri 44. evolvere cupiat $(1 - z)^x$, ordine desumpto a Potestatibus non secundi Nominis z , verum Indicis x , obtinebit hanc Aequationem

$$\begin{array}{r}
 + \frac{z^6 x^6}{1.2.3.4.5.6} - \frac{z^7 x^7}{1.2.3.4.5.6.7} + \frac{z^8 x^8}{1.2.3.4.5.6.7.8} - \frac{z^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{z^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c. \\
 + \frac{z^7 x^6}{1.2.4.5.6} - \frac{z^8 x^7}{1.2.3.5.6.8} + \frac{z^9 x^8}{1.2.3.5.6.7.8} - \frac{z^{10} x^9}{1.2.3.4.6.7.8.10} \&c. \\
 + \frac{23z^8 x^6}{1.3.4.5.6.8} - \frac{13z^9 x^7}{1.4.5.6.8.9} + \frac{29z^{10} x^8}{1.4.6.7.8.9.10} \&c. \\
 + \frac{z^9 x^6}{1.2.5.8} - \frac{z^{10} x^7}{1.2.4.6.8} \&c. \\
 + \frac{3013z^{10} x^6}{1.2.4.5.6.8.9.10} \&c.
 \end{array}$$

cuius

gno ditatis. Binomium $1 - z$ exhibere intelligo Numerum positivum, scilicet esse z Numerum fractum, ideoque ad normam numeri 43. Unitate minorem. Discrepantiam autem Signorum innumerabiles Serierum horizontalium, ac verticalium terminos afficientium oriri a negativo tantum Numero

K

— z

— z non est qui non videat, quum haec signorum diversitas Potestates z Indicis solummodo imparis perstringat, non illas Indicis paris; quod ita fieri debere Algebra docet, et geminae Seriei comparatio patefacit. Necessaria signorum oppositio quoad Potestates impari Exponente praeditas absque praesidio actualis evolutionis novi Binomii ipsamet sufficeret huic alteri Formulae Exponentiali inveniendae, quae tunc priori nuper recensitae in consuetarii morem necteretur. Nulla quoque indiget opera demonstratio Theorematis momentosissimi ad novae Formulae Coefficientes numero terminorum Infinitos pertinentis, in eoque positi „ *quod tertii termini Coefficientis dimidium sit Quadrati Magnitudinis Infinitinomialis* —

$$\frac{z^1}{1} - \frac{z^2}{1.2} -$$

$$\frac{z^3}{1.3} - \frac{z^4}{1.4} - \frac{z^5}{1.5} - \frac{z^6}{1.6} - \frac{z^7}{1.7} - \frac{z^8}{1.8} - \frac{z^9}{1.9} - \frac{z^{10}}{1.10} - \&c.; \text{ quar}$$

ti, quinti, sexti, septimi, octavi, noni, decimi &c: terminorum Coefficientes sint praedicti Infinitinomiali ad tertiam, quartam, quintam, sextam, septimam, octavam, nonam &c: Potestatem elevati pars respective sexta, vigesimaquarta, vigesima supra centesimam, vigesima supra septingentesi-

mam, quadragesima supra quintum millenarium, $\frac{ma}{40320}$, $\frac{ma}{362880}$,

$\frac{ma}{3628800}$ „ neque novit Natura Limitem. Et re quidem vera, quum istam proprietatem Coefficientium latissime ostenderimus num°. 50. prima in Serie Exponentiali, quisque per se incomparabili iucunditate experietur, quod signis Potestatum omnium imparium z in adversa immutatis eadem valeant adamussim proportionales, atque praesertim Coefficientium universorum derivatio ab unico Infinitinomio elegantissimo primam τ x Potestatem multiplicante.

66. Dum itaque distinctionis causa ab altero $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} -$

$$\frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c: = \Delta \text{ hoc no-}$$

$$\text{vum Infinitinomialium } \frac{z^1}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} + \frac{z^6}{6} + \frac{z^7}{7} +$$

$$\frac{z^8}{8} + \frac{z^9}{9} + \frac{z^{10}}{10} + \&c: \text{ cyphra } \Gamma \text{ denotetur, exfurret } (1 - z)^x = 1^x$$

(1 —

$$\left(1 - \frac{\Gamma x^1}{1} + \frac{\Gamma^2 x^2}{1.2} - \frac{\Gamma^3 x^3}{1.2.3} + \frac{\Gamma^4 x^4}{1.2.3.4} - \frac{\Gamma^5 x^5}{1.2.3.4.5} + \frac{\Gamma^6 x^6}{1.2.3.4.5.6} - \frac{\Gamma^7 x^7}{1.2.3.4.5.6.7} + \frac{\Gamma^8 x^8}{1.2.3.4.5.6.7.8} - \frac{\Gamma^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{\Gamma^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c: \right),$$

signis nimirum in Serie ita alternatim dispositis, ut terminis numero paribus, seu Potestatibus Infinitinomii Γ imparem Exponentem habentibus negativum, ac numero imparibus terminis, vel e contra potestatibus eiusdem Γ Indice pari praeditis positivum adscribi debeat. Inde enascitur usitata compendii forma

$$\frac{1 - z^x}{1^x} = \left(1 - \frac{\Gamma x}{\infty} \right)^{\infty}, \text{ ac propterea } 1 - z = 1 - \frac{\Gamma}{\infty},$$

$$\text{et denique } z = 1 - \left(1 - \frac{\Gamma}{\infty} \right)^{\infty}, \text{ necnon } \infty \left(1 - \frac{1}{\infty} (1 - z)^{\frac{1}{\infty}} \right)$$

$= \Gamma$, quae singulae Aequationes praesidio consuetae Binomii Formulae Newtonianae perquam facillime confirmantur, inferiusque maxima universalitate resolventur.

67. Nec immorari in Infinitinomii $\frac{z^1}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} + \frac{z^6}{6} + \frac{z^7}{7} + \frac{z^8}{8} + \frac{z^9}{9} + \frac{z^{10}}{10} + \&c:$ convergentia paranda operae pretium esset, quia hypothesis ipsa, a qua sumus digressi, z Unitate minoris, reddit Seriem istam sponte veluti sua convergentem, et eo magis, quo $1 - z$ minus differat ab Unitate, sive z minorem habeat valorem. Porro in hac Geometrico-Harmonica Progressione Numeratores, ac Denominatores eodem semper affecti signo, primi infinities decrescentes, alii sine limite crescentes, in convergentiam terminorum adstruendam cumulatim concurrunt. Convergentiae huiusmodi Limitem statuit hypothesis ultima $z = 1$ tum, quum Functio $\frac{z^1}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} + \frac{z^6}{6} + \frac{z^7}{7} + \frac{z^8}{8} + \frac{z^9}{9} + \frac{z^{10}}{10} + \&c:$ efficitur $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c:$, scilicet iam familiare Infinitum-Harmonicum. Sed inu-

tilem navasse laborem non censerem eum, qui ratiocinando iuxta numerum praecedentem 57. directo tramite intulerit, posita universaliter Serie

$$\frac{z^1}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} + \frac{z^6}{6} + \frac{z^7}{7} + \frac{z^8}{8} + \frac{z^9}{9} +$$

$\frac{z^{10}}{10} + \&c: = y$, alteram huius inversam Aequationem a generalissima Exponentiali

Serie numeri 65. prognatam $1 - z = 1 - \frac{y^1}{1} + \frac{y^2}{1.2} - \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4}$

$$- \frac{y^5}{1.2.3.4.5} + \frac{y^6}{1.2.3.4.5.6} - \frac{y^7}{1.2.3.4.5.6.7} + \frac{y^8}{1.2.3.4.5.6.7.8} - \frac{y^9}{1.2.3.4.5.6.7.8.9}$$

$$+ \frac{y^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c:, \text{ vel immutatis Signis } z = \frac{y^1}{1} - \frac{y^2}{1.2} +$$

$$\frac{y^3}{1.2.3} - \frac{y^4}{1.2.3.4} + \frac{y^5}{1.2.3.4.5} - \frac{y^6}{1.2.3.4.5.6} + \frac{y^7}{1.2.3.4.5.6.7} - \frac{y^8}{1.2.3.4.5.6.7.8}$$

$$+ \frac{y^9}{1.2.3.4.5.6.7.8.9} - \frac{y^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c:. \text{ Ex hac oritur nova conclusio,}$$

quod ab Aequatione priori $m + \frac{z^1}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} +$

$$\frac{z^6}{6} + \frac{z^7}{7} + \frac{z^8}{8} + \frac{z^9}{9} + \frac{z^{10}}{10} + \&c: = 0 \text{ altera fluat } z = \frac{m^1}{1} +$$

$$\frac{m^2}{1.2} - \frac{m^3}{1.2.3} + \frac{m^4}{1.2.3.4} - \frac{m^5}{1.2.3.4.5} + \frac{m^6}{1.2.3.4.5.6} - \frac{m^7}{1.2.3.4.5.6.7} +$$

$$\frac{m^8}{1.2.3.4.5.6.7.8} - \frac{m^9}{1.2.3.4.5.6.7.8.9} + \frac{m^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c: = 0, \text{ nimirum,}$$

si m valor sit uniformis, respondeat primae Aequationi Infinitinomiali unica

Radix z Realis expressa ab Infinitinomio $\frac{m^1}{1} - \frac{m^2}{1.2} + \frac{m^3}{1.2.3} - \frac{m^4}{1.2.3.4} +$

$$\frac{m^5}{1.2.3.4.5} - \frac{m^6}{1.2.3.4.5.6} + \frac{m^7}{1.2.3.4.5.6.7} - \frac{m^8}{1.2.3.4.5.6.7.8} + \frac{m^9}{1.2.3.4.5.6.7.8.9}$$

$$- \frac{m^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c:, \text{ five } 1 - z = 1 - \frac{m^1}{1} + \frac{m^2}{1.2} - \frac{m^3}{1.2.3} + \frac{m^4}{1.2.3.4}$$

$$- \frac{m^5}{1.2.3.4.5} + \frac{m^6}{1.2.3.4.5.6} - \frac{m^7}{1.2.3.4.5.6.7} + \frac{m^8}{1.2.3.4.5.6.7.8} - \frac{m^9}{1.2.3.4.5.6.7.8.9}$$

$$+ \frac{m^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c: \text{ in Infinitum.}$$

rus positivus minor sit Unitate, nimirum $1^x \left(1 - \frac{\omega x^1}{1} + \frac{\omega^2 x^2}{1.2} - \right.$

$$\frac{\omega^3 x^3}{1.2.3} + \frac{\omega^4 x^4}{1.2.3.4} - \frac{\omega^5 x^5}{1.2.3.4.5} + \frac{\omega^6 x^6}{1.2.3.4.5.6} - \frac{\omega^7 x^7}{1.2.3.4.5.6.7} + \frac{\omega^8 x^8}{1.2.3.4.5.6.7.8} \\ \left. - \frac{\omega^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{\omega^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c: \right), \text{ continere Notam } \omega \text{ Functio-}$$

nem huiusmodi $\tau \tilde{v}$, quae exhibeatur ab Infinitinomio positivo $\frac{1-v^1}{1} +$

$$\frac{1-v^2}{2} + \frac{1-v^3}{3} + \frac{1-v^4}{4} + \frac{1-v^5}{5} + \frac{1-v^6}{6} + \frac{1-v^7}{7} + \frac{1-v^8}{8} + \\ \frac{1-v^9}{9} + \frac{1-v^{10}}{10} + \&c: . \text{ Liquet hoc comparatione inita aequalium } v,$$

atque $1 - z$; etenim sponte sua derivatur $z = 1 - v$. Hoc autem evidentius

$$\frac{v-1^1}{1} - \frac{v-1^2}{2} + \frac{v-1^3}{3} - \frac{v-1^4}{4} + \frac{v-1^5}{5} - \frac{v-1^6}{6} + \frac{v-1^7}{7} -$$

$$\frac{v-1^8}{8} + \frac{v-1^9}{9} - \frac{v-1^{10}}{10} + \&c:, \text{ quod Membrum posterius, si } v < 1,$$

$$\text{aequipollet } \frac{(1-v)^1}{1} - \frac{(1-v)^2}{2} + \frac{(1-v)^3}{3} - \frac{(1-v)^4}{4} + \frac{(1-v)^5}{5} -$$

$$\frac{(1-v)^6}{6} + \frac{(1-v)^7}{7} - \frac{(1-v)^8}{8} + \frac{(1-v)^9}{9} - \frac{(1-v)^{10}}{10} + \&c: = \omega \text{ in In-}$$

finitum. Accedit eadem affectio, cuius in alia Formula Exponentiali consideranda mentionem iam fecimus, scilicet Analyticam expressionem illam Infinitinomiam

$$\text{ita esse universaliter comparatam, ut Potestas quaelibet } \left(1 - \frac{\omega^1}{1} + \frac{\omega^2}{1.2} - \right.$$

$$\frac{\omega^3}{1.2.3} + \frac{\omega^4}{1.2.3.4} - \frac{\omega^5}{1.2.3.4.5} + \frac{\omega^6}{1.2.3.4.5.6} - \frac{\omega^7}{1.2.3.4.5.6.7} + \frac{\omega^8}{1.2.3.4.5.6.7.8} \\ \left. - \frac{\omega^9}{1.2.3.4.5.6.7.8.9} + \frac{\omega^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c: \right)^x = \left(\frac{1}{C} \right)^x \text{ si } \omega = 1, \text{ vel}$$

generalius

$$\text{generalius } v^x = 1 - \frac{x^1 \omega^1}{1} + \frac{x^2 \omega^2}{1.2} - \frac{x^3 \omega^3}{1.2.3} + \frac{x^4 \omega^4}{1.2.3.4} - \frac{x^5 \omega^5}{1.2.3.4.5} +$$

$$\frac{x^6 \omega^6}{1.2.3.4.5.6} - \frac{x^7 \omega^7}{1.2.3.4.5.6.7} + \frac{x^8 \omega^8}{1.2.3.4.5.6.7.8} - \frac{x^9 \omega^9}{1.2.3.4.5.6.7.8.9} +$$

$$\frac{x^{10} \omega^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c., \text{ quicunque fit valor } \omega \text{ a praedefinita Functione}$$

$$- \left(\frac{(1-v)^1}{1} + \frac{(1-v)^2}{2} + \frac{(1-v)^3}{3} + \frac{(1-v)^4}{4} + \frac{(1-v)^5}{5} + \&c. \right)$$

indigitatus. Nec difficile foret ostendere respectu novae Functionis Γ , quod

suffecta in Infinitinomiae Aequationis priori Membro $\frac{z^1}{1} + \frac{z^2}{2} + \frac{z^3}{3}$

$$+ \frac{z^4}{4} + \frac{z^5}{5} + \frac{z^6}{6} + \frac{z^7}{7} + \frac{z^8}{8} + \frac{z^9}{9} + \frac{z^{10}}{10} + \&c. = \Gamma \text{ eius}$$

$$\text{inventa Radice } \frac{\Gamma^1}{1} - \frac{\Gamma^2}{1.2} + \frac{\Gamma^3}{1.2.3} - \frac{\Gamma^4}{1.2.3.4} + \frac{\Gamma^5}{1.2.3.4.5} - \&c. \text{ veri-}$$

tas Radicis istius confirmetur. Profecto ea substitutio praebet facillimi Cal-

$$\text{culi ope } \Gamma = \frac{\Gamma^1}{1} - \frac{\Gamma^2}{1.2} + \frac{\Gamma^3}{1.2.3} - \frac{\Gamma^4}{1.2.3.4} + \frac{\Gamma^5}{1.2.3.4.5} - \&c.,$$

$$+ \frac{\Gamma^2}{1.2} - \frac{\Gamma^3}{1.2} + \frac{\Gamma^4}{1.2.4} - \frac{\Gamma^5}{1.3.4}$$

$$+ \frac{\Gamma^3}{1.3} + \frac{\Gamma^4}{1.2.3} - \frac{\Gamma^5}{1.2.3.4}$$

$$- \frac{\Gamma^4}{1.2} + \frac{\Gamma^5}{1.4}$$

$$+ \frac{\Gamma^4}{1.4} + \frac{\Gamma^5}{1.2.3}$$

$$- \frac{\Gamma^5}{1.2}$$

$$+ \frac{\Gamma^5}{1.5}$$

uti in num°. 60. Functioni Δ contigisse etiam visum.

70. In universali Aequatione num°. praecedente 65. prolata, dum sup-
ponatur

ponatur Index $x = 1$, enascetur $1 - z = 1 - \frac{z^1}{1} = 1 - z$, quum ceteri omnes termini, facili experimento solito suadente singularum Diagonalium Linearum post $\frac{z^1 x^1}{1}$ in illa hypothesi evanescentium, se invicem destruant,

$$= 1 - \frac{\Gamma^1}{1} + \frac{\Gamma^2}{1.2} - \frac{\Gamma^3}{1.2.3} + \frac{\Gamma^4}{1.2.3.4} - \frac{\Gamma^5}{1.2.3.4.5} + \frac{\Gamma^6}{1.2.3.4.5.6} - \frac{\Gamma^7}{1.2.3.4.5.6.7} + \frac{\Gamma^8}{1.2.3.4.5.6.7.8} - \frac{\Gamma^9}{1.2.3.4.5.6.7.8.9} + \frac{\Gamma^{10}}{1.2.3.4.5.6.7.8.9.10} -$$

$$\&c.; \text{ atque idcirco } -z = -\frac{\Gamma^1}{1} + \frac{\Gamma^2}{1.2} - \frac{\Gamma^3}{1.2.3} + \frac{\Gamma^4}{1.2.3.4} -$$

$$\frac{\Gamma^5}{1.2.3.4.5} + \frac{\Gamma^6}{1.2.3.4.5.6} - \frac{\Gamma^7}{1.2.3.4.5.6.7} + \frac{\Gamma^8}{1.2.3.4.5.6.7.8} - \frac{\Gamma^9}{1.2.3.4.5.6.7.8.9} +$$

$$\frac{\Gamma^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c.; \text{ posita Functione } \Gamma = \frac{z^1}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4}$$

$$+ \frac{z^5}{5} + \frac{z^6}{6} + \frac{z^7}{7} + \frac{z^8}{8} + \frac{z^9}{9} + \frac{z^{10}}{10} + \&c.; \text{ Duplex ergo mo-}$$

dus prostat valorem z Unitate minoris Transcendentaliter exhibendi: pri-

$$\text{mus ope expressionis } z = \frac{\Delta^1}{1} + \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} + \frac{\Delta^4}{1.2.3.4} + \frac{\Delta^5}{1.2.3.4.5} \&c.;$$

$$\text{et alter } z = \frac{\Gamma^1}{1} - \frac{\Gamma^2}{1.2} + \frac{\Gamma^3}{1.2.3} - \frac{\Gamma^4}{1.2.3.4} + \frac{\Gamma^5}{1.2.3.4.5} \&c.; \text{ iuxta nume-}$$

ros 62. ac praesentem, Functionibus Δ , atque Γ quoad terminorum signa inter se discrepantibus. Sicuti autem hoc evidenter efficitur quia in Analyticis Trigonis numerorum 44. et 65. universi post secundum termini se invicem destruant, ambae illae Aequationes dubio omni, praestigiisve carebunt. Hinc $\frac{\Delta^1}{1}$

$$+ \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} + \frac{\Delta^4}{1.2.3.4} + \frac{\Delta^5}{1.2.3.4.5} \&c. = \frac{\Gamma^1}{1} - \frac{\Gamma^2}{1.2} + \frac{\Gamma^3}{1.2.3} -$$

$$\frac{\Gamma^4}{1.2.3.4} + \frac{\Gamma^5}{1.2.3.4.5} \&c.; \text{ quod vix in tanta Formularum perplexitate, et ap-}$$

paratu credendum, nisi vere identicam Aequalitatem $z = z$ ibidem abscondi

$$\text{detectum supra fuisset. Si igitur in Formula generalissima } -\frac{x^1}{1} + \frac{x^2}{1.2} -$$

$$\frac{x^3}{1.2.3}$$

$$\frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} - \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} - \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^8}{1.2.3.4.5.6.7.8} \\ - \frac{x^9}{1.2.3.4.5.6.7.8.9} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} \text{ \&c: substituamus vice } x \text{ notam}$$

eius Functionem $\frac{x^1}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} +$
 $\frac{x^8}{8} + \frac{x^9}{9} + \frac{x^{10}}{10} + \text{\&c.} \text{ proveniet tantum } -x: \text{ qua de re, quum } \frac{1}{C^x}$

$$= \left(\frac{1}{C}\right)^x = C^{-x} = 1 - \frac{x^1}{1} + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} - \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} - \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^8}{1.2.3.4.5.6.7.8} - \frac{x^9}{1.2.3.4.5.6.7.8.9} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} \text{ \&c: , obtinebitur post effectam substitutionem}$$

$$\frac{1}{C} \frac{x^1}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} + \frac{x^{10}}{10} \text{ \&c: ,}$$

vel C $\frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \frac{x^9}{9} - \frac{x^{10}}{10} \text{ \&c: }$

$$- \frac{x^{10}}{10} \text{ \&c: } = 1 - x, \text{ seu, quod idem est, } C = \left(\frac{1-x^1}{1} + \frac{1-x^2}{2} + \right.$$

$$\frac{1-x^3}{3} + \frac{1-x^4}{4} + \frac{1-x^5}{5} + \frac{1-x^6}{6} + \frac{1-x^7}{7} + \frac{1-x^8}{8} + \frac{1-x^9}{9} +$$

$$\left. \frac{1-x^{10}}{10} \text{ \&c: } \right) = 1 - (1-x) = +x =$$

$$C \frac{1-x^1}{1} + \frac{1-x^2}{2} + \frac{1-x^3}{3} + \frac{1-x^4}{4} + \frac{1-x^5}{5} + \frac{1-x^6}{6} + \frac{1-x^7}{7} + \frac{1-x^8}{8} + \frac{1-x^9}{9} + \frac{1-x^{10}}{10} \text{ \&c: ,}$$

ubi x minor est Unitate. Nec obstupescat Lector bis exprimi posse, eodem

fundamentali manente Numero C , valorem x , nimirum aut $= C^{\frac{(x-1)^1}{L}}$

$$\frac{(x-1)^0}{1} - \frac{(x-1)^1}{2} + \frac{(x-1)^2}{3} - \frac{(x-1)^3}{4} + \frac{(x-1)^4}{5} \&c: \quad \frac{(1-x)^0}{1}$$

, aut potius = C

$$\frac{(1-x)^1}{2} - \frac{(1-x)^2}{3} + \frac{(1-x)^3}{4} - \frac{(1-x)^4}{5} \&c:$$

, eoquod duo Indices

Exponentialium revera coincidunt.

71. Hisce compendiosiore forma, ac diverso quidem homologorum antecedentium Theorematum ordine praenotatis, ut argumenti foecunditas, et simul demonstrationum, quae praesto sunt, flexilis nativa vis elucescat, libenti animo omitto cetera minoris momenti fusius persequi, quae nova Formula Exponentialis paucis rite immutatis communia habet cum altera, cui longiorem illustrationem superius dicavimus. Sublata nunc omni mora differere necesse erit de singulari, nec tam obvia hypothefi Binomii $1 - z$ negativi,

$$(-1)^x \times (1 \pm z)^x = (-1)^x \times \left(1 \pm \frac{zx^1}{1} - \frac{z^2x^2}{1.2} \pm \frac{z^3x^3}{1.2.3} - \frac{z^4x^4}{1.2.3.4} \pm \frac{z^5x^5}{1.2.3.4.5} \right.$$

$\frac{z^2x^1}{1.2}$	$\frac{z^3x^2}{1.2}$	$\frac{z^4x^3}{1.4}$	$\frac{z^5x^4}{1.3.4}$	$\frac{z^6x^5}{1.2.4.6}$
$\pm \frac{z^3x^1}{1.3}$	$\pm \frac{11z^4x^2}{1.2.3.4}$	$\pm \frac{7z^5x^3}{1.2.3.4}$	$\pm \frac{17z^6x^4}{1.2.3.4.6}$	$\pm \frac{5z^7x^5}{1.2.3.4.6}$
$\frac{z^4x^1}{1.4}$	$\frac{5z^5x^2}{1.3.4}$	$\frac{5z^6x^3}{1.4.4}$	$\frac{7z^7x^4}{1.2.4.6}$	$\frac{7z^8x^5}{1.3.6.8}$
$\pm \frac{z^5x^1}{1.5}$	$\pm \frac{137z^6x^2}{1.3.4.5.6}$	$\pm \frac{29z^7x^3}{1.3.5.6}$	$\pm \frac{967z^8x^4}{1.2.3.4.5.6.8}$	$\pm \frac{1069z^9x^5}{1.2.4.5.6.8.9}$
$\frac{z^6x^1}{1.6}$	$\frac{7z^7x^2}{1.4.5}$	$\frac{469z^8x^3}{1.2.3.5.6.8}$	$\frac{89z^9x^4}{1.3.4.5.8}$	$\frac{19z^{10}x^5}{1.2.4.4.8}$
$\pm \frac{z^7x^1}{1.7}$	$\pm \frac{363z^8x^2}{1.4.5.7.8}$	$\pm \frac{29531z^9x^3}{1.2.3.5.6.7.8.9}$	$\pm \frac{4523z^{10}x^4}{1.3.4.5.6.7.9}$	$\&c.$
$\frac{z^8x^1}{1.8}$	$\frac{761z^9x^2}{1.5.7.8.9}$	$\frac{1303z^{10}x^3}{1.3.4.6.7.8}$	$\&c.$	
$\pm \frac{z^9x^1}{1.9}$	$\pm \frac{7129z^{10}x^2}{1.5.7.8.9.10}$	$\&c.$		
$\frac{z^{10}x^1}{1.10}$	$\&c.$			

$\&c.$

gativi, five z Unitatem positivam superantis, quod opus maximopere foret periculosum, et fallax, nisi antea acti Capitis luce favente universa, quae irreperant, menda sedulo corrigantur. Sat notum Geometris evolutionem τ_8 $(1 - y)^x$, seu $(-v)^x$ in falsum abire quominus Potestas ipsa per $(-1)^x$ $(y - 1)^x$, aut $(-1)^x (v)^x$ explicetur. Ad hoc autem, ut in Seriem numero terminorum Infinitam, sed Exponentialibus nullo modo perturbatam redigatur Formula ipsa Potentialis, oportet τ_0 $(y - 1)^x$ vertere in $(-2 + y + 1)^x$, five meliori ordine in $(1 + (y - 2))^x$, videlicet, posita specie $z = y - 2$, in commodiorem $(1 + z)^x$; qua in transformatione monendum z indicare magnitudinem positivam, quoties y sit maius Binario, at numerum e-contra negativum, dum y Unitatem, non vero Binarium excedat. Quibus ad maiorem rei claritatem praemissis evidens fit usitatam nunc methodum repetentibus sequens facillima Aequatio $(1 - y)^x =$

$$\begin{aligned}
 & - + \frac{z^6 x^6}{1.2.3.4.5.6} = \pm \frac{z^7 x^7}{1.2.3.4.5.6.7} - + \frac{z^8 x^8}{1.2.3.4.5.6.7.8} = \pm \frac{z^9 x^9}{1.2.3.4.5.6.7.8.9} - + \frac{z^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.) \\
 & = \frac{z^7 x^6}{1.2.4.5.6} - \frac{z^8 x^7}{1.2.3.5.6.8} + \frac{z^9 x^8}{1.2.3.5.6.7.8} - \frac{z^{10} x^9}{1.2.3.4.6.7.8.10} \&c. \\
 & - + \frac{23z^8 x^6}{1.3.4.5.6.8} = \pm \frac{13z^9 x^7}{1.4.5.6.8.9} - + \frac{29z^{10} x^8}{1.4.6.7.8.9.10} \&c. \\
 & = \frac{z^9 x^6}{1.2.5.8} - \frac{z^{10} x^7}{1.2.4.6.8} \&c. \\
 & + \frac{3013z^{10} x^6}{1.2.4.5.6.8.9.10} \&c.
 \end{aligned}$$

Hac in expressione analytica ob variabilem $z = y - 2$ gemino signo affectam utraque Formulae numerorum 44. et 65. simul continentur, proindeque nulla difficultate Infinitinomialium istud laborat. Palam autem fit casu signi negativi, sive y Binario minoris, sed ex hypothese Unitate maioris, fore z Numerum fractum; quapropter Coefficientes numero terminorum Infiniti erunt universim Series admodum convergentes, nec ulteriora artificia desiderabunt. Iisdem insuper regulis iam num^o. 54. memoratis subiicietur casus z Numeri positivi Unitate maioris, aut y excedentis Ternarium, et idcirco novos non requireret canones pro divergentibus Functionibus z in dictas convergentes Series mutandis, veluti prima fronte nonnemo poterat suspicari. Periuicundum equidem nobis obversatur symptoma ex valore ortum $y = 0$: tunc enim emergit casu $x = 1$ Aequatio $1 = -1(1 - 2)$, quum experimento facto alii omnes termini in singulis Hypothenusis Trianguli Algebraici annihilentur, necnon posito Indice eodem $x = 2, 3, 4$ &c: proveniat $1 = 1(1 - 4 - 4 + 8)$, $1 = -1(1 - 6 - 6 + 18 - 8 + 36 - 36)$, $1 = 1(1 - 8 - 8 + 32 - \frac{3^2}{3} + 64 - \frac{256}{3} - 16 + \frac{352}{3} - 256 + \frac{512}{3})$, scilicet universaliter $1 = 1$. Quae nedum Formulae veritatem in difficillimo eius periculo comprobant, quin etiam specimen tribuunt

$$\text{non absurdi, sed determinati, ac finiti valoris Seriesi } 1 \pm \frac{V^1 x^1}{1} \pm \frac{V^2 x^2}{1.2} \pm \frac{V^3 x^3}{1.2.3} \pm \frac{V^4 x^4}{1.2.3.4} \pm \frac{V^5 x^5}{1.2.3.4.5} \pm \text{ \&c: tametsi } V = \left(-\frac{2^1}{1} - \frac{2^2}{2} - \frac{2^3}{3} - \frac{2^4}{4} - \frac{2^5}{5} - \frac{2^6}{6} - \frac{2^7}{7} - \frac{2^8}{8} - \frac{2^9}{9} \text{ \&c:} \right) = -\infty. \text{ Uno}$$

verbo adferam singula praedemonstrata symptomata in Potestatibus quibuscunque Binomii positivi $(1 \pm z)^x$ aptari quoque Basibus negativis $(-v)^x$, sive Binomii propositi $(1 - y)^x$ Potestatibus, subintellecto in iis $y > 1$, dummodo pro veritate Seriesi Neutoniana tuenda illud antea per *Similem* negativae Unitatis Potestatem $(-1)^x$ multiplicetur, subindeque in Binomium vertatur, cuius primum Nomen sit Unitas positiva. Universaliter profluet $(-v)^x = (-1)^x \cdot v^x = (-1)^x (1 + (v - 1))^x = (-1)^x (1 \pm z)^x = (-1)^x \left(1 \pm \frac{(\Delta \text{ seu } \Gamma)^1 x^1}{1} \right)$

+

$$\begin{aligned}
& + \frac{(\Delta \text{ seu } \Gamma)^2 x^2}{1 \cdot 2} \pm \frac{(\Delta \text{ seu } \Gamma)^3 x^3}{1 \cdot 2 \cdot 3} + \frac{(\Delta \text{ seu } \Gamma)^4 x^4}{1 \cdot 2 \cdot 3 \cdot 4} \pm \frac{(\Delta \text{ seu } \Gamma)^5 x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \\
& \frac{(\Delta \text{ seu } \Gamma)^6 x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \pm \frac{(\Delta \text{ seu } \Gamma)^7 x^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{(\Delta \text{ seu } \Gamma)^8 x^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \pm \frac{(\Delta \text{ seu } \Gamma)^9 x^9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \\
& \frac{(\Delta \text{ seu } \Gamma)^{10} x^{10}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} \pm \&c. \quad \left. \right) \text{ Qua in Progressione, cui admodum brevior obtingeret}
\end{aligned}$$

forma $(-v)^x = (-1)^x C^x = (-C)^x$ si $\Delta = 1$, necnon $= (-C)^{-x}$ si $\Gamma = 1$, respondent huic $+z$ termini omnes signo positivo praediti, ac $-z$ alternatim positivo et negativo distincti, eidemque primae hypothesei convenit Δ

Functio huiusmodi $\text{est } z$, ut sit $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} +$

$$\frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c.; \text{ vel } \frac{\sqrt{-1}}{v-1} - \frac{\sqrt{-1}^2}{v-1^2} + \frac{\sqrt{-1}^3}{v-1^3} -$$

$$\frac{\sqrt{-1}^4}{v-1^4} + \frac{\sqrt{-1}^5}{v-1^5} - \frac{\sqrt{-1}^6}{v-1^6} + \frac{\sqrt{-1}^7}{v-1^7} - \frac{\sqrt{-1}^8}{v-1^8} + \frac{\sqrt{-1}^9}{v-1^9} - \frac{\sqrt{-1}^{10}}{v-1^{10}} +$$

$\&c.$; dum in altera hypothesei, cui satisfacit signum inferius, est Γ Functio huiusmodi

$$\frac{z^1}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} + \frac{z^6}{6} + \frac{z^7}{7} + \frac{z^8}{8} + \frac{z^9}{9} + \frac{z^{10}}{10}$$

$$+ \&c.; \text{ sive } \frac{1-v^1}{1-v} + \frac{1-v^2}{1-v^2} + \frac{1-v^3}{1-v^3} + \frac{1-v^4}{1-v^4} + \frac{1-v^5}{1-v^5} + \frac{1-v^6}{1-v^6} +$$

$$\frac{1-v^7}{1-v^7} + \frac{1-v^8}{1-v^8} + \frac{1-v^9}{1-v^9} + \frac{1-v^{10}}{1-v^{10}} + \&c. \quad \text{Harumce Serierum nobis fa-}$$

miliarium superiusque perpensarum Δ et Γ indoles ea profecto est, ut unicam sane praebeant Analyticam formulam, quippe cum Δ in $- \Gamma$ abeat quando v ex positivo Numero Unitatem excedente in Fractum positivum vertitur, ac vicissim; nec ideo alterutra Series notissima amicam Analystae manum denuo expostulare videtur.

72. Remedio traducendi negativum Binomii valorem in Numerum positivum uti renuentes paradoxam mutamque veluti Formulam directo tramite acquisituros quispiam praevidere nunc poterit. Cum etenim evolutio usitata Binomii elargiatur

$$(1-y)^x = (1)^x \times \left(1 - \frac{y^1 x^1}{1} + \frac{y^2 x^2}{1.2} - \frac{y^3 x^3}{1.2.3} + \frac{y^4 x^4}{1.2.3.4} - \frac{y^5 x^5}{1.2.3.4.5} \right. \\
- \frac{y^2 x^1}{1.2} + \frac{y^3 x^2}{1.2} - \frac{y^4 x^3}{1.4} + \frac{y^5 x^4}{1.3.4} - \frac{y^6 x^5}{1.2.4.6} \\
- \frac{y^3 x^1}{1.3} + \frac{11y^4 x^2}{1.2.3.4} - \frac{7y^5 x^3}{1.2.3.4} + \frac{17y^6 x^4}{1.2.3.4.6} - \frac{5y^7 x^5}{1.2.3.4.6} \\
- \frac{y^4 x^1}{1.4} + \frac{5y^5 x^2}{1.3.4} - \frac{5y^6 x^3}{1.4.4} + \frac{7y^7 x^4}{1.2.4.6} - \frac{7y^8 x^5}{1.3.6.8} \\
- \frac{y^5 x^1}{1.5} + \frac{137y^6 x^2}{1.3.4.5.6} - \frac{29y^7 x^3}{1.3.5.6} + \frac{967y^8 x^4}{1.2.3.4.5.6.8} - \frac{1069y^9 x^5}{1.2.4.5.6.8.9} \\
- \frac{y^6 x^1}{1.6} + \frac{7y^7 x^2}{1.4.5} - \frac{469y^8 x^3}{1.2.3.5.6.8} + \frac{89y^9 x^4}{1.3.4.5.8} - \frac{19y^{10} x^5}{1.2.4.4.8} \\
- \frac{y^7 x^1}{1.7} + \frac{363y^8 x^2}{1.4.5.7.8} - \frac{29531y^9 x^3}{1.2.3.5.6.7.8.9} + \frac{4523y^{10} x^4}{1.3.4.5.6.7.9} - \text{\&c.} \\
- \frac{y^8 x^1}{1.8} + \frac{761y^9 x^2}{1.5.7.8.9} - \frac{1303y^{10} x^3}{1.3.4.6.7.8} - \text{\&c.} \\
- \frac{y^9 x^1}{1.9} + \frac{7129y^{10} x^2}{1.5.7.8.9.10} - \text{\&c.} \\
- \frac{y^{10} x^1}{1.10} - \text{\&c.} \right)$$

nimirum $(1-y)^x = (1)^x \left(1 - \frac{Y^1 x^1}{1} + \frac{Y^2 x^2}{1.2} - \frac{Y^3 x^3}{1.2.3} + \frac{Y^4 x^4}{1.2.3.4} - \frac{Y^5 x^5}{1.2.3.4.5} + \frac{Y^6 x^6}{1.2.3.4.5.6} - \frac{Y^7 x^7}{1.2.3.4.5.6.7} + \frac{Y^8 x^8}{1.2.3.4.5.6.7.8} - \frac{Y^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{Y^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} - \text{\&c.} \right)$, Sigla Y denotante Functionem $\frac{y^1}{1} + \frac{y^2}{2} +$

$\frac{y^3}{3} + \frac{y^4}{4} + \frac{y^5}{5} + \frac{y^6}{6} + \frac{y^7}{7} + \frac{y^8}{8} + \frac{y^9}{9} + \frac{y^{10}}{10} + \text{\&c.}$, vide-

licet secundi termini Coefficientem, qui Coefficiens propter y Unitate maiorem pretium superat infinitum Harmonicae Progressionis $1 + \frac{1}{2} + \frac{1}{3} +$

$\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \text{\&c.}$, ideoque eo magis Infinitum

tum

$$\begin{aligned}
 & + \frac{y^6 x^6}{1.2.3.4.5.6} - \frac{y^7 x^7}{1.2.3.4.5.6.7} + \frac{y^8 x^8}{1.2.3.4.5.6.7.8} - \frac{y^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{y^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.), \\
 & + \frac{y^7 x^6}{1.2.4.5.6} - \frac{y^8 x^7}{1.2.3.5.6.8} + \frac{y^9 x^8}{1.2.3.5.6.7.8} - \frac{y^{10} x^9}{1.2.3.4.6.7.8.10} \&c. \\
 & + \frac{23y^8 x^6}{1.3.4.5.6.8} - \frac{13y^9 x^7}{1.4.5.6.8.9} + \frac{29y^{10} x^8}{1.4.6.7.8.9.10} \&c. \\
 & + \frac{y^9 x^6}{1.2.5.8} - \frac{y^{10} x^7}{1.2.4.6.8} \&c. \\
 & + \frac{3013y^{10} x^6}{1.2.4.5.6.8.9.10} \&c.
 \end{aligned}$$

tum valorem constituit, excitabitur tandem $(1-y)^x = (1)^x \left(1 - \frac{\infty^1 x^1}{1} + \frac{\infty^2 x^2}{1.2} \right.$

$$\begin{aligned}
 & - \frac{\infty^3 x^3}{1.2.3} + \frac{\infty^4 x^4}{1.2.3.4} - \frac{\infty^5 x^5}{1.2.3.4.5} + \frac{\infty^6 x^6}{1.2.3.4.5.6} - \frac{\infty^7 x^7}{1.2.3.4.5.6.7} + \frac{\infty^8 x^8}{1.2.3.4.5.6.7.8} \\
 & - \frac{\infty^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{\infty^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.) \text{ sine ullo Limite. Alterum hu-}
 \end{aligned}$$

iusce Aequationis Membrum, quicumque sit Index x , abscondere Nullitatem sensimus in Capite primo dum $y = 1$, seu Nota ∞ Infinitum punctualiter Harmonicum repraesentet; quae de re nihil mirum, si, y Unitatem excedente, et Nota ∞ in praesentia Harmonicum illud Inassignabile superante, non amplius termini secundum Membrum componentes mutuo se destruant, verumtamen, si minus valorem, tesseram saltem praebeant cuiusdam magnitudinis finitae ac determinatae, nunc positivae, nunc negativae prouti Exponens x par vel impar,

quod

quod liquido patuit in singulari specimine numeri 71. quando fuit $z = -2$, excepto casu Indicis x quocunque Numero assignabili maioris, aut Infiniti. Quinimmo quantitatem etiam Imaginariam alterum illud Membrum significat quoties sit x Numerus fractus vel fractionarius simplicissimis cyphris expositus, et Numeratore impare ac Denominatore pari donatus. Facilis artificii est adinventio Functionis finitae, cuius Emblema

offert indeterminatum Infinitinomialium $(1)^x \left(1 - \frac{\infty^1 x^1}{1} + \frac{\infty^2 x^2}{1.2} - \frac{\infty^3 x^3}{1.2.3} \right.$

$\left. + \frac{\infty^4 x^4}{1.2.3.4} - \frac{\infty^5 x^5}{1.2.3.4.5} + \&c: \right)$, Enimvero praesidio numeri 71. illud tra-

ducitur in $(-1)^x \left(1 + \frac{\Pi^1 x^1}{1} + \frac{\Pi^2 x^2}{1.2} + \frac{\Pi^3 x^3}{1.2.3} + \frac{\Pi^4 x^4}{1.2.3.4} + \right.$

$\left. \frac{\Pi^5 x^5}{1.2.3.4.5} + \&c: \right)$, ibidemque est $\Pi = \frac{(y-2)^1}{1} - \frac{(y-2)^2}{2} + \frac{(y-2)^3}{3} -$

$\frac{(y-2)^4}{4} + \frac{(y-2)^5}{5} - \frac{(y-2)^6}{6} + \frac{(y-2)^7}{7} - \frac{(y-2)^8}{8} + \frac{(y-2)^9}{9}$

$- \frac{(y-2)^{10}}{10} + \&c:$, quae Series eadem ratione efformatur ab $y = 1$, quemadmodum ex v altera profilit $\frac{(v-1)^1}{1} - \frac{(v-1)^2}{2} + \frac{(v-1)^3}{3} - \frac{(v-1)^4}{4} +$

$\frac{(v-1)^5}{5} - \&c:$ supra explanata, ac omnimodam in Capite III. determinationem finitam receptura. Hypothesis Series $\Pi = 1$ restituit, ut in 68^{mo}.

num^o., $y = 1 - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} - \frac{1}{1.2.3.4.5.6} + \&c:$,

aut $1 - y = \frac{1}{C}$, scilicet Numerum Positivum, eodem penitus pacto, quo Positivum ipsummet Numerum C^{-1} gigneret casus τ_2 y Unitate minoris; dum ex adverso Series emendata in num^o. 71. parit Numerum Negativum $1 - y$

$= -(C^{\pm 1})$, vel $(1 - y)^x = (-1)^x (C^{\pm x})$.

73. Quae omnia ne forte supra eorum vires posita Geometriae candidatis nimium imponant, et quasi miraculo terreant, illos oro obsecroque, ut memoria repetant Formulas universas ab evolutione Potestatis Binomii progenitas, in quo pars secunda negativo signo affecta priorem excedat, falsas mendosaeque fieri eoquod computo accedant Residua semper semperque in

Inf-

Infinitum crescentia. Cum haec itaque Residua neglecta, Magnitudinis Infinitae, terminis Analyticae Progressionis $1 - \frac{\infty^1 x^1}{1} + \frac{\infty^2 x^2}{1.2} - \frac{\infty^3 x^3}{1.2.3} + \frac{\infty^4 x^4}{1.2.3.4} -$

$\frac{\infty^5 x^5}{1.2.3.4.5} + \frac{\infty^6 x^6}{1.2.3.4.5.6} - \&c.$, excepto primo, Infinitatem affingant consequens

est, quod tunc Infiniti Nota nullo modo sublata Formulam totam linquat indeterminati valoris, nempe $\infty - \infty$, qua in indeterminatione latet Numerus ille finitus, positivus, vel negativus, et quandoque imaginarius sive impossibilis varios iuxta casus Indicis x Binomium exaltandum afficientis. Nullam in aperto est determinatam Magnitudinem in medio ponere expressionem istam ab Infinitis invicem additis subtractisque illaqueatam, veluti foret de alia quacumque Nullitates mutuo divisas, Producta Infinitorum per Nullitates, sexcentasque confimiles Algebraici Algorithmi indeterminationis seu reticentiae formas continente: verumtamen illa symbolis monet assumptae communis Analyseos inapplicabilitatem silentiumve inchoare tum cum Problematum *Data* Limitem quemdam exsuperent, quo facto neutiquam solutionibus, sed solutionum spectris inductis viam methodumque Algebra signat meliora sequendi. Iucunditate aliqua ceteroquin non caret Infiniti Ordinis Potestatem more Neutroniano enucleatam

$$(1-x)^\infty = 1 - \frac{\infty^1 x^1}{1} + \frac{\infty^2 x^2}{1.2} - \frac{\infty^3 x^3}{1.2.3} + \frac{\infty^4 x^4}{1.2.3.4} - \frac{\infty^5 x^5}{1.2.3.4.5} + \frac{\infty^6 x^6}{1.2.3.4.5.6} - \frac{\infty^7 x^7}{1.2.3.4.5.6.7} + \frac{\infty^8 x^8}{1.2.3.4.5.6.7.8} - \frac{\infty^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{\infty^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c.,$$

dum Index ∞ oculis subiiciat Infinitum Harmonicum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

$+ \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c.$, donec τ x minus aut maius infra

Binarium sit Unitate, vel isti Unitati ad unguem aequale, Nullitatem, prope dices personatam, completi ex num°. 39., quamvis eadem Potestas $(1-x)^\infty$, statim ac x vel tantulum excedat Binarium, protinus nunciet Infinitum aequivocum, positivum nimirum nutu Geometrae, seu negativum, sin vero haec Nota ∞ stet pro Infinito in immensum maiore quoad

praedictam Harmonicam Seriem, videlicet referat $\frac{y^1}{1} + \frac{y^2}{2} + \frac{y^3}{3} +$

$\frac{y^4}{4} + \frac{y^5}{5} + \frac{y^6}{6} + \frac{y^7}{7} + \frac{y^8}{8} + \frac{y^9}{9} + \frac{y^{10}}{10} + \&c.$, ubi y vel paul-

to etenim Binomio, ordinatisque terminis universis in morem Formularum antecedentium, idest gradu servato Potestatum Indicis — x , eadem Series ad amissim orientur, quarum supra mentionem inivimus. Dum hypotheticum Binomium sit $1 + z$ Progressio Exponentialis quaesita $= (1 + z)^{-x}$ evadet

$$\text{profecto } (1)^{-x} \cdot \left(1 - \frac{\Delta^1 x^1}{1} + \frac{\Delta^2 x^2}{1.2} - \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^4 x^4}{1.2.3.4} - \frac{\Delta^5 x^5}{1.2.3.4.5} + \right. \\ \left. \frac{\Delta^6 x^6}{1.2.3.4.5.6} - \frac{\Delta^7 x^7}{1.2.3.4.5.6.7} + \frac{\Delta^8 x^8}{1.2.3.4.5.6.7.8} - \frac{\Delta^9 x^9}{1.2.3.4.5.6.7.8.9} + \right. \\ \left. \frac{\Delta^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c: \right), \text{ manente } \Delta = \frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} \\ + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c: . \text{ Opposito autem in}$$

$$\text{casu Binomii } 1 - z \text{ erit } (1 - z)^{-x} = (1)^{-x} \cdot \left(1 + \frac{\Gamma^1 x^1}{1} + \frac{\Gamma^2 x^2}{1.2} + \right. \\ \left. \frac{\Gamma^3 x^3}{1.2.3} + \frac{\Gamma^4 x^4}{1.2.3.4} + \frac{\Gamma^5 x^5}{1.2.3.4.5} + \frac{\Gamma^6 x^6}{1.2.3.4.5.6} + \frac{\Gamma^7 x^7}{1.2.3.4.5.6.7} + \frac{\Gamma^8 x^8}{1.2.3.4.5.6.7.8} \right. \\ \left. + \frac{\Gamma^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{\Gamma^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: \right), \text{ indicante } \Gamma \text{ Functionem} \\ \frac{z^1}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} + \frac{z^6}{6} + \frac{z^7}{7} + \frac{z^8}{8} + \frac{z^9}{9} + \frac{z^{10}}{10} + \&c:;$$

feduloque notandum est, quoties z Unitatem superaverit, locum pariter fieri remedio analytico transformationis Binomii $(1 - z)^{-x}$ in $(-1)^{-x}$.

$$(1 + (z - 2))^{-x} = (-1)^{-x} \times \left(1 + \frac{\Phi^1 x^1}{1} + \frac{\Phi^2 x^2}{1.2} + \frac{\Phi^3 x^3}{1.2.3} + \frac{\Phi^4 x^4}{1.2.3.4} \right. \\ \left. + \frac{\Phi^5 x^5}{1.2.3.4.5} + \frac{\Phi^6 x^6}{1.2.3.4.5.6} + \frac{\Phi^7 x^7}{1.2.3.4.5.6.7} + \frac{\Phi^8 x^8}{1.2.3.4.5.6.7.8} + \frac{\Phi^9 x^9}{1.2.3.4.5.6.7.8.9} \right. \\ \left. + \frac{\Phi^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: \right), \text{ ubi, si } z - 2 \text{ Numerum positivum adaequet,}$$

$$\text{habebitur } \Phi = \frac{z-2^1}{1} - \frac{z-2^2}{2} + \frac{z-2^3}{3} - \frac{z-2^4}{4} + \frac{z-2^5}{5} - \frac{z-2^6}{6} \\ + \frac{z-2^7}{7} - \frac{z-2^8}{8} + \frac{z-2^9}{9} - \frac{z-2^{10}}{10} + \&c:, \text{ atque e-contra hypo-}$$

thesi facta $z = 2$ negativi Functio Φ significabit — $\left(\frac{2-z^1}{1} + \frac{2-z^2}{2} + \frac{2-z^3}{3} + \frac{2-z^4}{4} + \frac{2-z^5}{5} + \frac{2-z^6}{6} + \frac{2-z^7}{7} + \frac{2-z^8}{8} + \frac{2-z^9}{9} + \frac{2-z^{10}}{10} + \&c. \right)$. Quod algebraicum artificium rite recteque Potestati Nega-

tivi Binomii $(1 - z)^{-x}$ applicatum expressionem aufert indeterminatam ac nihil loquentem, quae solitae methodi vi oriretur illi numeri 72. non oppido dif-

similis $1 + \frac{\infty^1 x^1}{1} + \frac{\infty^2 x^2}{1.2} + \frac{\infty^3 x^3}{1.2.3} + \frac{\infty^4 x^4}{1.2.3.4} + \frac{\infty^5 x^5}{1.2.3.4.5} + \frac{\infty^6 x^6}{1.2.3.4.5.6} + \frac{\infty^7 x^7}{1.2.3.4.5.6.7} + \frac{\infty^8 x^8}{1.2.3.4.5.6.7.8} + \frac{\infty^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{\infty^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.$, ac

certe, nisi Residua computarentur, fallaciter Infinita. Huic nullas addo melioris notae considerationes; nam coram me Paradoxa non peramante omnis pene Algebra obmutescit statim ac a finitis Magnitudinibus ad Infinitum, seu Nihilum transitus locum habuerit.

75. Quae omnia facillime comprobantur; siquidem sunt consecutaria quatuor Formularum Exponentialium numeris praecedentibus 44. 65. 71. ac 72. absolutarum, nec aliud exposcunt praeterquamquod Index x in $-x$ immutetur. Proderit etiam subiungere monitum $\tau d (1 + z)^{-x} \cdot (1 + z)^x$ generaliter Unitatem peraequare; quapropter ex iamdudum cognita Formula Potestatum $(1 + z)^x$ altera statim profluet incognita Potestatum $(1 + z)^{-x}$ dividendo Unitatem per primam, scilicet, quod eodem recidit, Infinitinomium quaerendo adeo comparatum, ut Factum eius in primum nunquam non Unitati par fiat. Verum, si Exponentialem Progressionem $1 + \frac{\Delta^1 x^1}{1} + \frac{\Delta^2 x^2}{1.2}$

$+ \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^4 x^4}{1.2.3.4} + \frac{\Delta^5 x^5}{1.2.3.4.5} + \frac{\Delta^6 x^6}{1.2.3.4.5.6} + \&c.$ per aliam multipli-

ces $1 - \frac{\Delta^1 x^1}{1} + \frac{\Delta^2 x^2}{1.2} - \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^4 x^4}{1.2.3.4} - \frac{\Delta^5 x^5}{1.2.3.4.5} + \frac{\Delta^6 x^6}{1.2.3.4.5.6} -$

$\&c.$, facile obtinebis

$$\begin{aligned}
 1 &+ \frac{\Delta^1 x^1}{1} + \frac{\Delta^2 x^2}{1.2} + \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^4 x^4}{1.2.3.4} + \frac{\Delta^5 x^5}{1.2.3.4.5} + \frac{\Delta^6 x^6}{1.2.3.4.5.6} + \&c: = 1. \\
 &- \frac{\Delta^1 x^1}{1} - \frac{\Delta^2 x^2}{1} - \frac{\Delta^3 x^3}{1.2} - \frac{\Delta^4 x^4}{1.2.3} - \frac{\Delta^5 x^5}{1.2.3.4} - \frac{\Delta^6 x^6}{1.2.3.4.5} + \&c: \\
 &+ \frac{\Delta^2 x^2}{1.2} + \frac{\Delta^3 x^3}{1.2} + \frac{\Delta^4 x^4}{1.4} + \frac{\Delta^5 x^5}{1.3.4} + \frac{\Delta^6 x^6}{1.2.4.6} + \&c: \\
 &- \frac{\Delta^3 x^3}{1.2.3} - \frac{\Delta^4 x^4}{1.2.3} - \frac{\Delta^5 x^5}{1.3.4} - \frac{\Delta^6 x^6}{1.6.6} + \&c: \\
 &+ \frac{\Delta^4 x^4}{1.2.3.4} + \frac{\Delta^5 x^5}{1.2.3.4} + \frac{\Delta^6 x^6}{1.2.4.6} + \&c: \\
 &- \frac{\Delta^5 x^5}{1.2.3.4.5} - \frac{\Delta^6 x^6}{1.2.3.4.5} + \&c: \\
 &+ \frac{\Delta^6 x^6}{1.2.3.4.5.6} + \&c:
 \end{aligned}$$

Subintelligas velim, ut parcius de hisce agam quam supra rei novitas tule-
rit, τὸ z nunc praeferre omnes Numeros positivos, ac negativos, ideoque
Functionem Δ unius formae Analyticae in Γ verti, et vicissim, ob univer-
salitatem compendio Geometris sat familiari inconcussam servandam.

76. Perbella equidem, et commentatione digna Infinitinomialum $1 +$
 $\frac{\Delta^1 x^1}{1} + \frac{\Delta^2 x^2}{1.2} + \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^4 x^4}{1.2.3.4} + \frac{\Delta^5 x^5}{1.2.3.4.5} + \frac{\Delta^6 x^6}{1.2.3.4.5.6} + \&c:$, nec

non $1 - \frac{\Delta^1 x^1}{1} + \frac{\Delta^2 x^2}{1.2} - \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^4 x^4}{1.2.3.4} - \frac{\Delta^5 x^5}{1.2.3.4.5} + \frac{\Delta^6 x^6}{1.2.3.4.5.6} -$

$\&c:$ singularis affectio, Productum videlicet unius in alterum Unitatem reddere,
sive Quotum Unitatis per alterutrum divisae Infinitinomialium pro Divisore
non sumptum restituere. Quam societatem mutuae earum Formularum reprodu-
ctionis capacem ne putes tantummodo existere dum Δ sit Functio formam habens

hactenus usitatam $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \&c:$, vel

$\left(\frac{z^1}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} + \frac{z^6}{6} + \&c: \right)$, quinimmo ipsius po-

moeria ita latissime amplificantur, ut si rei periculum facias in qualibet alia

Functio-

Functione variabilium aut constantium, Algebraica seu Transcendente, author tibi sum semper experturum eiusdem Theorematis veritatem, propterea quod a terminorum unica Lege, non a Functionum indole pendeat. Posita enim loco Δx quacumque diversa Functione Π demonstratio generalissima in praecedente num^o. adhibita suadebit fore

$$\left(1 + \frac{\Pi^1}{1} + \frac{\Pi^2}{1.2} + \frac{\Pi^3}{1.2.3} + \frac{\Pi^4}{1.2.3.4} + \frac{\Pi^5}{1.2.3.4.5} + \frac{\Pi^6}{1.2.3.4.5.6} \&c.\right) \left(1 - \frac{\Pi^1}{1} + \frac{\Pi^2}{1.2} - \frac{\Pi^3}{1.2.3} + \frac{\Pi^4}{1.2.3.4} - \frac{\Pi^5}{1.2.3.4.5} + \frac{\Pi^6}{1.2.3.4.5.6} \&c.\right) = 1, \text{ quod idem est atque adserere}$$

$$\frac{1}{1 + \frac{\Pi^1}{1} + \frac{\Pi^2}{1.2} + \frac{\Pi^3}{1.2.3} + \frac{\Pi^4}{1.2.3.4} + \frac{\Pi^5}{1.2.3.4.5} + \frac{\Pi^6}{1.2.3.4.5.6} \&c.} = 1 - \frac{\Pi^1}{1} + \frac{\Pi^2}{1.2} - \frac{\Pi^3}{1.2.3} + \frac{\Pi^4}{1.2.3.4} - \frac{\Pi^5}{1.2.3.4.5} + \frac{\Pi^6}{1.2.3.4.5.6} \&c., \text{ ac}$$

viceversa. Istud ne incassum mireris, quum satis pateat fons et origo huiusce Infinitinomiorum elegantissimae conversionis. Primum Infinitinomialium namque suppeditat evolutio directa Neutroniani Binomii $\left(1 + \frac{\Delta x}{\infty}\right)^\infty$, vel ge-

neraliter $\left(1 + \frac{\Pi}{\infty}\right)^\infty$, et alterum $\tau_z \left(1 - \frac{\Delta x}{\infty}\right)^\infty$, sive $\left(1 - \frac{\Pi}{\infty}\right)^\infty$. Alge-

brae autem communes, seu *topici* canones docent esse $\left(1 + \frac{\Pi}{\infty}\right)^\infty \times \left(1 - \frac{\Pi}{\infty}\right)^\infty$

$$= \left(1 - \frac{\Pi^2}{\infty^2}\right)^\infty = 1 - \frac{\Pi^2}{1.\infty} + \frac{\Pi^4}{1.2.\infty^2} - \frac{\Pi^6}{1.2.3.\infty^3} + \frac{\Pi^8}{1.2.3.4.\infty^4} -$$

$\frac{\Pi^{10}}{1.2.3.4.5.\infty^5} + \&c. = 1$, quemadmodum novis sine fucis, at sua veluti sponte elucescit.

77. Antequam manum de tabula retraham non ingratis Lectoribus fore confido dum sciant Theorice ipsam derivari etiam a praeostenso memorati Infinitinomii charactere, cuius ergo per numerum 62. aequalitas constat inter

$$\left(1 + \frac{\Pi^1}{1} + \frac{\Pi^2}{1.2} + \frac{\Pi^3}{1.2.3} + \frac{\Pi^4}{1.2.3.4} + \frac{\Pi^5}{1.2.3.4.5} + \frac{\Pi^6}{1.2.3.4.5.6} \&c.\right)$$

$$\frac{\Pi^6}{1.2.3.4.5.6} + \&c: \Big)^{-1}, \text{ et } 1 - \frac{\Pi^1}{1} + \frac{\Pi^2}{1.2} - \frac{\Pi^3}{1.2.3} + \frac{\Pi^4}{1.2.3.4} - \frac{\Pi^5}{1.2.3.4.5} + \frac{\Pi^6}{1.2.3.4.5.6} - \&c:, \text{ substituendo nimirum } -1 \text{ vice } x \text{ Indicis}$$

universalissime appositi. Nec immoror frustra monens terminorum Seriei omni-
genae Exponentialis convergentiam, de qua mentio facta in num°. 56. re-
spectu Potestatis $(1+z)^x$, repeti quoque, et valere eadem demonst ratio-
ne manente quoad ceteras Potestates $(1-z)^x$, et $(1 \pm z)^{-x}$. Summa
attamen cura perlustrandum erit utillimum iis, quae sequentur, Theorema
perpetuae Aequationis, aut potius *conversionis* Exponentialis $\tau \approx (1 \pm z)^x$,

vel generalius v^x in $\frac{C^{(\Delta \text{feu} \Gamma)x}}{\Delta \text{feu} \Gamma}$, hypothesi facta x positivi seu negativi, ac Si-

$$\text{glae } (\Delta \text{feu} \Gamma) \text{ variis in casibus designantis } \frac{\overline{v-1}^1}{1} - \frac{\overline{v-1}^2}{2} + \frac{\overline{v-1}^3}{3} - \frac{\overline{v-1}^4}{4} + \frac{\overline{v-1}^5}{5} - \frac{\overline{v-1}^6}{6} \&c:, \text{ five } - \left(\frac{1-v^1}{1} + \frac{1-v^2}{2} + \frac{1-v^3}{3} + \frac{1-v^4}{4} + \frac{1-v^5}{5} + \frac{1-v^6}{6} \&c: \right).$$

De valore C nihil addo, cum aperte ille
constet numerum 61. cuilibet perlegenti. Revera $C^{(\Delta \text{feu} \Gamma)x} = 1^x \times 1^{\Delta \text{feu} \Gamma} \times$
 $\left(1 + \frac{(\Delta \text{feu} \Gamma)^1 x^1}{1} + \frac{(\Delta \text{feu} \Gamma)^2 x^2}{1.2} + \frac{(\Delta \text{feu} \Gamma)^3 x^3}{1.2.3} + \frac{(\Delta \text{feu} \Gamma)^4 x^4}{1.2.3.4} + \frac{(\Delta \text{feu} \Gamma)^5 x^5}{1.2.3.4.5} + \frac{(\Delta \text{feu} \Gamma)^6 x^6}{1.2.3.4.5.6} \&c: \right) = 1^{\Delta \text{feu} \Gamma} \cdot v^x$ ex iam demonstratis.

78. Multas hinc adipiscimur Series numero pariter terminorum infini-
tas si eruta Exponentialia quoquomodo summantes, vel subtrahentes vario or-

$$\text{dine disponamus. Ita elicietur } \frac{z^x + z^{-x}}{2} = 1 + \frac{\Delta^2 x^2}{1.2} + \frac{\Delta^4 x^4}{1.2.3.4} + \frac{\Delta^6 x^6}{1.2.3.4.5.6} + \frac{\Delta^8 x^8}{1.2.3.4.5.6.7.8} + \frac{\Delta^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c:, \text{ necnon eadem pe-}$$

nitus ratione Progressiones aliae Infinitinomiales originem ducent facillimam

$$\frac{z^x - z^{-x}}{2}$$

$$\begin{aligned}
\frac{z^x - z^{-x}}{2} &= \frac{\Delta^1 x^1}{1} + \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^5 x^5}{1.2.3.4.5} + \frac{\Delta^7 x^7}{1.2.3.4.5.6.7} + \\
&\frac{\Delta^9 x^9}{1.2.3.4.5.6.7.8.9} + \&c.; \text{ atque ex opposito fluet } \frac{z^{-x} - z^x}{2} = -\frac{\Delta^1 x^1}{1} - \frac{\Delta^3 x^3}{1.2.3} \\
&- \frac{\Delta^5 x^5}{1.2.3.4.5} - \frac{\Delta^7 x^7}{1.2.3.4.5.6.7} - \frac{\Delta^9 x^9}{1.2.3.4.5.6.7.8.9} \&c.; \text{ Sigla } \Delta \text{ iuxta morem si-} \\
\text{gnificante nunc } &\frac{\overline{z-1}^1}{1} - \frac{\overline{z-1}^2}{2} + \frac{\overline{z-1}^3}{3} - \frac{\overline{z-1}^4}{4} + \frac{\overline{z-1}^5}{5} - \\
&\frac{\overline{z-1}^6}{6} \&c.; \text{ nunc } - \left(\frac{\overline{1-z}^1}{1} + \frac{\overline{1-z}^2}{2} + \frac{\overline{1-z}^3}{3} + \frac{\overline{1-z}^4}{4} + \frac{\overline{1-z}^5}{5} \right. \\
&\left. + \frac{\overline{1-z}^6}{6} \&c. \right). \text{ Consimili iure invenientur } \frac{C^x + C^{-x}}{2} = 1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} \\
&+ \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.; \text{ insuper } \frac{C^x - C^{-x}}{2} \\
&= \frac{x^1}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^9}{1.2.3.4.5.6.7.8.9} \&c.; \text{ atque} \\
&\frac{C^{-x} - C^x}{2} = - \left(\frac{x^1}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \frac{x^7}{1.2.3.4.5.6.7} + \right. \\
&\left. \frac{x^9}{1.2.3.4.5.6.7.8.9} \&c. \right). \text{ Hae Series, in quibus evidenter apparent Potestates}
\end{aligned}$$

Indicis x , Functionisque Δ , ubi adsint, alternis tantum Exponentibus prae-ditae paribus vel imparibus, saepius infra recurrent, maximoque erunt usui in Trigonometricae praesertim doctrinae genesi patefacienda. Commodo in-serviens, cum de postremis Formulis agerem, omisi Factorem 1^x , quem nun-quam ab Exponentialibus dissociatum, tametsi quandoque silentio praeteri-tum, una vice solummodo liceat advertisse.

79. Neminem arbitror, Radices retroactarum Serierum investigantem, dubium haerere: namque superius explicita docent esse $C^x + C^{-x} = \left(1 + \frac{x}{\infty}\right)^\infty + \left(1 - \frac{x}{\infty}\right)^\infty$, $C^x - C^{-x} = \left(1 + \frac{x}{\infty}\right)^\infty - \left(1 - \frac{x}{\infty}\right)^\infty$, $C^{-x} - C^x = \left(1 - \frac{x}{\infty}\right)^\infty - \left(1 + \frac{x}{\infty}\right)^\infty$, $z^x + z^{-x} = \left(1 + \frac{\Delta x}{\infty}\right)^\infty +$
(1 —

$\left(1 - \frac{\Delta x}{\infty}\right)^{\infty}$, $z^x - z^{-x} = \left(1 + \frac{\Delta x}{\infty}\right)^{\infty} - \left(1 - \frac{\Delta x}{\infty}\right)^{\infty}$, et tandem $z^{-x} - z^x = \left(1 - \frac{\Delta x}{\infty}\right)^{\infty} - \left(1 + \frac{\Delta x}{\infty}\right)^{\infty}$; adeo ut Radicum quae-
 fitarum inventio quoad Series Infinitas supra singillatim prolatae a postero-
 rum Binomiorum, formae $a^n \pm z^n$, resolutione pendeat, de qua, vel suppo-
 sito Indice $n = \infty$, in Capite VII. fusius differere occasio feret. Sed pe-
 culiari etiam methodo Radices eas in Capite V. exhibebo.

80. Qui secum animo reputaverit Formulas $1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.$, atque $\frac{x^1}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^9}{1.2.3.4.5.6.7.8.9} \&c.$, five universalissimas $1 + \frac{\Delta^2 x^2}{1.2} + \frac{\Delta^4 x^4}{1.2.3.4} + \frac{\Delta^6 x^6}{1.2.3.4.5.6} + \frac{\Delta^8 x^8}{1.2.3.4.5.6.7.8} + \frac{\Delta^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.$, ac $\frac{\Delta^1 x^1}{1} + \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^5 x^5}{1.2.3.4.5} + \frac{\Delta^7 x^7}{1.2.3.4.5.6.7} + \frac{\Delta^9 x^9}{1.2.3.4.5.6.7.8.9}$, et earum ge-
 nesis memor eas invicem comparaverit lucrabitur iucundum sane Theorema

„*Differentiam nempe Quadrati prioris Series limite carentis supra Quadra-
 tum alterius constantem fieri* „, videlicet Unitatem adaequare, quilibet valor
 variabili x , aut simul Functioni Δ tribuatur. Primae etenim Series secunda

Potestas, seu $\left(1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.\right)^2 = \tau \left(\frac{x^1}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^9}{1.2.3.4.5.6.7.8.9} \&c.\right)^2 = \frac{C^{2x} + 2 + C^{-2x} - C^{2x} + 2 - C^{-2x}}{4} = 1$; ipsa-

que methodo repetita $\left(1 + \frac{\Delta^2 x^2}{1.2} + \frac{\Delta^4 x^4}{1.2.3.4} + \frac{\Delta^6 x^6}{1.2.3.4.5.6} + \frac{\Delta^8 x^8}{1.2.3.4.5.6.7.8} + \frac{\Delta^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.\right)^2 = \left(\frac{\Delta^1 x^1}{1} + \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^5 x^5}{1.2.3.4.5} + \frac{\Delta^7 x^7}{1.2.3.4.5.6.7} + \frac{\Delta^9 x^9}{1.2.3.4.5.6.7.8.9} \&c.\right)^2 =$

O

 $\frac{2^{2x} + 1}{2}$

$$\frac{z^{2x} + 2 + z^{-2x} - z^{2x} + 2 - z^{-2x}}{4} = 1, \text{ veluti innuimus.}$$

81. Nec minus eadem Infinitinomialium peculiaris affectio confirmaretur usitatae ope Formulae Neutronianae ipsamet exaltando ad Quadratum. Est

$$\begin{aligned} \text{etenim } \left(1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} + \right. \\ \left. \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c. \right)^2 = \\ 1 + \frac{x^2}{1} + \frac{x^4}{4} + \frac{x^6}{2.3.4} + \frac{x^8}{4.9.16} + \frac{x^{10}}{2.9.16.5.6} + \frac{x^{12}}{4.9.16.25.36} \&c.; \\ + \frac{x^4}{3.4} + \frac{x^6}{3.4.5.6} + \frac{x^8}{2.3.4.5.6} + \frac{x^{10}}{2.3.4.5.6.7.8} + \&c.; \\ + \frac{x^8}{3.4.5.6.7.8} + \frac{x^{10}}{3.4.5.6.7.8.9.10} + \&c.; \end{aligned}$$

a qua expressione si dematur altera aequalis Quadrato secundae Seriei, nimirum ex tritis iam regulis

$$\begin{aligned} \frac{x^2}{3} + \frac{x^4}{3} + \frac{x^6}{4.9} + \frac{x^8}{2.9.4.5} + \frac{x^{10}}{4.9.16.25} + \&c.; \\ + \frac{x^6}{3.4.5} + \frac{x^8}{3.4.5.6.7} + \frac{x^{10}}{2.9.4.5.6.7} + \&c.; \\ + \frac{x^{10}}{3.4.5.6.7.8.9} + \&c. \end{aligned}$$

Residuum manebit tantummodo 1, universis se mutuo destruentibus terminis utriusque Progressionis affectis ab homologis $\tau\theta$ x Potestatibus. Hoc evidentissimum cum sit uberioris demonstrationis non eget. Similia similibus applicari possent Seriebus superius deductis, proptereaquod intima rei natura ab enucleatis non discrepent, sed solum vice x eiusque Potestatum, ordine ipso servato, habeant Δx , parique gradu istius Δx Potestates.

82. Illa vero omnia analytico oculi ictu intuenti constabit de mirabili

Analogia Formulas inter expositas $\frac{C^x + C^{-x}}{2}$, et $\frac{C^x - C^{-x}}{2}$, eademque ratione $\frac{z^x + z^{-x}}{2}$, ac $\frac{z^x - z^{-x}}{2}$, atque Coordinatas Hyperbolae Apollonianae, cui Aequilaterae nomen imposuerunt Geometrae. Profecto ignorari nequit Localis

lis huiusce Curvae proprietas, quae statuit constantem, quippe aequalem Quadrato Semi-Laterum Versi, Recti, seu Coniugati, differentiam inter Quadratum Abscissae cuiuslibet a Centro super Diametrum in Angulo Asymptotico iacentem computatae, et Quadratum Ordinatum-Semiapplicatae ad ipsam Diametrum, quod idem nuper contineri observavimus tam in $\left(\frac{C^x + C^{-x}}{2}\right) - \left(\frac{C^x - C^{-x}}{2}\right)^2$, quam in $\left(\frac{z^x + z^{-x}}{2}\right)^2 - \left(\frac{z^x - z^{-x}}{2}\right)^2$. Analogiae Hyperbolico-Exponentialis hic iacta semina felices admodum fructus edent statim ac filo eodem nusquam interrupto ducente universa detegere, novoque nitore perpolire infra licebit, quae doctrina praestantes viri Rogerus Cotesius, et Vincentius Riccatus in Rationum atque Angulorum Harmonice, nec non in Cosinum, Sinuumque Hyperbolicorum Scientia Pythagoricae Numerorum voluptatis quasi participes confecere.

83. Communio praenotatae affectionis Formulas Exponentiales Quadraticas respicientis $\left(\frac{C^x + C^{-x}}{2}\right)^2$, $\left(\frac{C^x - C^{-x}}{2}\right)^2$ aequae ac $\left(\frac{z^x + z^{-x}}{2}\right)^2$, et $\left(\frac{z^x - z^{-x}}{2}\right)^2$ nullam excitare debet insolitam curiositatem post mirifici illius Numeri C ostensam vim, atque potentiam, ob quam fit, ut eadem ratione tractetur $(z)^x$ perinde uti foret $(C)^{\Delta x}$, et insuper z^{-x} quemadmodum $C^{-\Delta x}$; ita ut variabili x , ceteris omnibus iisdem manentibus, in variabilem Δx immutata ipsi adamussim theoriae z^x eiusque derivata subiiciantur. Frustra etiam mentis aciem intenderemus de novis Seriebus vaticinium facturi, quae a Summatione Quadratorum praecedentium Exponentialium originem ducere possent, scilicet $\left(\frac{C^x + C^{-x}}{2}\right)^2 + \left(\frac{C^x - C^{-x}}{2}\right)^2$. Confectis enim

Quadratis nanciscimur $\frac{C^{2x} + C^{-2x}}{2}$, nimirum $1 + \frac{(2x)^2}{1.2} + \frac{(2x)^4}{1.2.3.4} + \frac{(2x)^6}{1.2.3.4.5.6} + \frac{(2x)^8}{1.2.3.4.5.6.7.8} + \frac{(2x)^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.$, quae Series nihil differt a Progressionibus supra evolutis, nec potest quidem differre, quum Quadratorum eorumdem Summa ab hisce pendeat singulis elementis $1 + x^2 + \frac{x^4}{3} +$

$\frac{2x^6}{45} + \frac{x^8}{315} + \&c.$, nec non $x^2 + \frac{x^4}{3} + \frac{2x^6}{45} + \frac{x^8}{315} + \&c.$, quibus

ex Infinitinomiis in unum coalescentibus expressio prius tradita existit.

84. Sed huc illuc percurrens, omniaque libans aut in nugas inciderem, aut taedii, non succi, plenam orationem longius, quam par esset, protrahere consulto viderer. Sit satis novisse, non solum Exponentialium magnitudinum Facultatem cum Potestatum Calculo amice coniurare, quinimmo intimius cum ipso confundi, iniuriaque usque adhuc seiunctam, perinde ac si usitati Algorithmi experts, tenuisse Analyseos Scriptores. Antiquo quasi diaphragmate iam interciso ea, quae sequentur, lacunas perpaucae a Logometriae doctrina perficiendas complebunt, universamque penitus exhaurient desideratam Theoricen, veneresque, et flores ipsos cursim, ac leviter ob oculos ponent.



C A P U T III.

EVOLUTIO SERIERUM QUAE NUMERORUM LOGARITHMOS

E X P R I M U N T.

85.  Uod erat in votis circa Series numero terminorum Infinitas, quae Coefficientes evolutarum Exponentialium Magnitudinum constituunt, Siglis Δ, Γ vel aliis indeterminationis Symbolis obiter inscitiae tunc caussa notatos, ad exitum ducere aggredimur. Latam regiamque viam praeteritae investigationes huic tractationi paravere: tam arcto etenim affinitatis vinculo Exponentialia atque Logarithmi continentur, ut mutuam sibi, etiam invito Geometra, lucem praestent, difficultatesve invicem solvant. Haec sane ipsa subiecti argumenti, in quo sumus, affinitas inevitabiles quasdam pariet Formularum, Regularumque repetitiones, quas non mihi imputandas autumo; nam peculiari rei Logometrico-Exponentialis naturae debentur, et, pro virili mea parte, limando, perpoliendo, amplificandoque, novitatem aliquam repetitis impertiri, aut saltem ea novis coloribus exornare curabo.

86. Primum autem vellem altius tyronum animo infixum, Potestates gemina variabili affectas, uti foret z^x , Ordinatum-Applicatas Loci *Supersolidi* ad Superficiem Aequatione trium variabilium $z^x = y$ definitam significare. Nec una ex variabilibus eliminabitur artificio superius allato transformationis z^x in $C^{(\Delta \text{ seu } \Gamma)x}$: namque ob $(\Delta \text{ seu } \Gamma)$ Functionem variabilis z , quemadmodum toties perspeximus, nova Aequatio $C^{(\Delta \text{ seu } \Gamma)x} = y$ triplicem nihiloscus variabilem instar prioris complecteretur. Nisi itaque z fuerit Numerus constans, quem tunc Analystae vocant *Basin*, videlicet, nisi Aequatio $z^x = y$ simpliciore formam induerit $a^x = y$, tertia variabilis nullo aliter modo auferetur, Aequatio ad Superficiem in Aequationem ad Curvam unius curvaturae non immutabitur, nec duarum variabilium, quae superessent, x et y relatio obtineri unquam poterit ita comparata, ut, dum prima illarum Arithmetice procedat, altera evidentissime Geometricam Progressionem sequatur. Erit igitur universaliter x Logarithmus, et y Numerus ipsi respondens; ideo-

ideoque geminae Seriei Numericae vinculi Logarithmos, ac Numeros five potius Antilogarithmos exhibentis caussa et origo oecumenica quaerenda in generalissima $z^x = y$ Exponentiali Aequatione. Sicuti vero, quomodocunque z fiat constans, nil vetat quominus ad nutum Geometrae infinite varios induere possit singillatim valores, inde oritur innumerabilium Systematum Logometricorum diversitas, non equidem substantialis, quum omnia comprehendantur ab unica expressione Transcendente $z^x = y$, sed ad Basis z aut in variationem habita singulis in casibus proportionem. Ea Basis Logarithmo gaudet 1, vel in latiori etiam significatione statuit Systematis *Protonumerum* pro Logarithmo.

87. Dum Potestatem indefinitam $(1 \pm z)^x$, cui sub Indice x negativi et positivi omnes adsunt Exponentes, evolvere occasio fuit vidimus, ac praefertim num^o. 57., quod, posito Numero $1 \pm z = 1 + \frac{y^1}{1} + \frac{y^2}{1.2} +$

$$+ \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} + \frac{y^5}{1.2.3.4.5} + \frac{y^6}{1.2.3.4.5.6} + \frac{y^7}{1.2.3.4.5.6.7} + \frac{y^8}{1.2.3.4.5.6.7.8} \\ + \frac{y^9}{1.2.3.4.5.6.7.8.9} + \frac{y^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c., \text{ extat simul Aequatio inver-}$$

$$\text{sa } y = \pm \frac{z^1}{1} \pm \frac{z^2}{2} \pm \frac{z^3}{3} \pm \frac{z^4}{4} \pm \frac{z^5}{5} \pm \frac{z^6}{6} \pm \frac{z^7}{7} \pm \frac{z^8}{8} \pm \frac{z^9}{9} \pm \frac{z^{10}}{10} \pm \&c., \text{ nec non universalissime ab Aequatione } (1 \pm z)^x = 1$$

$$+ \frac{y^1 x^1}{1} + \frac{y^2 x^2}{1.2} + \frac{y^3 x^3}{1.2.3} + \frac{y^4 x^4}{1.2.3.4} + \frac{y^5 x^5}{1.2.3.4.5} + \frac{y^6 x^6}{1.2.3.4.5.6} + \frac{y^7 x^7}{1.2.3.4.5.6.7} + \frac{y^8 x^8}{1.2.3.4.5.6.7.8} + \frac{y^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{y^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c. \text{ simili-}$$

$$\text{ter derivatur } yx = x \left(\pm \frac{z^1}{1} \pm \frac{z^2}{2} \pm \frac{z^3}{3} \pm \frac{z^4}{4} \pm \frac{z^5}{5} \pm \frac{z^6}{6} \pm \frac{z^7}{7} \pm \frac{z^8}{8} \pm \frac{z^9}{9} \pm \frac{z^{10}}{10} \pm \&c. \right), \text{ quae, ob Factorem } x \text{ communem, antea ex-}$$

$$\text{positae congruit. Sit itaque } 1 + \frac{y^1}{1} + \frac{y^2}{1.2} + \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} +$$

$$+ \frac{y^5}{1.2.3.4.5} + \frac{y^6}{1.2.3.4.5.6} + \frac{y^7}{1.2.3.4.5.6.7} + \frac{y^8}{1.2.3.4.5.6.7.8} + \frac{y^9}{1.2.3.4.5.6.7.8.9} +$$

$$+ \frac{y^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.$$

$\frac{y^{10}}{1.2.3.4.5.6.7.8.9.10}$ &c.; ex praeostensis speciatim in num°. 61. ac sequentibus,

$= C^y = 1 \pm z$; tum evadet, ut supra, $1 \pm z$ Numerus, atque y vel Series

Infinita $\pm \frac{z^1}{1} - \frac{z^2}{2} \pm \frac{z^3}{3} - \frac{z^4}{4} \pm \frac{z^5}{5} - \frac{z^6}{6} \pm \frac{z^7}{7} - \frac{z^8}{8} \pm \frac{z^9}{9} - \frac{z^{10}}{10} \pm \&c.$ illius Logarithmus, idemque confirmatur animadvertendo,

quod, selecto ac determinato valore z ad nutum Analystrarum, $\tau\delta (1 \pm z)^x$ procedat Geometrice dum $x \left(\pm \frac{z^1}{1} - \frac{z^2}{2} \pm \frac{z^3}{3} - \frac{z^4}{4} \pm \frac{z^5}{5} - \frac{z^6}{6} \pm \frac{z^7}{7} - \frac{z^8}{8} \pm \frac{z^9}{9} - \frac{z^{10}}{10} \pm \&c. \right)$ in Arithmetica fit Progreſſione.

Summo itaque iure Halleii Formula legitur in *Philosophicis Londinensibus Actis*, qua Logarithmo praecognito L datae Rationis $\frac{a}{b}$, cuius $a < b$, fit

$$a = b \left(1 - \frac{L^1}{1} + \frac{L^2}{1.2} - \frac{L^3}{1.2.3} + \frac{L^4}{1.2.3.4} - \frac{L^5}{1.2.3.4.5} + \frac{L^6}{1.2.3.4.5.6} - \&c. \right), \text{ et } b = a \left(1 + \frac{L^1}{1} + \frac{L^2}{1.2} + \frac{L^3}{1.2.3} + \frac{L^4}{1.2.3.4} + \frac{L^5}{1.2.3.4.5} + \frac{L^6}{1.2.3.4.5.6} + \&c. \right),$$

quippe quum, posita specie $L = \pm y$, fit

$$\text{Numerus Unitate maior } \frac{b}{a} = 1 + z = 1 + \frac{y^1}{1} + \frac{y^2}{1.2} + \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} + \frac{y^5}{1.2.3.4.5} + \frac{y^6}{1.2.3.4.5.6} + \&c.,$$

$$\text{ideoque } b = a \left(1 + \frac{y^1}{1} + \frac{y^2}{1.2} + \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} + \frac{y^5}{1.2.3.4.5} + \frac{y^6}{1.2.3.4.5.6} + \&c. \right), \text{ nec non Numerus Unitate minor } \frac{a}{b} = 1 - z = 1 - \frac{y^1}{1} + \frac{y^2}{1.2} - \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} - \frac{y^5}{1.2.3.4.5} + \frac{y^6}{1.2.3.4.5.6} - \&c.,$$

$$\text{nimirum } a = b \left(1 - \frac{y^1}{1} + \frac{y^2}{1.2} - \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} - \frac{y^5}{1.2.3.4.5} + \frac{y^6}{1.2.3.4.5.6} - \&c. \right),$$

$$\text{adeo ut eodem colliment expressiones}$$

pressiones

pressiones illae prima tantopere fronte dissimiles. Functio y Magnitudinis z Infinitinomia Geometrico-Harmonica sine ullo discrimine convenit cum ea, tunc ignota, quam in Coefficientibus universis Formulae Exponentialis iam evolutae saepius sumus contemplati, et in praesentiarum nihil aliud esse, nisi Logarithmum Numeri $1 \pm z$, summa facilitate invenisse delectamur. Eandem Transcendentem Functionem cum Logarithmo confundi Numeri variabilis $1 \pm z$ confirmat praeterea diverso tramite altera Aequalitas, superius in nu-

$$\pm \frac{z^1}{1} - \frac{z^2}{2} \pm \frac{z^3}{3} - \frac{z^4}{4} \pm \frac{z^5}{5} - \frac{z^6}{6} \pm \frac{z^7}{7} - \frac{z^8}{8} \pm \frac{z^9}{9} - \frac{z^{10}}{10} \pm \&c:$$

, ex qua luculenter apparet $1 \pm z$ Geometrice progredi simul ac Index Infinitinomialis, sive Exponens Basis Logometricae C, processerit Arithmetice. Universa autem haec idem sonant perinde ac si diceret Indicem ipsum τ Numeri $1 \pm z$ Logarithmum.

88. Placuit Geometris Logarithmum, aut Functionem istam, quae vires

$$\text{superat Algorithmi Algebraici, } \pm \frac{z^1}{1} - \frac{z^2}{2} \pm \frac{z^3}{3} - \frac{z^4}{4} \pm \frac{z^5}{5} - \frac{z^6}{6} \pm \frac{z^7}{7} - \frac{z^8}{8} \pm \frac{z^9}{9} - \frac{z^{10}}{10} \pm \&c: \text{ in Infinitum, aut, positis A, B,}$$

C, D, E, F, G, H, K &c: valoribus terminorum proximo loco praecedentium, $\pm$

$$\frac{z}{1} \mp \frac{Az}{2} \mp \frac{2Bz}{3} \mp \frac{3Cz}{4} \mp \frac{4Dz}{5} \mp \frac{5Ez}{6} \mp \frac{6Fz}{7} \mp \frac{7Gz}{8} \mp \frac{8Hz}{9} \mp$$

$$\frac{9Kz}{10} \mp \&c:, \text{ respectu Numeri } 1 \pm z, \text{ vel unisonam potius Expressionem } +$$

$$\frac{\sqrt{v-1}^1}{1} - \frac{\sqrt{v-1}^2}{2} + \frac{\sqrt{v-1}^3}{3} - \frac{\sqrt{v-1}^4}{4} + \frac{\sqrt{v-1}^5}{5} - \frac{\sqrt{v-1}^6}{6} + \frac{\sqrt{v-1}^7}{7} -$$

$$\frac{\sqrt{v-1}^8}{8} + \frac{\sqrt{v-1}^9}{9} - \frac{\sqrt{v-1}^{10}}{10} + \&c: \text{ relatam ad Numerum } v, \text{ Sigla}$$

L. initiali universe notare. Quo facto non ambigetur de omnimoda verita-

$$\text{te Aequationis } C^{Lx} = x, \text{ aut latius } C^{L\Phi(x, y, z, v \&c:)} = \Phi(x, y, z, v \&c:).$$

Detecta

Detecta insuper Aequalitas $C^y = 1 + \frac{y^1}{1} + \frac{y^2}{1.2} + \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} +$
 $\frac{y^5}{1.2.3.4.5} + \frac{y^6}{1.2.3.4.5.6} + \frac{y^7}{1.2.3.4.5.6.7} + \frac{y^8}{1.2.3.4.5.6.7.8} + \frac{y^9}{1.2.3.4.5.6.7.8.9} +$
 $\frac{y^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.$, eadem cum altera $\left(\frac{1}{C}\right)^y = 1 - \frac{y^1}{1} + \frac{y^2}{1.2} -$
 $\frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} - \frac{y^5}{1.2.3.4.5} + \frac{y^6}{1.2.3.4.5.6} - \frac{y^7}{1.2.3.4.5.6.7} +$
 $\frac{y^8}{1.2.3.4.5.6.7.8} - \frac{y^9}{1.2.3.4.5.6.7.8.9} + \frac{y^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c.$ quia $= C^{-y}$.

quod sane memoratu dignum, parit elegantissimam, ac contortis methodis
 saepenumero a nuperrimis quoque Scriptoribus enucleatam $C^{Lx} = 1 +$

$$\frac{(Lx)^1}{1} + \frac{(Lx)^2}{1.2} + \frac{(Lx)^3}{1.2.3} + \frac{(Lx)^4}{1.2.3.4} + \frac{(Lx)^5}{1.2.3.4.5} + \frac{(Lx)^6}{1.2.3.4.5.6} +$$

$$\frac{(Lx)^7}{1.2.3.4.5.6.7} + \frac{(Lx)^8}{1.2.3.4.5.6.7.8} + \frac{(Lx)^9}{1.2.3.4.5.6.7.8.9} + \frac{(Lx)^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.$$

$= x$; eaque etiam fluit ab indole Formulae $Lx = + \frac{\overline{x-1}^1}{1} - \frac{\overline{x-1}^2}{2}$
 $+ \frac{\overline{x-1}^3}{3} - \frac{\overline{x-1}^4}{4} + \frac{\overline{x-1}^5}{5} - \frac{\overline{x-1}^6}{6} + \frac{\overline{x-1}^7}{7} - \frac{\overline{x-1}^8}{8} +$
 $\frac{\overline{x-1}^9}{9} - \frac{\overline{x-1}^{10}}{10} + \&c.$ Nam, si haec postrema Functio, eiusque Potestates in

exposito Infinitinomio substituantur, destructi, uti iam alias dictum, praeter
 & omnes termini facillime comprobant Aequationem conclusam. Hinc vice

usitatae Aequalitatis $C^x = 1 + \frac{x^1}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} +$
 $\frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^9}{1.2.3.4.5.6.7.8.9} +$
 $\frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.$ scribi semper poterit altera substantialiter identica,

utcunque diversimode expressa, $C^x = 1 + \frac{L(C^x)^1}{1} + \frac{L(C^x)^2}{1.2} + \frac{L(C^x)^3}{1.2.3} +$
 P

$$\begin{aligned}
& -\frac{z^1}{1} + \frac{z^2}{1} - \frac{z^3}{1} + \frac{z^4}{1} - \frac{z^5}{1} + \frac{z^6}{1} \&c: = -\frac{z^1}{1} + \frac{z^2}{2} - \frac{z^3}{3} + \frac{z^4}{4} - \frac{z^5}{5} + \frac{z^6}{6} \&c:, \\
& -\frac{z^2}{2} + \frac{2z^3}{2} - \frac{3z^4}{2} + \frac{4z^5}{2} - \frac{5z^6}{2} \&c: \\
& -\frac{z^3}{3} + \frac{3z^4}{3} - \frac{6z^5}{3} + \frac{10z^6}{3} \&c: \\
& -\frac{z^4}{4} + \frac{4z^5}{4} - \frac{10z^6}{4} \&c: \\
& -\frac{z^5}{5} + \frac{5z^6}{5} \&c: \\
& -\frac{z^6}{6} \&c:
\end{aligned}$$

$$\text{videlicet} = -\left(\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \&c:\right) =$$

$-L(1+z)$; quod demonstrandum suscepimus. Hinc, in praenotatis ad calcem numeri 87. Formulæ Exponentialibus Halleianis, y aut L eundem in utraque significant Numerum Logarithmicum, quia $\frac{a}{b} = \frac{1}{\frac{b}{a}}$, atque vicissim.

90. Exinde etiam petitur facilis comprobatio Aequalitatis non tam obviae Serierum Infinitarum $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \frac{1}{9.10} + \&c:$ at-

que $\frac{1}{1.2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} + \frac{1}{5.32} + \&c:$, vel potius $1 - \frac{1}{2}$

$+ \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \&c:$ et

alterius $\frac{1}{1.2^1} + \frac{1}{2.2^2} + \frac{1}{3.2^3} + \frac{1}{4.2^4} + \frac{1}{5.2^5} + \&c:$, Legis elegantia no-

tabilis, et ab Iacobo Bernoullio sub initium vertentis Saeculi proditae, cui in *Calculi Differentialis Elementis* confirmandae totam impendit Eulerus Transformationis Serierum Theoricen. Prior etenim Series suppeditat

$L(2)$, cum in ipsam vertatur Functio Limite carens $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} -$

$\frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} \&c$: si $z = 1$, tunc scilicet factò $L(1+z) = L(2)$.

Verum $L(2) = L\left(\frac{1}{2}\right)$ permutato eius Signo in oppositum, propterea quod

$\frac{1}{2}$ Reciprocus est Numeri 2. Igitur evadente $1+z = \frac{1}{2}$, aut $z = -$

$\frac{1}{2}$ fiet Series $\frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \frac{1}{160} + \&c = \frac{1}{2} + \frac{1}{12}$

$+ \frac{1}{30} + \frac{1}{56} + \frac{1}{90} + \&c$; quae enunciato Theoremati congruunt (*).

91. Mirabilis equidem Series Infinitae $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} +$
 $\frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c$: natura videbitur, quae
 ita fit comparata, ut aequaliter crescat, minuaturque dum valores z adeo
 temperentur in Numeris $1+z$, ut isti Progressione Geometrica incedant.
 Eo magis haec emicat Geometrico-Harmonicae Progressionis illius univer-
 salis affectio si sedulo perscrutemur, eandem quoque Seriem Arithmetice
 progredi donec Reciprocus Numerus $\frac{1}{1+z}$, vel altera Series numero ter-
 minorum Infinita $1 - z^1 + z^2 - z^3 + z^4 - z^5 + z^6 - z^7 + z^8 - z^9$
 $+ z^{10} - \&c$: servet Geometricam Progressionem. Ternae itaque Progressio-
 nes Geometrica, Arithmetica, Harmonica, tam in terminorum integrantium
 Lege, quam in ipsorum complexus valore, vinculum illud societatemque si-
 mul cumulatae constituunt Logarithmos inter, et Numeros, unaque conspi-
 rant in Logarithmicis Systematibus efformandis. Sin quaeras intimam evi-
 dentiore rationem mirifici huiusce veluti foederis Arithmetico-Geometri-
 ci, cuius vi proditae Series sine carentes Logometriam ferme universam
 concludunt, illa in eo sita est, quod $1+z = 1 + \frac{\Delta^1}{1} + \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3}$
 $+ \dots$

(*) Eulerus affirmat in Capite I. Partis secundae *Institutionum Calculi Differentia-
 lis* se demonstrasse in *Introductione* &c: $L_2 = \frac{1}{2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} +$
 $\frac{1}{5.32} + \&c$. Mihi locum illum quaerenti nunquam arrisit fortuna.

$$+ \frac{\Delta^4}{1.2.3.4} + \frac{\Delta^5}{1.2.3.4.5} + \frac{\Delta^6}{1.2.3.4.5.6} \&c: = \left(1 + \frac{\Delta}{\infty}\right)^{\infty}, \text{ atque } \frac{1}{1+z} =$$

$$(1+z)^{-1} = \left(1 + \frac{\Delta^1}{1} + \frac{\Delta^2}{1.2} + \frac{\Delta^3}{1.2.3} + \frac{\Delta^4}{1.2.3.4} + \frac{\Delta^5}{1.2.3.4.5} + \right.$$

$$\left. \frac{\Delta^6}{1.2.3.4.5.6} + \&c: \right)^{-1} = 1 - \frac{\Delta^1}{1} + \frac{\Delta^2}{1.2} - \frac{\Delta^3}{1.2.3} + \frac{\Delta^4}{1.2.3.4} -$$

$$\frac{\Delta^5}{1.2.3.4.5} + \frac{\Delta^6}{1.2.3.4.5.6} \&c: = \left(1 - \frac{\Delta}{\infty}\right)^{\infty} \text{ ex numeris 75. 76., praesup-}$$

$$\text{posita Functione } \Delta = \frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} \&c.. \text{ Pa-}$$

$$\text{tet exinde fore } (1+z)^n = \left(1 + \frac{\Delta}{\infty}\right)^{n.\infty}, \text{ nec non } \frac{1}{1+z} =$$

$$\left(1 - \frac{\Delta}{\infty}\right)^{n.\infty}. \text{ Porro tam } \left(1 + \frac{\Delta}{\infty}\right)^{n.\infty}, \text{ quam } \left(1 - \frac{\Delta}{\infty}\right)^{n.\infty} \text{ eo-}$$

$$\text{dem redeunt discriminatim ac } \left(1 + \frac{n\Delta}{\infty}\right)^{\infty}, \text{ et } \left(1 - \frac{n\Delta}{\infty}\right)^{\infty}. \text{ Igitur Ari-}$$

$$\text{thmetice crescente vel decrescente } + n\Delta, \text{ aut } - n\Delta, \text{ nimirum Serie, de}$$

$$\text{qua fit fermo, } \frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} \&c: \text{ progredien-}$$

$$\text{te per aequalia intervalla, ex adverso incedit Geometrice } (1+z)^n, \text{ et}$$

$$\left(\frac{1}{1+z}\right)^n, \text{ vel } (1 - z^1 + z^2 - z^3 + z^4 - z^5 + z^6 \&c:)^n. \text{ Aequatio ergo}$$

$$\text{caussalis universae Logometriae } \left(1 \pm \frac{\Delta}{\infty}\right)^{n.\infty} = \left(1 \pm \frac{n.\Delta}{\infty}\right)^{\infty} \text{ haudquaquam}$$

subsistere posset nisi unico remedio Indicis Infiniti adiuvante; qua de re statim intelligitur summa necessitas in Infinitatis labyrintho Logarithmos inter ac Numeros cognationem universali Formula contentam hauriendi, et insuper Transcendentem hanc Formulam tali artificio concinnandi, ut eliminato In-

finito evadat Finita. Hoc equidem obtinetur in $\left(1 \pm \frac{\Delta}{\infty}\right)^{n.\infty}$, sive

$$\left(1 \pm \frac{n.\Delta}{\infty}\right)^{\infty} \text{ ope multiplicationis et divisionis Infinitum quasi elidentium, ad}$$

imitationem casus percelebris, quem in Aequationibus tertii Gradus nuncupare solent *irreductibilem*, ubi destructa, praeter spem, expressionis Imagina-

rietate

rietate quaelibet Radix realem valorem designat. Videas quomodo, Algebraicae Analyseos iure nec minimum perturbato, ex ipsomet Calculi sinu promere liceat non spernendam originem primitivam uberrimamque geminae Magnitudinum Progressionis, quae extat totius rei Logometricae fundamentum, absque eo quod recondita haec ratio Geometricae Syntheseos praesidio ab Hyperbolicis Sectoribus seu Quadrilineis necessario elici debeat.

92. Nullus nescit, quod peculiari non egeat demonstratione summus Logometriae Canon, in quo tota haec Scientia consistit, liberum scilicet apertumque patere ab Exponentiali Aequatione $z^x = v$ transitum ad comitem Logarithmicam $xLz = Lv$. Hinc deducitur veritas Aequationis Exponentialis

$a^{\frac{Lx}{La}} = x$, quippe ad Logarithmos transeundo in identicam vertitur $\frac{Lx}{La} \cdot La = Lx$: immo generalissime ob $\frac{Lx}{Lz} \cdot Lz = Lx$ erit pariter, regressu facto ad Exponentialia, $z^{\frac{Lx}{Lz}} = x$. Quod itaque fuit singulariter assumptum in num°. 87. ex praecostensis respectu solius Numeri C, nunc infinities amplificatum occurrit, cum sit

$$\begin{aligned} & \frac{\frac{x-1^1}{1} - \frac{x-1^2}{2} + \frac{x-1^3}{3} - \frac{x-1^4}{4} + \frac{x-1^5}{5} - \frac{x-1^6}{6} \&c:}{\frac{z-1^1}{1} - \frac{z-1^2}{2} + \frac{z-1^3}{3} - \frac{z-1^4}{4} + \frac{z-1^5}{5} - \frac{z-1^6}{6} \&c:} \\ & = x, \text{ ac subinde } a^{\frac{\frac{x-1^1}{1} - \frac{x-1^2}{2} + \frac{x-1^3}{3} - \frac{x-1^4}{4} + \frac{x-1^5}{5} - \frac{x-1^6}{6} \&c:}{\frac{a-1^1}{1} - \frac{a-1^2}{2} + \frac{a-1^3}{3} - \frac{a-1^4}{4} + \frac{a-1^5}{5} - \frac{a-1^6}{6} \&c:}} = x, \end{aligned}$$

quae omnia traducuntur in Aequationes seiunctis Indicibus praeditas Infinitinomiis

$$\begin{aligned} & \frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \frac{x^9}{9} - \frac{x^{10}}{10} \&c: \\ (1+z) & \frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} \&c: \\ = (1+x) & \frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \frac{x^9}{9} - \frac{x^{10}}{10} \&c: \\ \text{et } (1+a) & \end{aligned}$$

=

$$\frac{a^1}{1} - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \frac{a^5}{5} - \frac{a^6}{6} + \frac{a^7}{7} - \frac{a^8}{8} + \frac{a^9}{9} - \frac{a^{10}}{10} \&c: \\ = (1+x)$$

videlicet, alternationis adfectu non parum iucundo, nunc $(1+z)^{L(1+x)}$ $=$
 $(1+x)^{L(1+z)}$ et $(1+a)^{L(1+x)} = (1+x)^{L(1+a)}$, tunc autem z
 $= x$ et $a = x$.

93. II, qui alacriori mentis acie praestantes introspexerint canonicam Aequationem $(1 \pm z)^x = 1 + \frac{\Delta^1 x^1}{1} + \frac{\Delta^2 x^2}{1.2} + \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^4 x^4}{1.2.3.4} +$
 $\frac{\Delta^5 x^5}{1.2.3.4.5} + \frac{\Delta^6 x^6}{1.2.3.4.5.6} + \frac{\Delta^7 x^7}{1.2.3.4.5.6.7} + \frac{\Delta^8 x^8}{1.2.3.4.5.6.7.8} + \frac{\Delta^9 x^9}{1.2.3.4.5.6.7.8.9} +$
 $\frac{\Delta^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c:$, novae nunc lucis super Formulam $\Delta = L(1 \pm z)$

diffusae participes perfecto calculo hucusque optato Exponentialia prius mutila complere poterunt, atque idcirco adipisci plenissimam Aequalitatem z^x

$$= 1 + \frac{(Lz)^1 x^1}{1} + \frac{(Lz)^2 x^2}{1.2} + \frac{(Lz)^3 x^3}{1.2.3} + \frac{(Lz)^4 x^4}{1.2.3.4} + \frac{(Lz)^5 x^5}{1.2.3.4.5} +$$

$$\frac{(Lz)^6 x^6}{1.2.3.4.5.6} + \frac{(Lz)^7 x^7}{1.2.3.4.5.6.7} + \frac{(Lz)^8 x^8}{1.2.3.4.5.6.7.8} + \frac{(Lz)^9 x^9}{1.2.3.4.5.6.7.8.9} +$$

$$\frac{(Lz)^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c:, \text{ et praeterea, si velint, } z^{-x} = 1 - \frac{(Lz)^1 x^1}{1} +$$

$$\frac{(Lz)^2 x^2}{1.2} - \frac{(Lz)^3 x^3}{1.2.3} + \frac{(Lz)^4 x^4}{1.2.3.4} - \frac{(Lz)^5 x^5}{1.2.3.4.5} + \frac{(Lz)^6 x^6}{1.2.3.4.5.6} -$$

$$\frac{(Lz)^7 x^7}{1.2.3.4.5.6.7} + \frac{(Lz)^8 x^8}{1.2.3.4.5.6.7.8} - \frac{(Lz)^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{(Lz)^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c:, \text{ uni-}$$

versas complectentes Exponentialium familias. Quibus confectis surgit facillime simplicior Aequatio, unicam tantum continens variabilem, $m^x = 1 +$

$$\frac{(Lm)^1 x^1}{1} + \frac{(Lm)^2 x^2}{1.2} + \frac{(Lm)^3 x^3}{1.2.3} + \frac{(Lm)^4 x^4}{1.2.3.4} + \frac{(Lm)^5 x^5}{1.2.3.4.5} + \frac{(Lm)^6 x^6}{1.2.3.4.5.6}$$

$$+ \frac{(Lm)^7 x^7}{1.2.3.4.5.6.7} + \frac{(Lm)^8 x^8}{1.2.3.4.5.6.7.8} + \frac{(Lm)^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{(Lm)^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} +$$

$\&c:$

$\&c: = C^{(Lm)x}$, et omnium sine numero Logisticorum Systematum, eorumque diversarum Basium, ut in num^o. 61., propemodum mater. Indeterminatus etenim Numerus m quamcumque infinities discrepantem Basin Logarithmorum nutu Geometrae eligendam significabit; quumque Numerus quilibet m^x , five $C^{(Lm)x}$ pro Logarithmo habeat in casu Baseos m Indicem x , et in hypothese superius explicata singularis Baseos C Indicem $(Lm)x$, erit hic Logarithmus ad alterum in Ratione $Lm:1$, quae Ratio fit constans dum quaerantur eiusdem Numeri Logarithmi in Logarithmorum variis Basibus gaudentium peculiari quovis Systemate. Exinde fluit alter Canon, summae quidem in Logarithmotechnia utilitatis, Logarithmos nimirum eiusdem Numeri diverso utcunque Systemate, vel, quod idem est, diversis cum Basibus computatos, se habere in constante semper Ratione, utpote reciproca illius Logarithmorum ipsis Basibus respondentium in Systemate Antonomasticae Baseos C , supra omnes vere extollendo. Duae enim Bases sint m , et n ; Numerus quilibet y ; atque ulterius esto $m^x = y = C^{(Lm)x}$, nec non $n^z = y = C^{(Ln)z}$: hisce suppositis Aequatio excitabitur $C^{(Lm)x} = C^{(Ln)z}$, vel $(Lm)x = (Ln)z$, five $x:z::\frac{1}{(Lm)}:\frac{1}{(Ln)}$, quod luculenter ostendit theoreticam, cui nunc sacra facimus, propositionem. Cognitis itaque unius Systematis, Basi singulari C gaudentis, Numerorum Logarithmis illos quoque universorum Systematum facile adipiscemur, si primos per constantem Logarithmum datae Baseos partiamur. Hoc aperte liquet ab Aequatione $y = C^x = a = C^{(La)z}$, quae in aliam resolvitur Logometricam $x = (La)z$, scilicet $z = \frac{x}{(La)}$. Versa autem vice, dum Logarithmos cuiusque Systematis, quod Basin m habeat, tessera l distinguamus, ne cum iis Sigla L ditatis respectu Baseos C confundantur, valebit undequaque Proportio $ly:Ly::1:Lm$. Ista exhibet Aequalitatem $Ly = (Lm)ly$, cuius ope, si Basi m conveniens Logarithmus Numeri cuiuscumque in Logarithmum Antonomastici Systematis, seu Baseos C verti debeat, Problemati satisfiet priore multiplicato per Baseos m in posteriore Systemate Logarithmum. Rebus sic stantibus non modo z^x

$$\begin{aligned}
&= 1 + \frac{(Lz)^1 x^1}{1} + \frac{(Lz)^2 x^2}{1.2} + \frac{(Lz)^3 x^3}{1.2.3} + \frac{(Lz)^4 x^4}{1.2.3.4} + \frac{(Lz)^5 x^5}{1.2.3.4.5} + \\
&\frac{(Lz)^6 x^6}{1.2.3.4.5.6} + \frac{(Lz)^7 x^7}{1.2.3.4.5.6.7} + \frac{(Lz)^8 x^8}{1.2.3.4.5.6.7.8} + \frac{(Lz)^9 x^9}{1.2.3.4.5.6.7.8.9} + \\
&\frac{(Lz)^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c., \text{ verum etiam } = 1 + \frac{(Lm)^1 (lz)^1 x^1}{1} + \frac{(Lm)^2 (lz)^2 x^2}{1.2} + \\
&\frac{(Lm)^3 (lz)^3 x^3}{1.2.3} + \frac{(Lm)^4 (lz)^4 x^4}{1.2.3.4} + \frac{(Lm)^5 (lz)^5 x^5}{1.2.3.4.5} + \frac{(Lm)^6 (lz)^6 x^6}{1.2.3.4.5.6} + \frac{(Lm)^7 (lz)^7 x^7}{1.2.3.4.5.6.7} \\
&+ \frac{(Lm)^8 (lz)^8 x^8}{1.2.3.4.5.6.7.8} + \frac{(Lm)^9 (lz)^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{(Lm)^{10} (lz)^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c., \text{ qua in expref-}
\end{aligned}$$

sione, utcumque Logarithmi Basi m respondentes superveniant, indissolubili tamen lege Logarithmi quoque ex Basi C dependentes commisceantur oportet.

94. Numerus C Basem constituens Systematis Logarithmorum prae cunctis eximii, atque omnium, quasi dixeris, generatoris, Basis est signanter eorum, quibus nomen praebuere Naturalium vel Hyperbolicorum Analyseos sublimioris cultores. Communem autem ceteris Basibus innumerorum Systematum habet ea Systematis Hyperbolici affectionem, quia Numerum in Systemate denotat, cuius Logarithmus sit Unitas. Usitato etiam nomine Proto-Numerus Hyperbolici, aliorumque Systematum in Aequatione comprehensorum $m^x = y$ est ipsamet Unitas. Selecta qualibet Basi invenire cuiusvis Numeri Logarithmum, aut Logarithmi Numerum sive Anti-Logarithmum, et vicissim dato cuiuscumque Numeri determinati Logarithmo Basin assignare, sunt Problemata fere ludicra post Aequationes praecognitas. Haec etenim omnia comple-

$$\begin{aligned}
&\text{atur unica Aequalitas } v = C^{\frac{Lv}{C}} = 1 + \frac{(Lv)^1}{1} + \frac{(Lv)^2}{1.2} + \frac{(Lv)^3}{1.2.3} + \\
&\frac{(Lv)^4}{1.2.3.4} + \frac{(Lv)^5}{1.2.3.4.5} + \frac{(Lv)^6}{1.2.3.4.5.6} + \frac{(Lv)^7}{1.2.3.4.5.6.7} + \frac{(Lv)^8}{1.2.3.4.5.6.7.8} + \\
&\frac{(Lv)^9}{1.2.3.4.5.6.7.8.9} + \frac{(Lv)^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c. = m^x, \text{ in qua, dum Logari-}
\end{aligned}$$

thmus detur, erit $x = b$, et idcirco $Lv = bLm$, videlicet, quum datus sit ex hypothese in casu secundo etiam Numerus $v = a$, habebitur $La = bLm$;

Q

quamob-

quamobrem denique fiet $Lm = \frac{La}{b}$, vel $m = C^{\frac{La}{b}} = 1 + \frac{\left(\frac{La}{b}\right)^1}{1} +$
 $\frac{\left(\frac{La}{b}\right)^2}{1.2} + \frac{\left(\frac{La}{b}\right)^3}{1.2.3} + \frac{\left(\frac{La}{b}\right)^4}{1.2.3.4} + \frac{\left(\frac{La}{b}\right)^5}{1.2.3.4.5} + \frac{\left(\frac{La}{b}\right)^6}{1.2.3.4.5.6} + \frac{\left(\frac{La}{b}\right)^7}{1.2.3.4.5.6.7} +$
 $\frac{\left(\frac{La}{b}\right)^8}{1.2.3.4.5.6.7.8} + \frac{\left(\frac{La}{b}\right)^9}{1.2.3.4.5.6.7.8.9} + \frac{\left(\frac{La}{b}\right)^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.$ Iisdem insistentes

principiis eliciemus ulterius, si dentur Basis Systematis $m = d$, et Numerus $v = a$ iuxta hypothesein primam, $x = la = \frac{La}{Ld}$, atque profluet e-contra Lv

$= la.Ld = bLd$, seu potius Numerus $v = C^{bLd} = 1 + \frac{(bLd)^1}{1} +$
 $\frac{(bLd)^2}{1.2} + \frac{(bLd)^3}{1.2.3} + \frac{(bLd)^4}{1.2.3.4} + \frac{(bLd)^5}{1.2.3.4.5} + \frac{(bLd)^6}{1.2.3.4.5.6} + \frac{(bLd)^7}{1.2.3.4.5.6.7} +$
 $\frac{(bLd)^8}{1.2.3.4.5.6.7.8} + \frac{(bLd)^9}{1.2.3.4.5.6.7.8.9} + \frac{(bLd)^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.$, ubi pro x , et b

substituuntur (lv) , atque (la) respectu habito ad Systema Logarithmorum Basi m , vel d superstructum.

95. Huc certatim veluti confluunt Corollariorum examina maximae molis in Logometricis disciplinis. Modulus ille Logarithmorum cuiusque Systematis Cotesio tam praedilectus nihil est aliud, quam Unitas per Naturalem vel Hyperbolicum Logarithmum Baseos Systematis m divisa, nempe $\frac{1}{Lm}$.

Eodem recidit Numerus, quem Subtangente vocant singularis cuiuscumque Systematis Logarithmici: quapropter non obstupesces, si Modulorum Ratio duo quaelibet distinguunt Systemata et facillime reperiatur, et Rationi inter Subtangentes congruat, et adaequet Logarithmorum determinati cuiusvis Numeri Rationem. Aequatio ipsa, unde oritur, nimirum $(Lm) ly = Ly = (Ln) l'y$, satis ostendit Modulorum, sive eiusdem Numeri Logarithmorum Rationem aequiparari Rationi reciprocae Nat.^{ium} Logarithmorum Basium utrique Systemati respondentium. Modularis autem Ratio stylo Cotesii ea est, ad quam

pro

pro Logarithmo aut Mensura pertinet Systematis Modulus $\frac{1}{L_m}$, vel potius

Numerus in Geometrica continua Progreffione postsequens Logarithmum Hyperbolicum Baseos, et Unitatem; ex quo fit, Modularem Rationem eandem haberi in quocumque Logarithmorum Systemate. Nam, cum dati cuiusvis Numeri in diversis Systematibus Logarithmi proportionem teneant Modulorum, hi Moduli erunt unius eiusdem Numeri supra vel infra Unitatem Logari-

thmi. Idipsum concluditur ab Aequatione $m^{\frac{1}{L_m}} = n^{\frac{1}{L_n}}$, scilicet $1 + \frac{(\frac{L_m}{L_m})^1}{1} + \frac{(\frac{L_m}{L_m})^2}{1.2} + \frac{(\frac{L_m}{L_m})^3}{1.2.3} + \frac{(\frac{L_m}{L_m})^4}{1.2.3.4} + \frac{(\frac{L_m}{L_m})^5}{1.2.3.4.5} + \frac{(\frac{L_m}{L_m})^6}{1.2.3.4.5.6} + \&c. = 1 + \frac{(\frac{L_n}{L_n})^1}{1} + \frac{(\frac{L_n}{L_n})^2}{1.2} + \frac{(\frac{L_n}{L_n})^3}{1.2.3} + \frac{(\frac{L_n}{L_n})^4}{1.2.3.4} + \frac{(\frac{L_n}{L_n})^5}{1.2.3.4.5} + \frac{(\frac{L_n}{L_n})^6}{1.2.3.4.5.6} + \&c.$, quae

positis Systematum Basibus m, n evidenter identica adparet, decernitque constantem semper Modularem Rationem evadere $\approx 2,71828182845904523536028 \&c.$ ad 1, vel Modulum cuiuslibet Systematis confundi cum Logarithmo invariabilis Numeri 2,718281 $\&c.$ superius conscripti, in singulis Systematibus computando. Numerus denique k in Euleri *Introductione* $\&c.$ tanta Calculi vi, atque elegantia excitatus inverse se habet ad Modulum Cotefianum, scilicet Baseos Systematis Logarithmum Naturalem peraequat; quamobrem, posita Basi m , erit $k = \frac{m-1^1}{1} - \frac{m-1^2}{2} + \frac{m-1^3}{3} - \frac{m-1^4}{4} + \frac{m-1^5}{5} - \frac{m-1^6}{6} + \frac{m-1^7}{7} - \frac{m-1^8}{8} + \frac{m-1^9}{9} - \frac{m-1^{10}}{10} + \&c.$, dum Subtangens

feu Modulus $=$

$$\frac{1}{1} - \frac{m-1^2}{2} + \frac{m-1^3}{3} - \frac{m-1^4}{4} + \frac{m-1^5}{5} - \frac{m-1^6}{6} + \frac{m-1^7}{7} - \frac{m-1^8}{8} + \frac{m-1^9}{9} - \frac{m-1^{10}}{10} + \&c.$$

Eadem facilitate Aequatio percelebris derivatur Euleriana valorem exhibens Baseos m cuiusque Systematis ope cogniti Numeri k : namque Aequalitas sane familiaris usque a praecedente num°. 57., nimirum inversa superius iam

$$\text{praeftitae Exponentialis } m = 1 + \frac{k^1}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} + \frac{k^4}{1.2.3.4} + \frac{k^5}{1.2.3.4.5} + \frac{k^6}{1.2.3.4.5.6} + \frac{k^7}{1.2.3.4.5.6.7} + \frac{k^8}{1.2.3.4.5.6.7.8} + \frac{k^9}{1.2.3.4.5.6.7.8.9} + \frac{k^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: \text{ novae huic quaestioni luculenter satisfacit. Haec}$$

cum ita sint, nunquam mirari definient Analystae, tot tantaque inventa universam Logometriam respicientia ab unica pendere Aequatione $m = 1 +$

$$\frac{(Lm)^1}{1} + \frac{(Lm)^2}{1.2} + \frac{(Lm)^3}{1.2.3} + \frac{(Lm)^4}{1.2.3.4} + \frac{(Lm)^5}{1.2.3.4.5} + \frac{(Lm)^6}{1.2.3.4.5.6} + \frac{(Lm)^7}{1.2.3.4.5.6.7} + \frac{(Lm)^8}{1.2.3.4.5.6.7.8} + \frac{(Lm)^9}{1.2.3.4.5.6.7.8.9} + \frac{(Lm)^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: ;$$

quae, tametsi Transcendens ad universalitatem tuendam, substantialiter tamen ac directo tramite derivatur a Formula Neutoniani Binomii, et, quemadmodum antecedentia docent potissimum in num°. 44, dum rite evolvatur Functio Lm consentit cum identica Aequalitate $m = m$.

96. Logarithmi igitur Naturales sive Hyperbolici omnem penitus complent Logometricam doctrinam, quum et sibi ipsis soli sufficiant, et universa alia Systemata moderantes ea sibi devinciant, et in cetera facillime vertantur Systemata Logarithmica. Profecto obtinemus adiumento multiplicationis Logarithmorum Hyperbolicorum per Modulum, aut Subtangente, seu Reciprocum Hyperbolici datae Baseos Logarithmi, vel ope divisionis per k, Logarithmorum directum valorem in quolibet alio Systemate. Inde noscitur esse $\left(1 + \frac{1}{\infty}\right)^{\infty} = C$ Basi Logarithmorum Hyperbolicorum, in quibus Modulus, Subtangens, ac Numerus k Unitatem adaequant. Pari modo altera Aequatio $\left(1 + \frac{Lm}{\infty}\right)^{\infty} = m$ Basi alterius cuiusque Sy-

stematis significare valebit; atque vicissim $\infty \left(1^{\frac{1}{\infty}} (1 + z)^{\frac{1}{\infty}} - 1\right) = Lz$,

et $\frac{\infty}{Lm} \left(1^{\frac{1}{\infty}} (1 + z)^{\frac{1}{\infty}} - 1\right) = lz$ in quocumque ex propria Basi seu Characterem m distincto Systemate. Nullam posthac difficultatem praesefereunt Elementa Systematis Logarithmorum, quos Briggianos seu Tabulares appellare mos

est:

est: Basis etenim m , ex qua pendent, aequatur 10, ideoque Systemati competens Numerus $k = Lm = 2.3025850929940456840179914546843642076$
 $01101488628772976033328 \&c.$, nec non Systematis Modulus five Subtan-

gens $\frac{1}{Lm} = 0.43429448190325182765112891891660508229439700580366$

$6566114454 \&c.$, non $2.30258509299 \&c.$ quemadmodum errant Shervinii *Tabulae Mathematicae* quintae Editionis. Logarithmus ergo cuiusvis Numeri $1+z$ in Tabularium Systemate fit $0.434294481903251827651128918916$

$605082294397005803666566114454 \&c.$ $\left(\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} \right)$, et Numerus cuilibet Loga-

rithmo x respondens par est 10^x , five potius $\left(1 + \frac{(L10)^x}{\infty} \right)^\infty$, videlicet

$$1 + \frac{(2.30258 \&c: x)^1}{1} + \frac{(2.30258 \&c: x)^2}{1.2} + \frac{(2.30258 \&c: x)^3}{1.2.3} + \frac{(2.30258 \&c: x)^4}{1.2.3.4} \\ + \frac{(2.30258 \&c: x)^5}{1.2.3.4.5} + \frac{(2.30258 \&c: x)^6}{1.2.3.4.5.6} + \frac{(2.30258 \&c: x)^7}{1.2.3.4.5.6.7} + \frac{(2.30258 \&c: x)^8}{1.2.3.4.5.6.7.8} + \\ \frac{(2.30258 \&c: x)^9}{1.2.3.4.5.6.7.8.9} + \frac{(2.30258 \&c: x)^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.: \text{Systema autem illud, cuius Ba-}$$

sis foret 2, Protonumero manente 1, Logarithmum Numeri ipsius $1+z$

praebebit aequalem $1.44269504 \&c.$ $\left(\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} \right.$

$\left. - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} \&c. \right)$, five $1.44269504 \&c. \times \infty$

$\left(\left(1 + \frac{1}{\infty} \cdot 1 + z^{\frac{1}{\infty}} \right) - 1 \right)$, et Numerum vel Antilogarithmum tribuet

Logarithmi x ope Formulae Infinitinomialis $2^x = 1 + \frac{(0.69314 \&c: x)^1}{1} +$

$$\frac{(0.69314 \&c: x)^2}{1.2} + \frac{(0.69314 \&c: x)^3}{1.2.3} + \frac{(0.69314 \&c: x)^4}{1.2.3.4} + \frac{(0.69314 \&c: x)^5}{1.2.3.4.5} +$$

$$\frac{(0.69314 \&c: x)^6}{1.2.3.4.5.6} + \frac{(0.69314 \&c: x)^7}{1.2.3.4.5.6.7} + \frac{(0.69314 \&c: x)^8}{1.2.3.4.5.6.7.8} + \frac{(0.69314 \&c: x)^9}{1.2.3.4.5.6.7.8.9} +$$

$$\frac{(0.69314 \&c: x)^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.$$

$\frac{(0,69314 \&c: x)^{10}}{1.2.3.4.5.6.7.8.9.10} \&c:$, ubi brevitatis causa compendiose notavimus $L_2 =$

$0,6931471805599453094172321 \&c:$.

97. Exemplis istis accedat aliud Paradigma, quod prae caeteris attentionem meretur, Logarithmorum nempe Systematis consideratio, cuius Modulus, Subtangens, et characteristicus Numerus k sint Unitas negativa. Basis m huiusce peculiaris Systematis, seu $\left(1 - \frac{1}{\infty}\right)^\infty = 1 - \frac{1}{1} + \frac{1}{1.2} - \frac{1}{1.2.3}$

$$+ \frac{1}{1.2.3.4} - \frac{1}{1.2.3.4.5} + \frac{1}{1.2.3.4.5.6} - \frac{1}{1.2.3.4.5.6.7} + \frac{1}{1.2.3.4.5.6.7.8} - \frac{1}{1.2.3.4.5.6.7.8.9} + \frac{1}{1.2.3.4.5.6.7.8.9.10} \&c: = \frac{1}{C} = 0,367879441 \&c: = C^{-1}.$$

Numerus insuper Logarithmo x respondens erit $\left(\frac{1}{C}\right)^x = 1 - \frac{x^1}{1} +$

$$\frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} - \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} - \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^8}{1.2.3.4.5.6.7.8} - \frac{x^9}{1.2.3.4.5.6.7.8.9} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c: = C^{-x} =$$

$\left(1 - \frac{x}{\infty}\right)^\infty$. Denique universaliter expressio Transcendens

$$= \infty \left(1 - \frac{1}{\infty} \cdot \frac{1}{1+z} - 1\right) = - \left(\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} \&c:\right) \text{ Logarithmum Numeri } 1+z \text{ praestabit. Cum}$$

igitur postremi hi Logarithmi eundem habeant valorem, sed signo tantum immutato, Logarithmorum Hyperbolicorum, sintque positivi pro Numeris Unitate minoribus, atque vicissim pro Unitate maioribus negativi, nomen Systemati adscriptum Logarithmorum Antinaturalium vel Antihyperbolicorum, quibus eadem Tabulae inserviant reciprocos Numeros assumendo, non sine quadam veritatis specie videbitur. Hoc interim quaeso, quominus obli-

vionem patiat Logarithmorum in omni Systemate comes $\tau \delta \left(1 - \frac{1}{\infty}\right)$, dum nobis brevitati consulentibus istud non tantum quandoque omittere contigit, prouti Fa-

ctorem

ctorem $(\pm 1)^x$ Magnitudinum Exponentialium, verum etiam in posterum, nec raro, continget.

98. Saepius in antea actis monuimus Numerum z considerari posse positivum vel negativum, posteriorique casu Formulam $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} \&c.$, Logarithmum Numeri $1 + z$ exhibentem, evadere $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} \&c.$, ut elementaris Algebra docet; quapropter expressionis Logarithmicae Symbola bisgemina $\pm \left(\pm \frac{z^1}{1} - \frac{z^2}{2} \pm \frac{z^3}{3} - \frac{z^4}{4} \pm \frac{z^5}{5} - \frac{z^6}{6} \pm \frac{z^7}{7} - \frac{z^8}{8} \pm \frac{z^9}{9} - \frac{z^{10}}{10} \&c. \right)$ reapse in unam vere oecu-

menicam redigitur, quae sola variatione Signorum diversas species, non formas, acquirat. Operae itaque pretium erit non uni singillatim speciei Formulae ipsius sequentia aptare argumenta, quin potius universalitatem Analyticam Logometrici Infinitinomii simul complecti. Quod qui effecerit omnia sentiet Logarithmorum Systemata veluti in unum coalescere Moduli, vel Subtangents, aut eius Reciproci adiumento utens; nec difficultate laborabit forte aliquando interpretaturus quae Leibnitio, Bernoulliis, Halleio, aliisque

Geometris ita scribere placuit, $l(1 \pm x) = S \left(\pm \frac{x^1}{1} - \frac{x^2}{2} \pm \frac{x^3}{3} - \frac{x^4}{4} \pm \frac{x^5}{5} - \frac{x^6}{6} \pm \frac{x^7}{7} - \frac{x^8}{8} \pm \frac{x^9}{9} - \frac{x^{10}}{10} \&c. \right)$, five $= \frac{1}{m}$ vel potius $M \left(\pm \frac{x^1}{1} - \frac{x^2}{2} \pm \frac{x^3}{3} - \frac{x^4}{4} \pm \frac{x^5}{5} - \frac{x^6}{6} \pm \frac{x^7}{7} - \frac{x^8}{8} \pm \frac{x^9}{9} - \frac{x^{10}}{10} \&c. \right)$, ulteriusque y , seu Numerus Logarithmo $\pm x$ respondens in Systemate, cui S et M pro Subtangente ac Modulo dedicentur, $= 1 \pm$

$$\frac{x^1}{1.S} + \frac{x^2}{1.2.S^2} \pm \frac{x^3}{1.2.3.S^3} + \frac{x^4}{1.2.3.4.S^4} \pm \frac{x^5}{1.2.3.4.5.S^5} + \frac{x^6}{1.2.3.4.5.6.S^6} \pm \frac{x^7}{1.2.3.4.5.6.7.S^7} \pm \dots$$

$$\begin{aligned}
& \frac{x^7}{1.2.3.4.5.6.7.S^7} + \frac{x^8}{1.2.3.4.5.6.7.8.S^8} + \frac{x^9}{1.2.3.4.5.6.7.8.9.S^9} + \\
& \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10.S^{10}} \&c.; \text{ aut } 1 = \frac{m^1 x^1}{1} + \frac{m^2 x^2}{1.2} + \frac{m^3 x^3}{1.2.3} + \frac{m^4 x^4}{1.2.3.4} \\
& + \frac{m^5 x^5}{1.2.3.4.5} + \frac{m^6 x^6}{1.2.3.4.5.6} + \frac{m^7 x^7}{1.2.3.4.5.6.7} + \frac{m^8 x^8}{1.2.3.4.5.6.7.8} + \frac{m^9 x^9}{1.2.3.4.5.6.7.8.9} \\
& + \frac{m^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.; \text{ videlicet } 1 = \frac{x^1}{1.M} + \frac{x^2}{1.2.M^2} + \frac{x^3}{1.2.3.M^3} + \\
& \frac{x^4}{1.2.3.4.M^4} + \frac{x^5}{1.2.3.4.5.M^5} + \frac{x^6}{1.2.3.4.5.6.M^6} + \frac{x^7}{1.2.3.4.5.6.7.M^7} + \\
& \frac{x^8}{1.2.3.4.5.6.7.8.M^8} + \frac{x^9}{1.2.3.4.5.6.7.8.9.M^9} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10.M^{10}} \&c.; \text{ si memi-}
\end{aligned}$$

nerit eorum, de quibus iam differuimus. Quinimmo longius progredi ipsi licebit ad constituendum valorem Subtangens, Moduli, aut $\frac{1}{m}$ quocumque in

Systemate, quum $\frac{1}{m}$, Modulus, vel Subtangens aequalis sit Reciproco Logarithmi Hyperbolici Baseos Systematis, scilicet Logarithmi Hyperbolici illius Numeri pro Logarithmo in dato Systemate Unitatem habentis, vel clarius

fit semper $S = M = \frac{1}{m} = \frac{1}{k} = \frac{1x}{Lx}$, qua in Formula $\tau\delta$ x eundem

quemlibet Numerum denotet. Theorema igitur ab Iacobo Bernoullio prodictum in Parte IV^a. *Positionum de Infinitis Seriebus*, nempe $S = M =$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \&c. = \frac{1}{L2}$$

immaniter amplificari poterit, dum Binario quivis Numerus substituatur.

99. Qui sui ipsius ingenii vim in Mathematicarum Facultatum curriculo exercere fuetus Formulam Exponentialem rite perpendat $z^x = 1 +$

$$\frac{(Lz)^1 x^1}{1} + \frac{(Lz)^2 x^2}{1.2} + \frac{(Lz)^3 x^3}{1.2.3} + \frac{(Lz)^4 x^4}{1.2.3.4} + \frac{(Lz)^5 x^5}{1.2.3.4.5} + \frac{(Lz)^6 x^6}{1.2.3.4.5.6} + \&c.;$$

$$\text{vel } z^x = 1 + \frac{\left(\frac{1z}{M}\right)^1}{1} + \frac{\left(\frac{1z}{M}\right)^2}{1.2} + \frac{\left(\frac{1z}{M}\right)^3}{1.2.3} + \frac{\left(\frac{1z}{M}\right)^4}{1.2.3.4} + \frac{\left(\frac{1z}{M}\right)^5}{1.2.3.4.5} +$$

$+ \frac{\left(\frac{Lz}{M}\right)^6}{1.2.3.4.5.6} + \&c:$ in ipsa leget abunde satis et evidenter universa,

quae supra huc illuc paullatim sparsa inveniantur. Dum $Lz = \Delta = k$ in Unitatem vertatur, etiam $M = \frac{1}{Lz} = \frac{1}{k}$ Unitatem peraequabit, fietque Ba-

sis $z = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} + \frac{1}{1.2.3.4.5.6}$

$\&c:$, Numerus $z^x = C^x = 1 + \frac{x^1}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} +$

$\frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} \&c:$, et Logarithmus Naturalis vel Hyperbolicus

expressus Littera x , atque ad Numerum N pertinens $= \frac{N-1^1}{1} - \frac{N-1^2}{2} +$

$\frac{N-1^3}{3} - \frac{N-1^4}{4} + \frac{N-1^5}{5} - \frac{N-1^6}{6} \&c:$ in Infinitum. Posito autem $Lz = \Delta$

$= k = L10$, ideoque $M = \frac{1}{L10}$, Basis erit τ^0 Denarium, ac Logarithmo-

rum Systema mutabitur in Briggianum aut Tabulare; quamobrem $(10)^x$

$= 1 + \frac{(L10)^1 x^1}{1} + \frac{(L10)^2 x^2}{1.2} + \frac{(L10)^3 x^3}{1.2.3} + \frac{(L10)^4 x^4}{1.2.3.4} + \frac{(L10)^5 x^5}{1.2.3.4.5} +$

$\frac{(L10)^6 x^6}{1.2.3.4.5.6} + \&c:$, et $x = l((10)^x) = \frac{1}{L10} \left(\frac{N-1^1}{1} - \frac{N-1^2}{2} + \frac{N-1^3}{3} -$

$\frac{N-1^4}{4} + \frac{N-1^5}{5} - \frac{N-1^6}{6} \&c:$). Si Basis concepti cuiusdam Systematis effin-

geretur $= 7$ esset Δ , sive $k = L7 = 1,9459101490553133051054639 \&c:$,

Modulusque $M = \frac{1}{1,9459101490553133051054639 \&c:}$. Ex adverso, si Modulus

Systematis alicuius foret $\frac{1}{3}$, Basis evaderet $= 1 + \frac{3}{1} + \frac{9}{1.2} + \frac{27}{1.2.3}$

$+ \frac{81}{1.2.4} + \frac{243}{1.2.4.5} + \frac{729}{1.2.4.5.6} + \frac{2187}{1.2.4.5.6.7} \&c:$, eademque methodo

in Infinitum. Universaliter $\left(1 + \frac{kx}{\infty}\right)^\infty$, sive $\left(1 + \frac{x}{M.\infty}\right)^\infty = N$, et

R

 $\infty.M$

$${}_M \left(1^{\frac{1}{\infty}} \left(\frac{1}{1+z} \right)^{\frac{1}{\infty}} - 1 \right), \text{ vel } \frac{\infty}{k} \left(1^{\frac{1}{\infty}} \left(\frac{1}{1+z} \right)^{\frac{1}{\infty}} - 1 \right) = 1z. \text{ Ele-}$$

gans quoque affectio videbitur, quod $C^{lx} = C^{M(Lx)} = C^{L(x^M)} = x^M = x^{\frac{1}{k}}$,
quaecumque Functio sub specie x intelligatur abscondita; et pari ratione

$${}_m^{lx} = C^{k \cdot \frac{1}{k} Lx} = C^{Lm \cdot M Lx} = C^{\frac{1}{M} M Lx} = x, \text{ videlicet in quovis Lo-}$$

garithmorum Systemate eadem proprietas valebit prouti in Hyperbolico
iam demonstravimus, quae fundamentum per se est universae Exponentialium
doctrinae. Insuper Basis cuiuslibet Logarithmorum Systematis non tantum-

$$\text{modo Infinitinomial praefatum } 1 + \frac{k^1}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} + \frac{k^4}{1.2.3.4} +$$

$$\frac{k^5}{1.2.3.4.5} + \frac{k^6}{1.2.3.4.5.6} \&c: \text{ adaequabit, quin etiam } 1 + \frac{1}{1.M^1} + \frac{1}{1.2.M^2}$$

$$+ \frac{1}{1.2.3.M^3} + \frac{1}{1.2.3.4.M^4} + \frac{1}{1.2.3.4.5.M^5} + \frac{1}{1.2.3.4.5.6.M^6} \&c: . \text{ Nec immora-}$$

ri opus erit in facillima Coronide deducenda quoad magnitudinem Expo-

$$\text{nentialem } x^x, \text{ utpote quae ob } z=x \text{ fit aequalis } 1 + \frac{(Lx)^1 x^1}{1} + \frac{(Lx)^2 x^2}{1.2}$$

$$+ \frac{(Lx)^3 x^3}{1.2.3} + \frac{(Lx)^4 x^4}{1.2.3.4} + \frac{(Lx)^5 x^5}{1.2.3.4.5} + \frac{(Lx)^6 x^6}{1.2.3.4.5.6} + \frac{(Lx)^7 x^7}{1.2.3.4.5.6.7} +$$

$$\frac{(Lx)^8 x^8}{1.2.3.4.5.6.7.8} + \frac{(Lx)^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{(Lx)^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c: , \text{ vel } \left(1 + \frac{(Lx)x}{\infty} \right)^{\infty}$$

dum maiore compendio frui libeat. Est autem Aequationes inter innumeras

Exponentiales elegantissima $x^{\frac{1}{Lx}} = C$, cuius vix aliquod specimen in Ana-
lyseos Fastis occurrit.

100. Ne tanto Logometricorum Canonum pondere fatiscat Lector, otium
quiddam ei nunc aliquantisper sublevando paramus nonnihil ab incoepta se-
mita deflectentes. Quam iucundum etenim foret, si, post diuturnum iter huc
illuc perductis, ac centena per Theoremata diversimode semper actis, a Logari-
thmorum Formula mox retrogredientibus, veluti circuli in se redeuntis symbolum

genera-

generatricem Binomii Newtoniani expressionem denuo licuerit amplecti? Hoc

signanter in praesentia molimur. Perspiciatur namque $L(1+z) = \frac{z^1}{1} -$

$$\frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \&c., \text{ ac simul obversabitur}$$

$$mL(1+z) = L(1+z^m) = \frac{mz^1}{1} - \frac{mz^2}{2} + \frac{mz^3}{3} - \frac{mz^4}{4} + \frac{mz^5}{5} - \frac{mz^6}{6}$$

$$+ \frac{mz^7}{7} - \&c. = L\left(1 + \frac{mz^1}{1} + \frac{m.m-1.z^2}{1.2} + \frac{m.m-1.m-2.z^3}{1.2.3} + \right.$$

$$\left. \frac{m.m-1.m-2.m-3.z^4}{1.2.3.4} + \frac{m.m-1.m-2.m-3.m-4.z^5}{1.2.3.4.5} + \right.$$

$$\left. \frac{m.m-1.m-2.m-3.m-4.m-5.z^6}{1.2.3.4.5.6} + \&c. \right), \text{ quae aequipollentia dummodo uni-}$$

versaliter pateat pro indeterminato Potestatis Indice m , veritatem Formulae ipsius generalis Binomii confirmari, et Exponentialium ac Logarithmorum inde repetendum nexum nullus dubito arctius perstringi. Atqui Polynomio, instar Binomii, regulis Analyseos subacto desideratam primo intuitu Aequalita-

$$\text{tem nanciscimur } L\left(1 + \frac{mz^1}{1} + \frac{(m^2-m)z^2}{1.2} + \frac{(m^3-3m^2+2m)z^3}{1.2.3} + \right.$$

$$\left. \frac{(m^4-6m^3+11m^2-6m)z^4}{1.2.3.4} + \frac{(m^5-10m^4+35m^3-50m^2+24m)z^5}{1.2.3.4.5} + \right.$$

$$\left. \frac{(m^6-15m^5+85m^4-225m^3+274m^2-120m)z^6}{1.2.3.4.5.6} + \&c. \right) =$$

$$\begin{aligned}
& \frac{mz^1}{1} + \frac{(m^2-m)}{1.2} z^2 + \frac{(m^3-3m^2+2m)}{1.2.3} z^3 + \frac{(m^4-6m^3+11m^2-6m)}{1.2.3.4} z^4 + \frac{(m^5-10m^4+35m^3-50m^2+24m)}{1.2.3.4.5} z^5 + \\
& - \frac{m^2}{1.2} z^2 - \frac{m^3}{1.2} z^3 - \frac{m^4}{1.2.4} z^4 - \frac{(m^5-3m^4+2m^3)}{1.3.4} z^5 \\
& + \frac{m^2}{1.2} z^3 + \frac{m^3}{1.4} z^4 + \frac{(m^4-3m^3+2m^2)}{1.3.4} z^5 \\
& + \frac{m^3}{1.3} z^3 - \frac{m^2}{1.2.4} z^4 + \frac{(m^5-2m^4+m^3)}{1.4} z^5 \\
& - \frac{m^4}{1.2.3} z^4 - \frac{(m^5-m^4)}{1.2} z^5 \\
& + \frac{m^3}{1.2.3} z^4 - \frac{(m^5-6m^4+11m^3-6m^2)}{1.2.3.4} z^5 \\
& - \frac{m^2}{1.3} z^4 + \frac{(m^5-3m^4+2m^3)}{1.2.3} z^5 \\
& + \frac{m^4}{1.2} z^4 + \frac{m^5}{5} z^5 \\
& - \frac{m^3}{1.2} z^4 \\
& - \frac{m^4}{1.4} z^4
\end{aligned}$$

nimirum perfecte invenimus Aequationem illam identicam, quam fueramus polliciti.

101. Formulam universalis Logarithmi $\pm \left(\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c: \right)$ divergentiae vitio perturbari quum z superet Unitatem, vel Numerus $1 + z$ fit major Binario saepius in Capite praecedente, ac praesertim num°. 54., huc respiciens indigitavi. Huic malo remedium curantes valde defudarunt Geometrae, et in arenam descendere summi quoque in re Mathematica Viri non sunt dedignati. Operam oleumque perderem si omnes explicare methodos haecenus investigatas mihi animus foret; quamobrem tantummodo recenferi primigenias, ac celebriores non fit qui obliquo oculo aspiciat. Cum Logarithmorum Tabulae

pro

$$\frac{(m^6 - 15m^5 + 85m^4 - 225m^3 + 274m^2 - 120m)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} z^6 \&c: = m \left(\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} \&c: \right),$$

&c:

&c:

&c:

&c:

nimirum

pro Systemate tam Briggiano, quam Naturali in promptu habeantur, silentio praetermittens ad divergentiam illam medendam omnia minoris momenti par-
ciorisque usus artificia nullum Theorices universalitati, meliora ac generalio-
ra solummodo persequenti, naevum apponam. Methodorum demonstrationes
principiis superius expositis concinnatas adjiciam etiamdum in Auctorum Sche-
dis pleraeque vatem desiderent; et majorem saepius perfundere adlaborabo
facilitatem. Non autem sperandum heic repertum iri Theoremata fere innume-
ra, quae non nativis, sed peregrinis coloribus Logarithmorum ducunt conver-
gentiam a diversis Approximationum, ut vocant, regulis, seu Calculi Infini-
te parvorum praesidium adprecantur, vel opem Algorithmi Interpolationum aut
descriptionis Curvarum Parabolici generis sive Finitarum differentiarum Cal-
culi, cui Differentialis nomen impositum usque ab inventoris aetate Neutoni.

102. Agmen incipiat methodus ab Halleio sub finem elapsi saeculi,
deindeque

deindeque ab omnibus pene Analystis adhibita. Indubium est $L\left(\frac{1+z}{1-z}\right)$

$$\begin{aligned} &= L(1+z) - L(1-z) = \frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \\ &\frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} \&c: + \frac{z^1}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} \\ &+ \frac{z^6}{6} + \frac{z^7}{7} + \frac{z^8}{8} + \frac{z^9}{9} + \frac{z^{10}}{10} \&c: = 2 \left(\frac{z^1}{1} + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \right. \\ &\frac{z^9}{9} + \frac{z^{11}}{11} \&c: \left. \right), \text{ parique pacto } l\left(\frac{1+z}{1-z}\right) = 2M \left(\frac{z^1}{1} + \frac{z^3}{3} + \frac{z^5}{5} + \right. \\ &\frac{z^7}{7} + \frac{z^9}{9} + \frac{z^{11}}{11} \&c: \left. \right). \text{ Igitur, praesupposita Aequatione } \frac{1+z}{1-z} = x \text{ cui-} \end{aligned}$$

$$\begin{aligned} &\text{libet Numero, erit } z = \frac{x-1}{x+1}, \text{ ac propterea } Lx = 2 \left(\frac{\left(\frac{x-1}{x+1}\right)^1}{1} + \right. \\ &\frac{\left(\frac{x-1}{x+1}\right)^3}{3} + \frac{\left(\frac{x-1}{x+1}\right)^5}{5} + \frac{\left(\frac{x-1}{x+1}\right)^7}{7} + \frac{\left(\frac{x-1}{x+1}\right)^9}{9} + \frac{\left(\frac{x-1}{x+1}\right)^{11}}{11} \&c: \left. \right), \text{ nec} \\ &\text{non } lx = 2M \left(\frac{\left(\frac{x-1}{x+1}\right)^1}{1} + \frac{\left(\frac{x-1}{x+1}\right)^3}{3} + \frac{\left(\frac{x-1}{x+1}\right)^5}{5} + \frac{\left(\frac{x-1}{x+1}\right)^7}{7} + \frac{\left(\frac{x-1}{x+1}\right)^9}{9} \right. \\ &+ \frac{\left(\frac{x-1}{x+1}\right)^{11}}{11} \&c: \left. \right). \text{ Hinc fit, quod } \frac{x-1}{1} - \frac{x-1}{2} + \frac{x-1}{3} - \frac{x-1}{4} + \\ &\frac{x-1}{5} - \frac{x-1}{6} + \frac{x-1}{7} - \frac{x-1}{8} + \frac{x-1}{9} - \frac{x-1}{10} + \frac{x-1}{11} \&c: = \\ &2 \left(\frac{\left(\frac{x-1}{x+1}\right)^1}{1} + \frac{\left(\frac{x-1}{x+1}\right)^3}{3} + \frac{\left(\frac{x-1}{x+1}\right)^5}{5} + \frac{\left(\frac{x-1}{x+1}\right)^7}{7} + \frac{\left(\frac{x-1}{x+1}\right)^9}{9} + \right. \\ &\frac{\left(\frac{x-1}{x+1}\right)^{11}}{11} \&c: \left. \right), \text{ cuius alterum Membrum est Series semper convergens pro} \end{aligned}$$

inveniendò Lx five lx , etiam cum Numerus x Binarium quoquomodo excedat, aut divergentia suboriens prioris Series Logometriae commodo adverte-

tur.

tur. Quinimmo prior geminarum Serierum, quando convergat usuique Analyse melius auspiciata se praeestet, scilicet quando x Binario sit minus, lentius ac fastidiosius, respectu habito ad alteram, quaesito $\tau \approx Lx$ valori appro-

pinquat, tam ob Signa alternatim diversa, quam ob terminos $\frac{\left(\frac{x-1}{x+1}\right)^{n+1}}{n+1}$

secundae Seriei semper minores $\frac{\frac{x-1}{n+1}}{n+1}$ primae homologis terminis. Dum vero loco unius Numeri x assignantur gemini quilibet indefiniti, maior m , minor n , horumque Summa sit S , differentia D , evadet $L\left(\frac{m}{n}\right) =$

$$L\left(\frac{S+D}{S-D}\right) = 2\left(\frac{\left(\frac{D}{S}\right)^1}{1} + \frac{\left(\frac{D}{S}\right)^3}{3} + \frac{\left(\frac{D}{S}\right)^5}{5} + \frac{\left(\frac{D}{S}\right)^7}{7} + \frac{\left(\frac{D}{S}\right)^9}{9} + \&c.\right)$$

proptereaquod $\frac{S+D}{S-D} - 1 = 2D$, et $\frac{S+D}{S-D} + 1 = 2S$. Obtinebitur itaque

$$Lm = Ln + \frac{2D}{S} + \frac{D^2 A}{3S^2} + \frac{3D^2 B}{5S^2} + \frac{5D^2 C}{7S^2} + \frac{7D^2 E}{9S^2} + \frac{9D^2 F}{11S^2} + \&c.;$$

$$\text{ac vicissim } Ln = Lm - \frac{2D}{S} - \frac{D^2 A}{3S^2} - \frac{3D^2 B}{5S^2} - \frac{5D^2 C}{7S^2} - \frac{7D^2 E}{9S^2} -$$

$$\frac{9D^2 F}{11S^2} - \&c.; \text{ speciebus } A, B, C, E, F \&c. \text{ vicem terminorum proxime ante-}$$

cedentium tenentibus, quae Series passim demonstratione carentes penes varios inveniuntur Trigonometricae facultatis Scriptores. Exinde oritur Lm

$$= L(m-1) + 2\left(\frac{1}{(2m-1)^1} + \frac{1}{3 \cdot 2m-1^3} + \frac{1}{5 \cdot 2m-1^5} + \frac{1}{7 \cdot 2m-1^7} + \frac{1}{9 \cdot 2m-1^9} + \&c.\right),$$

$$\text{recipiens usitatior } Lm = L(m-1) + \frac{2}{2m-1}\left(1 + \frac{1}{3(2m-1)^2} + \frac{1}{5(2m-1)^4} + \frac{1}{7(2m-1)^6} + \frac{1}{9(2m-1)^8} + \frac{1}{11(2m-1)^{10}} + \&c.\right),$$

$$\text{communi segregato Factore.}$$

103. Aliae Series ex hisce praemissis derivantur Numericae, elegantia nullis equidem

equidem posthabendae. Si Numerus $x = C$ Basi Logarithmorum Hyperbolicorum, a quibus Univerſa Systemata pendent, erit $L(C) = 1 = 2 \left(\frac{(C-1)^1}{C+1} \right.$

$$+ \frac{(C-1)^3}{3} + \frac{(C-1)^5}{5} + \frac{(C-1)^7}{7} + \frac{(C-1)^9}{9} + \frac{(C-1)^{11}}{11} \&c. \left. \right),$$

quae Series Unitati aequalis et admodum convergens obtineri non posset, propter $C = 2,718281828459 \&c.$, subsidio usitatae Logarithmicae Formulae

$$1 = \frac{C-1^1}{1} - \frac{C-1^2}{2} + \frac{C-1^3}{3} - \frac{C-1^4}{4} + \frac{C-1^5}{5} - \frac{C-1^6}{6} + \frac{C-1^7}{7} - \frac{C-1^8}{8} + \frac{C-1^9}{9} - \frac{C-1^{10}}{10} \&c.,$$

utpote enormiter divergentis.

Valor Numeri k , five $\frac{1}{M}$, quodlibet ex innumeris Logarithmorum Systematibus tessera veluti quadam distinguens, ex praestentis adaequat $L(B) = \frac{B-1^1}{1}$

$$- \frac{B-1^2}{2} + \frac{B-1^3}{3} - \frac{B-1^4}{4} + \frac{B-1^5}{5} - \frac{B-1^6}{6} + \frac{B-1^7}{7} - \frac{B-1^8}{8} + \frac{B-1^9}{9} - \frac{B-1^{10}}{10} + \&c.,$$

si littera B Basin Systematis denotet. Saepius autem occurrit, quod evadat haec Series divergens, cui vitio remedium

$$\text{affertur scribendo } L(B) = \frac{1}{M} = k = 2 \left(\frac{(B-1)^1}{B+1} + \frac{(B-1)^3}{3} + \frac{(B-1)^5}{5} + \frac{(B-1)^7}{7} + \frac{(B-1)^9}{9} + \frac{(B-1)^{11}}{11} \&c. \right).$$

Erit itaque in Brig-

$$\text{giano Systemate } k, \text{ vel Reciprocum Moduli } = 2 \left(\frac{9}{11^1} + \frac{9^3}{3 \cdot 11^3} + \frac{9^5}{5 \cdot 11^5} + \frac{9^7}{7 \cdot 11^7} + \frac{9^9}{9 \cdot 11^9} + \frac{9^{11}}{11 \cdot 11^{11}} \&c. \right) = \frac{9^1}{1} - \frac{9^2}{2} + \frac{9^3}{3} - \frac{9^4}{4} + \frac{9^5}{5} - \frac{9^6}{6} + \frac{9^7}{7} - \frac{9^8}{8} + \frac{9^9}{9} - \frac{9^{10}}{10} + \frac{9^{11}}{11} \&c.$$

divergentissimae

inutilique Series, propterea quod, ut omnes norunt, $B = 10$. Nam $B = 2$

proveniet

proveniet $\frac{1}{M}$ five $k = 2 \left(\frac{1}{1.3^1} - \frac{1}{3.3^3} + \frac{1}{5.3^5} - \frac{1}{7.3^7} + \frac{1}{9.3^9} - \frac{1}{11.3^{11}} + \dots \right)$ in Infinitum $= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} + \dots$ &c: , vel potius $\frac{1}{1.3^1} - \frac{1}{3.3^3} + \frac{1}{5.3^5} - \frac{1}{7.3^7} + \frac{1}{9.3^9} - \frac{1}{11.3^{11}} + \dots$ &c: $= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \frac{1}{18} - \frac{1}{20} + \frac{1}{22} + \dots$ &c: . Fiat denuo $B = \frac{1}{3}$; tunc nanciscemur $k = \frac{1}{M} = - \left(\frac{2^1}{1.3^1} + \frac{2^2}{2.3^2} + \frac{2^3}{3.3^3} + \frac{2^4}{4.3^4} + \frac{2^5}{5.3^5} + \frac{2^6}{6.3^6} + \frac{2^7}{7.3^7} + \frac{2^8}{8.3^8} + \frac{2^9}{9.3^9} + \frac{2^{10}}{10.3^{10}} + \frac{2^{11}}{11.3^{11}} + \dots \right) = - 2 \left(\frac{1}{1.2^1} + \frac{1}{3.2^3} + \frac{1}{5.2^5} + \frac{1}{7.2^7} + \frac{1}{9.2^9} + \frac{1}{11.2^{11}} + \dots \right)$ &c: , quas Series negativo vel positivo Signo praeditas $\frac{2^0}{1.3^1} + \frac{2^1}{2.3^2} + \frac{2^2}{3.3^3} + \frac{2^3}{4.3^4} + \frac{2^4}{5.3^5} + \frac{2^5}{6.3^6} + \frac{2^6}{7.3^7} + \frac{2^7}{8.3^8} + \frac{2^8}{9.3^9} + \frac{2^9}{10.3^{10}} + \frac{2^{10}}{11.3^{11}} + \dots$ &c: , nec non $\frac{1}{1.2^1} + \frac{1}{3.2^3} + \frac{1}{5.2^5} + \frac{1}{7.2^7} + \frac{1}{9.2^9} + \frac{1}{11.2^{11}} + \dots$ &c: numero terminorum Infinitas diversif-

mode, ut libet, ac sine ullo Limite novis exemplis locupletare fas esset.

104. Quae in Halleianis Formulis varia fortasse alioque ducentia prima fronte apparebunt, nihilo tamen minus adamussim congruunt superius iam dictis. Eximius enim Auctor nuncupavit in illum Numerum Logarithmici Systematis characteristicum, quem demonstravimus esse Moduli M reciprocum, et a Specie k generatim significari. Praeterea, dum ille quaerit Logarithmum Numeri five Rationis Geometricae $\frac{a}{b}$, praesupponit Aequalitates $x = a - b$, et $z = a + b$; quamobrem $x = D$, ac $z = S$ respectu habito ad Magnitudines a , et b , prouti in num. 102. ad alias m atque n . Hisce in antecessum adnotatis

Series $\frac{1}{m} \times 2 \left(\frac{x^1}{1.z^1} + \frac{x^3}{3.z^3} + \frac{x^5}{5.z^5} + \frac{x^7}{7.z^7} + \frac{x^9}{9.z^9} + \frac{x^{11}}{11.z^{11}} + \dots \right)$ Phi-
S loso-

Philosophicis Transactionibus anni MDCXCV. inserta eadem est atque illa in praecitato num°. 102. recensita cum Moduli $\frac{1}{m}$ additione; et casu $b=1$

$$\text{aequivaleret } la = 2M \left(\frac{\left(\frac{a-1}{a+1}\right)^1}{1} + \frac{\left(\frac{a-1}{a+1}\right)^3}{3} + \frac{\left(\frac{a-1}{a+1}\right)^5}{5} + \frac{\left(\frac{a-1}{a+1}\right)^7}{7} + \frac{\left(\frac{a-1}{a+1}\right)^9}{9} + \frac{\left(\frac{a-1}{a+1}\right)^{11}}{11} + \&c. \right), \text{ quemadmodum nostra methodo elicitur.}$$

Aliter quoque ipsius Seriei Logarithmicae veritas patet observantibus

$$\text{Aequalitates } a = \frac{z+x}{2}, b = \frac{z-x}{2}, \text{ ideoque } \frac{a}{b} = \frac{z+x}{z-x} = \frac{1+\frac{x}{z}}{1-\frac{x}{z}}, \text{ cuius Logarithmum in quolibet Systemate usque ab initio numeri}$$

$$102. \text{ constat suppeditari a Serie infinita } \frac{2}{m} \times \left(\frac{x^1}{z^1} + \frac{x^3}{3z^3} + \frac{x^5}{5z^5} + \frac{x^7}{7z^7} + \frac{x^9}{9z^9} + \frac{x^{11}}{11z^{11}} + \&c. \right). \text{ Ne vero maxima evidentia desit Ae-}$$

$$\text{qualitati adferendae Serierum } \frac{\frac{x-1}{x+1}^1}{1} - \frac{\frac{x-1}{x+1}^2}{2} + \frac{\frac{x-1}{x+1}^3}{3} - \frac{\frac{x-1}{x+1}^4}{4} + \frac{\frac{x-1}{x+1}^5}{5} - \frac{\frac{x-1}{x+1}^6}{6} + \frac{\frac{x-1}{x+1}^7}{7} - \frac{\frac{x-1}{x+1}^8}{8} + \frac{\frac{x-1}{x+1}^9}{9} - \frac{\frac{x-1}{x+1}^{10}}{10} + \&c.; \text{ atque}$$

$$2 \left(\frac{\left(\frac{x-1}{x+1}\right)^1}{1} + \frac{\left(\frac{x-1}{x+1}\right)^3}{3} + \frac{\left(\frac{x-1}{x+1}\right)^5}{5} + \frac{\left(\frac{x-1}{x+1}\right)^7}{7} + \frac{\left(\frac{x-1}{x+1}\right)^9}{9} + \frac{\left(\frac{x-1}{x+1}\right)^{11}}{11} + \&c. \right), \text{ quae tantam praesefert, utut mendosam, diversitatem,}$$

novum spartae huic inlustrandae accommodatum periculum faciam. Sit $x-1 = v$, et ope aequipollentiae terminorum ostendere oporteat Aequationem

$$\frac{v^1}{1} - \frac{v^2}{2} + \frac{v^3}{3} - \frac{v^4}{4} + \frac{v^5}{5} - \frac{v^6}{6} + \frac{v^7}{7} - \frac{v^8}{8} + \frac{v^9}{9} - \frac{v^{10}}{10} + \&c.$$

$$\frac{v^{10}}{10} \&c: = 2 \left(\frac{\left(\frac{v}{2+v}\right)^1}{1} + \frac{\left(\frac{v}{2+v}\right)^3}{3} + \frac{\left(\frac{v}{2+v}\right)^5}{5} + \frac{\left(\frac{v}{2+v}\right)^7}{7} + \frac{\left(\frac{v}{2+v}\right)^9}{9} \right.$$

$$\&c: \left. \right), \text{ scilicet } \frac{v^1}{2} - \frac{v^2}{4} + \frac{v^3}{6} - \frac{v^4}{8} + \frac{v^5}{10} - \frac{v^6}{12} + \frac{v^7}{14} \&c: =$$

$$\frac{\left(\frac{v}{2+v}\right)^1}{1} + \frac{\left(\frac{v}{2+v}\right)^3}{3} + \frac{\left(\frac{v}{2+v}\right)^5}{5} + \frac{\left(\frac{v}{2+v}\right)^7}{7} \&c: . \text{ Hoc statim conficie-}$$

$$\text{tur dummodo evolvamus } \frac{v}{2+v} \text{ in } \frac{v^1}{2} - \frac{v^2}{4} + \frac{v^3}{8} - \frac{v^4}{16} + \frac{v^5}{32}$$

$$- \frac{v^6}{64} + \frac{v^7}{128} \&c: \text{ ope notissimorum Canonum Neutroniani Binomii. Nam-}$$

que fiet, explicito totius Calculi Typo,

$$\begin{aligned}
\frac{v^1}{2} - \frac{v^2}{4} + \frac{v^3}{6} - \frac{v^4}{8} + \frac{v^5}{10} - \frac{v^6}{12} + \frac{v^7}{14} \&c. = \frac{v^1}{2} - \frac{v^2}{4} + \frac{v^3}{8} - \frac{v^4}{16} + \frac{v^5}{32} - \frac{v^6}{64} + \frac{v^7}{128} \&c. \\
&+ \frac{v^3}{24} - \frac{v^4}{16} + \frac{v^5}{32} - \frac{v^6}{192} + \frac{2v^7}{128} \\
&+ \frac{v^5}{32} - \frac{v^6}{32} + \frac{v^7}{64} \\
&+ \frac{v^5}{160} - \frac{v^6}{64} + \frac{v^7}{192} \\
&- \frac{v^6}{64} + \frac{3v^7}{128} \\
&+ \frac{v^7}{384} \\
&+ \frac{v^7}{896}
\end{aligned}$$

cui identitati curandae non inanem laborem arbitror impendisse.

105. Sin Summam potius quam Differentiam Logarithmorum Numeri $1+x$, atque $1-x$ absque alio artificio menti subiicere libeat, nulla praefecto erit iuvandae doctrinae Logometricae apta deductio. Vixdum etenim nacti fuerimus Aequationem $L(1+x) + L(1-x) = -\left(\frac{x^2}{1} + \frac{x^4}{2} + \frac{x^6}{3} + \frac{x^8}{4} + \frac{x^{10}}{5} + \frac{x^{12}}{6} + \frac{x^{14}}{7} + \&c.\right)$ nec hilum Algebra promovebitur,

ex eo, quod neminem lateat esse $L(1+x) + L(1-x) = L(1-x^2)$, ideoque in praecognitis Formulis Capitis II. pro x subrogando x^2 elici minime foecundatam Seriem superius expositam. Neque aliter sentiam quia dimidium Logarithmi Hyperbolici $\approx$ Numeri assumpti $1-x^2$, sive $L\sqrt{1-x^2}$ Seriem illam exhibeat $-\frac{x^2}{2} - \frac{x^4}{4} - \frac{x^6}{6} - \frac{x^8}{8} - \frac{x^{10}}{10} - \frac{x^{12}}{12} - \frac{x^{14}}{14} -$

..... $-\frac{x^n}{n}$ &c: a Leibnitio usque ab anno MDCXCI. in vulgus e-

ditam veluti Logarithmum Recti-Sinus, posito Cosinu x , et deinceps ab Iacobo Bernoullio longiore demonstrationis circuitu firmatam: nam, praeter summam huius Theorematis ostendendi facilitatem, vel nullo vel tenui Analytico usu ea Series clarescit. Quae vero insuper habet Halleius de ce-
teris

$$= \frac{v^1}{2} - \frac{v^2}{4} + \frac{v^3}{6} - \frac{v^4}{8} + \frac{v^5}{10} - \frac{v^6}{12} + \frac{v^7}{14} \&c.;$$

cui

teris appropinquationibus differens, omnia facili studio comprobantur. Ac primum a postrema Formula derivatur Aequatio percelebris, ob quam Logarithmus Fractionis Numeratorem habentis Medietatem Geometricam, ac Denominato-

rem Arithmeticam duos inter quoslibet Numeros n , et r , nimirum $L\left(\frac{\sqrt{nr}}{\left(\frac{n+r}{2}\right)}\right)$

$$= -\left(\frac{D^2}{2S^2} + \frac{D^4}{4S^4} + \frac{D^6}{6S^6} + \frac{D^8}{8S^8} + \frac{D^{10}}{10S^{10}} + \frac{D^{12}}{12S^{12}} \&c.\right), \text{ ubi } D \text{ de}$$

more Differentiam, atque S Summam Numerorum significat. Etenim $L\left(\frac{\sqrt{nr}}{\left(\frac{n+r}{2}\right)}\right)$

$$= \frac{1}{2} L\left(\frac{S^2 - D^2}{4}\right) - L\left(\frac{S}{2}\right) = \frac{1}{2} L\left(1 - \frac{D^2}{S^2}\right) + \frac{1}{2} L\left(\frac{S^2}{4}\right) -$$

$$L\left(\frac{S}{2}\right) = \frac{1}{2} L\left(1 - \frac{D^2}{S^2}\right) = -\left(\frac{D^2}{2S^2} + \frac{D^4}{4S^4} + \frac{D^6}{6S^6} + \frac{D^8}{8S^8} \&c. \&c.\right)$$

quemadmodum superius descripsimus. Eodem pacto, si addatur hypothesis

$$y^2 = \frac{\frac{S^2}{4} + nr}{\frac{S^2}{4} - nr} = \frac{\frac{S^2}{4} + nr}{\frac{D^2}{4}}, \text{ vel potius } \frac{\sqrt{1 - \frac{1}{y^2}}}{\sqrt{1 + \frac{1}{y^2}}} = \frac{\sqrt{nr}}{\left(\frac{n+r}{2}\right)}, \text{ obtinebitur}$$

L(

$$L\left(\frac{\sqrt{nr}}{\left(\frac{n+r}{2}\right)}\right) = \frac{1}{2}L\left(1 - \frac{1}{y^2}\right) - \frac{1}{2}L\left(1 + \frac{1}{y^2}\right), \text{ videlicet } = -$$

$$\left(\frac{1}{y^2} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} + \frac{1}{11y^{22}} + \frac{1}{13y^{26}} \&c.\right), \text{ quae}$$

Series tam cito convergit ob y Unitate maiorem, Coefficientes in Progressione naturali imparium Numerorum crescente, et Indices Potestatum quatuor Unitatibus perpetuo auctos, ut vix in Logometria castigatio optari, nedum confici, possit. Quin etiam, dum Numeri n, r forent $m, m+2$, obtineretur

$$\text{alia notissima Formula } L\left(\frac{\sqrt{m \cdot m+2}}{m+1}\right) = - \left(\frac{1}{v} + \frac{1}{3v^3} + \frac{1}{5v^5} + \right.$$

$$\left.\frac{1}{7v^7} + \frac{1}{9v^9} + \frac{1}{11v^{11}} + \frac{1}{13v^{13}} + \&c.\right), \text{ vel } L(m+1) =$$

$$\frac{1}{2}L(m \cdot m+2) + \frac{1}{v} + \frac{1}{3v^3} + \frac{1}{5v^5} + \frac{1}{7v^7} + \frac{1}{9v^9} + \frac{1}{11v^{11}} + \frac{1}{13v^{13}}$$

$$+ \&c., \text{ qua in expressione } v \text{ adaequat } \frac{\frac{S^2}{4} + nr}{\frac{D^2}{4}}, \text{ nimirum } 2m^2 + 4m + 1,$$

prouti constat post Numerorum datorum substitutiones effectas. Iucunde autem accidit geminas Series priores tunc, uti debent, evanescere quum $n=r$,

$$\text{ex eo quod } \left(\frac{\sqrt{nr}}{\left(\frac{n+r}{2}\right)}\right) = 1, D \text{ in nihilum abeat, fiatque } y^2 \text{ quantitas Infinita.}$$

Qui rigido Neperi more Logarithmos expriment, ii positivos scribunt Numeris Unitate minoribus respondentes, vicissimque maioribus negativos. Sed arbitraria haec Signorum varietas aut quamplurima praeli vitia ne Halleianam Diatriben, vel alias Lucubrationes quemque legat offendant; nam ceteroquin universa congruunt Hyperbolicis Logarithmis, et facilis deducitur transitus consideratione Factoris seu Moduli M ad Logarithmos utcunque diversi Systematis. Ita in Briggiano fit $M = \frac{0.8685889638 \&c.}{2} =$

$\frac{n}{2}$, qui Numerus n signanter adhibitus extat in Prolegomenis Tabularum Shervinii.

106. Elegancia atque Calculi commoditate nulli concedit Series a Keilio
 evulgata in Opusculo *de Natura, et Arithmetica Logarithmorum*. Hanc, mi-
 hi saltem, indemonstratam eo libentius aggredior, quod ipso Auctore fatente
 sit memoratae Halleianae Seriei facillimum Confectarium. Et re quidem vera
 investigandus proponatur Logarithmus cuiusque Numeri imparis Primi z ex
 iam cognitis Logarithmis parium Numerorum $z-1$, ac $z+1$; erit $\frac{S}{2} = z$,
 atque $D = 2$. Praeterea habebuntur Aequalitates $\sqrt{nr} = \sqrt{z^2-1} =$
 $z\sqrt{1-\frac{1}{z^2}}$, nec non $L\left(\frac{z-1}{z+1}\right) = L\left(\frac{1-\frac{1}{z}}{1+\frac{1}{z}}\right) = L\left(1-\frac{1}{z}\right) -$
 $L\left(1+\frac{1}{z}\right) = -2\left(\frac{1}{z} + \frac{1}{3z^3} + \frac{1}{5z^5} + \frac{1}{7z^7} + \frac{1}{9z^9} + \frac{1}{11z^{11}} \&c.\right)$
 $= y$, seu Magnitudini $L(z-1) - L(z+1)$ quocumque in casu propo-
 sito determinatae. Verum prior Series Halleii praebet $L\left(\frac{\sqrt{z^2-1}}{z}\right) =$
 $-\left(\frac{1}{2z^2} + \frac{1}{4z^4} + \frac{1}{6z^6} + \frac{1}{8z^8} + \frac{1}{10z^{10}} + \&c.\right)$, five =
 $y\left(\frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} + \frac{1903}{211340z^9} + \&c.\right)$, utpote
 excepto ultimo termino, in quo sane errat Keilius, ostenditur evidentissimi
 ope Algorithmi, cuius Typum, ne in nimiam molem excrescat, claritate
 quantum potero servata ita coarctabo. Erit nimirum

$$y\left(\frac{1}{4z}\right)$$

$$y \left(\frac{1}{4z^1} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} + \frac{1903}{24113400z^9} + \&c: \right) = - \left(\frac{2}{z} + \frac{2}{3z^3} + \frac{2}{5z^5} + \frac{2}{7z^7} + \frac{2}{9z^9} + \&c: \right) \times$$

$$- \left(\frac{1}{2z^2} + \frac{1}{4z^4} + \frac{1}{6z^6} + \frac{1}{8z^8} + \frac{1}{10z^{10}} + \&c: \right), \text{ cui conclusioni ob-}$$

tinendae Calculum istum vovere suscepi. Mira sane in condendis aut amplificandis Tabulis iamdudum confectis Logarithmorum, quos nonnulli iure vocant *Numerorum geometricae proportionalium aequidifferentes comites*, appropinquatio a postrema Serie hauriretur: nam saepius Logarithmus quaesitus Numeri imparis Primi z , si Myriadem superet, ab unico priori termino effluit $\frac{y}{4z}$,

qua in Fractione z, y dantur, ideoque, ut in Halleianis Seriebus etiam occurrit,

determinato Logarithmo Numeri $\frac{\sqrt{z^2-1}}{z}$, vel $L \sqrt{z^2-1} - Lz$, sive potius

$$\frac{L(z+1) + L(z-1)}{2} - Lz = p, \text{ valor quoque } \tau \approx Lz = \frac{1}{2} L(z+1) +$$

$$\frac{1}{2} L(z-1) - p \text{ elicietur.}$$

107. Selecto quolibet inter artificia Analytica superius emensa facillime redigetur in Seriem omnino convergentem Logarithmus cuiusque Numeri $1+z$. Qui secum ipse perpendat numerorum 54. 55. 57. 67. 70. 71. 72. &c: Capitis IIⁱ. sensum, illumque referre velit ad ea, quae nuper sunt tradita, oculi pene ictu intuebitur convergentiam Seriei Logarithmicae nativae $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} \&c:$ angustis limitibus circumscribi, nimirum in Curva Schematis XVII^{mi}. a puncto A, in quo Numerus $1+z=0$, vel $BA=z=-1$, posito B initio Abscissarum

$$\left(\frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} + \frac{1903}{226800z^9} + \&c \right) = - \left(\frac{1}{2z^2} + \frac{1}{6z^4} + \frac{1}{10z^6} + \frac{1}{14z^8} + \frac{1}{18z^{10}} + \&c \right) =$$

$$+ \frac{1}{12z^4} + \frac{1}{36z^6} + \frac{1}{60z^8} + \frac{1}{84z^{10}} + \frac{7}{180z^6} + \frac{7}{540z^8} + \frac{7}{900z^{10}} + \frac{181}{7560z^8} + \frac{181}{22680z^{10}} + \frac{1903}{113400z^{10}} -$$

scissarum z , usque ad alterum punctum C adeo situm, ut $BC = z = 1$, et

ideo $1 + z = 2$. Curva igitur, seu Locus ab Aequatione expressus $y = \frac{z^1}{1}$

$-\frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \&c$: delineari quamproxime, et

sub oculos poni nequit nisi per intervallum Perimetri $FBED$ versus D infinites protensae. Ars vero postremum adhibita, quum transformationibus rite institutis convergentem reddat Seriem illam Logarithmicam etiam ultra confinia puncti C , videlicet etiam dum $z > 1$, aut $1 + z > 2$, commodum praebebit

Curvam continuandi per G, H, L &c: e regione M sine Limite. Sed in regione opposita N nulla praesto nunc erit convergentiae methodus, uti ex datis Formulis patet, nec ibidem de Loci prosecutione, indole, affectionibus quae-

dam iure poterunt praedicari. Si loco Seriei $\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} -$

$\frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + \&c$: sufficiatur altera in praecedentibus num^{is}.

fusus perspecta, statim eliminabitur incommodum divergentiae, vel lentioris etiam convergentiae ab omni Formula Exponentiali $(1 + z)^x$, illaque quasi aggerabitur lacuna, quae adhuc supererat in complendam Exponentialium, et Logarithmorum Theoricen. Ne verbum quidem protulimus de Logarithmis Numerorum infra Unitatem, sive $1 - z$, tametsi varia Serierum expositarum symptomata aequae sint istis communia. Series etenim nativa, quemadmo-

T

dum

dum toties fuit monitum, — $\left(\frac{z^1}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} + \frac{z^6}{6} + \frac{z^7}{7} + \frac{z^8}{8} + \frac{z^9}{9} + \frac{z^{10}}{10} \&c. \right)$, semper semperque convergentia ditatur. Innumera

de eodem argumento promere ulterius possem dum alias vel minus commodas Series, vel ex dictis, etiam invita minerva, derivandas singillatim persequi liberet: verumtamen facis hisce reiectis pauca quaedam specimina superaddo. Caveant autem in Analyfi non apprime versati a nimio iniqui canonis usu decernentis Numerorum differentias iis Logarithmorum proportionales. Ut illi arctissimos noscant eius canonis limites, dentur duo quilibet Numeri $a + x$,

a , eritque $L(a + x) - La = L\left(\frac{a+x}{a}\right) = L\left(1 + \frac{x}{a}\right)$, quam Lo-

garithmorum differentiam specie v significari nunc velim. Igitur $C^v = 1 +$

$$\frac{v^1}{1} + \frac{v^2}{1.2} + \frac{v^3}{1.2.3} + \frac{v^4}{1.2.3.4} + \frac{v^5}{1.2.3.4.5} + \frac{v^6}{1.2.3.4.5.6} + \&c. = 1 +$$

$$\frac{x}{a}; \text{quamobrem Numerorum differentia } x = av \left(1 + \frac{v^1}{1.2} + \frac{v^2}{1.2.3} + \right.$$

$$\left. \frac{v^3}{1.2.3.4} + \frac{v^4}{1.2.3.4.5} + \&c. \right), \text{ five } =$$

$$\frac{av}{1 - \frac{v}{1.2} + \frac{v^2}{1.2.2.3} - \frac{v^3}{1.2.3.4.5.6} + \frac{v^4}{1.2.3.4.5.6.7} - \&c.}, \text{ idest nullo modo}$$

proportionem sequitur $\tau \approx v$ differentiae Logarithmorum, at proxime tantum, si liceat omnes Infinitinomii terminos praeter primum in Praxeos usu negligere. Huius Formulae commoditati summopere cognationis vinculo nequitur celeberrimum illud Newtoni Theorema, quo Ln aequalis statuitur

$$L(n - x) + \frac{1}{2} L\left(\frac{n+x}{n-x}\right) \left(1 + \frac{x^1}{2n} + \frac{x^3}{12n^3} + \frac{7x^5}{180n^5} + \&c. \right),$$

et idcirco, praecognitis Numerorum aequae distantium Logarithmis, intermedii Numeri Logarithmus emergit. Est enim $Ln = Ln + L\left(1 - \frac{x}{n}\right) -$

$$L\left(1 - \frac{x}{n}\right) = L\left(n.1 - \frac{x}{n}\right) - L\left(1 - \frac{x}{n}\right) = L(n - x) + \frac{1}{2} L$$

$$+ \frac{1}{2} L \left(\frac{1 + \frac{x}{n}}{1 - \frac{x}{n}} \right) \times \frac{-L \left(1 - \frac{x}{n} \right)}{\frac{1}{2} L \left(\frac{1 + \frac{x}{n}}{1 - \frac{x}{n}} \right)} = L(n-x) + \frac{1}{2} (L(n+x) -$$

$$- L(n-x)) \left[\frac{\frac{x}{n} + \frac{x^2}{2n^2} + \frac{x^3}{3n^3} + \frac{x^4}{4n^4} + \&c:}{\frac{x}{n} + \frac{x^3}{3n^3} + \frac{x^5}{5n^5} + \frac{x^7}{7n^7} + \&c:} \right] = L(n-x) + \left(\frac{L(n+x) - L(n-x)}{2} \right) \left(1 + \frac{x^1}{2n^1} + \frac{x^3}{12n^3} + \frac{7x^5}{180n^5} + \&c: \right),$$

pro-
ti habet Neuton, sed Hyperbolae tetragonismo demonstrationem suam in
Fluxionum ac Serierum Infinitarum Methodo superstruens.

108. Omnes norunt $Lz = -L \left(\frac{1}{z} \right)$, quod et Logometriae Institu-
tiones tyronibus ipsis suadent, et, si velis, numerus 89. novam istius Aequatio-
nis ac plenissimam continet demonstrationem. Quando igitur $L(1+x)$ in Se-
riei scopulum aut divergentis aut parum convergentis impingat, quare tunc
vicissim $L \left(\frac{1}{1+x} \right) = - \left(\left(\frac{x}{1+x} \right)^1 + \left(\frac{x}{1+x} \right)^2 + \left(\frac{x}{1+x} \right)^3 +$

$$\left(\frac{x}{1+x} \right)^4 + \left(\frac{x}{1+x} \right)^5 + \left(\frac{x}{1+x} \right)^6 + \&c: \right),$$

signoque in positivum tran-

$$\text{seunte ipsamet Series facile continuanda, et undeque convergens prae-}$$

$$\text{bebit } Lv = \left(\frac{v-1}{v} \right)^1 + \left(\frac{v-1}{v} \right)^2 + \left(\frac{v-1}{v} \right)^3 + \left(\frac{v-1}{v} \right)^4 +$$

$$\left(\frac{v-1}{v} \right)^5 + \left(\frac{v-1}{v} \right)^6 + \left(\frac{v-1}{v} \right)^7 + \left(\frac{v-1}{v} \right)^8 + \left(\frac{v-1}{v} \right)^9 +$$

$$\left(\frac{v-1}{v} \right)^{10} + \&c: \text{ in Infinitum. Nescio quo fato tot Analystarum Adver-}$$

faria, quae volutavi, de hac Serie Logarithmica nec obscurae nec sterilis originis fere penitus fileant.

109. Age nunc percurramus praecipuarum Serierum, de quibus supra facta fuit mentio, universitatem, grataque erit, nisi fallor, animadversio omnes inter se convenire dum Logarithmus Nihili vel Unitatis petatur, hunc Nihilo, atque illum Infinito negativo Progressionis Harmonicae —

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c: \right) \text{aequa-}$$

lem praebentes. Eodem itaque pacto, quo minus concinna prior Series —

$$\left(\frac{z^1}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} + \frac{z^6}{6} + \frac{z^7}{7} + \frac{z^8}{8} + \frac{z^9}{9} + \frac{z^{10}}{10} \right) \&c: = L(1 - z) \text{ evadit casu } z = 1, \text{ five } L(1 - z) =$$

$$Lo, \text{ uti facile patet, } - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c: \right), \text{ etiam Series in num}^\circ. 102. \text{ allata fit } - 2 \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \&c: \right) = Lo, \text{ eaeque in num}^{is} \text{ exhibitae postsequen-}$$

$$\text{tibus } 105. 106. 108. \text{ supponendo } n = 0, \text{ vel } y^2 = 1, \text{ five } z^2 = 1, \text{ aut } x = \infty \text{ dant } \frac{1}{2} Lo = - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \&c: \right),$$

$$\text{ac praeterea } \frac{1}{2} Lo = - \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \&c: \right), \text{ deinde-}$$

$$\text{que } \frac{1}{2} Lo = - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \&c: \right), \text{ et tandem}$$

$$Lo = - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c: \right).$$

$$\text{Quarum Serierum Harmonicarum, utut prima fronte discrepantium, mirabilis consensus elucescit si mente volvamus esse eiusdem valoris tam}$$

$$\frac{2}{2} + \frac{2}{4} + \frac{2}{6} + \frac{2}{8} + \frac{2}{10} + \frac{2}{12} + \frac{2}{14} + \&c:, \text{ quam } 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c:.$$

$$\text{Quarum Serierum Harmonicarum, utut prima fronte discrepantium, mirabilis consensus elucescit si mente volvamus esse eiusdem valoris tam}$$

$$\frac{2}{2} + \frac{2}{4} + \frac{2}{6} + \frac{2}{8} + \frac{2}{10} + \frac{2}{12} + \frac{2}{14} + \&c:, \text{ quam } 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c:.$$

$$\frac{2}{2} + \frac{2}{4} + \frac{2}{6} + \frac{2}{8} + \frac{2}{10} + \frac{2}{12} + \frac{2}{14} + \&c:, \text{ quam } 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c:.$$

$$\frac{2}{2} + \frac{2}{4} + \frac{2}{6} + \frac{2}{8} + \frac{2}{10} + \frac{2}{12} + \frac{2}{14} + \&c:, \text{ quam } 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c:.$$

$$\frac{2}{2} + \frac{2}{4} + \frac{2}{6} + \frac{2}{8} + \frac{2}{10} + \frac{2}{12} + \frac{2}{14} + \&c:, \text{ quam } 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c:.$$

$$\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \&c.; \text{ nec non } 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} \&c.:$$

peraequare medietatem Seriei $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$
 $+ \frac{1}{9} + \frac{1}{10} \&c.$ proptereaquod $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} \&c.$ fit al-

tera ipsius Seriei ex praeostensis medietas. Multis retro annis ambo seniores Bernoullii hanc ipsam veritatem arithmeticam expresserunt Aequationibus

$$\text{analogis } 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} \&c. = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} +$$

$$\frac{1}{10} + \frac{1}{12} \&c.; \text{ atque } \frac{2}{1} + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9} + \frac{2}{11} \&c. = \frac{1}{1} +$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \&c.; \text{ et absit quod putes ab ea aliquantulum in-}$$

firmari Theorema demonstratae Aequalitatis inter L_2 seu $\frac{12}{M}$, ac Seriem

$$\text{praecognitam } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10}$$

$$\&c.: \text{ Quomodocumque etenim sit de perfecta aequalitate } 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}$$

$$+ \frac{1}{9} + \frac{1}{11} \&c. \text{ ac } \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} \&c. \text{ cum utra-}$$

que Series limitis ignara Infinites magnum habeat valorem, redigaturque in Aequationem $\infty = \infty$, haec utpote indeterminata nullatenus excludit

$$\text{alteram } 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} \&c. - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} -$$

$$\frac{1}{10} - \frac{1}{12} \&c. \text{ nimirum } \infty - \infty = L_2, \text{ vel } \infty = \infty + L_2, \text{ quem-}$$

admodum docet numerus 40. et natura ipsa Infinitorum homogeneorum evincit absque ullius Gordiani nodi resolutione. Finitum autem Infinitorum, al-

$$\text{terius ab altero dempti, } \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} \&c. \right) - \left(\frac{1}{2} \right.$$

$$\left. + \frac{1}{4} \right)$$

$-\frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \dots$) haberi valorem satis adstruit diffe-

rentia ipsamet num^o. 90. evoluta, videlicet $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} +$

$\frac{1}{9.10} + \frac{1}{11.12} + \dots$, quae nullam patitur difficultatem, determinatque Ae-

quationem $L 2 = \infty - \infty$, secus indeterminatione turbatam. Nec ultra moror in Unitatis Logarithmo exponendo, quia universi proditarum Serierum termini iisdem hypothesibus, de quibus supra, singuli tunc permutantur in Numeros fractos Numeratoribus 0 gaudentes, atque idcirco evanescunt. Accedat demonstratio etiam nova Theorematis $\left(\frac{n}{m}\right)^{\infty} = 0$ dummodo $m >$

n ex Logometricis doctrinis petita, quandoquidem erit $\infty L\left(\frac{n}{m}\right) = \infty$ ($L n - L m$) $= -\infty = L 0$ iuxta superius enucleatas rationes.

110. De Logarithmis Unitatis, et Nihili edifferens supervacaneum non duco eosdem simplicius deducere ab Exponentialibus C^x in Hyperbolico, ac generatim B^x in quolibet alio Systemate, ut quae coniurant amice eo magis uberrimam veritatem confirment. Numerus C^x habens x pro suo L^{thmo} , nec non B^x gaudens x pro L^{thmo} vel $M.L^{\text{thmo}}$, tunc erit Unitas, cum Index $x = 0$. Affectio isthaec evanescentiae Logarithmi Unitatis, quam in Capite IV^o. Logometriae universaliter consideratae nullatenus necessariam videbimus, ducit originem ab Exponentialium evolutorum modo, quo factum est, ut $(1 + z)^x$

$$= 1 + \frac{\Delta^1 x^1}{1} + \frac{\Delta^2 x^2}{1.2} + \frac{\Delta^3 x^3}{1.2.3} + \frac{\Delta^4 x^4}{1.2.3.4} + \frac{\Delta^5 x^5}{1.2.3.4.5} + \dots, \text{ et ideo}$$

$$\Delta = L(1 + z) \text{ vel } = \frac{1(1+z)}{M} = 0 \text{ tum quum } z = 0, \text{ aut Numerus } 1 + z$$

$= 1$, Exponentiali, non autem Logometrica, necessitate id postulante. Numerus insuper ex adverso evanescet, positis Basibus Systematum Unitate maioribus, quum

$C^x = C^{-\infty}$, ac $B^x = B^{-\infty}$. Quod si Logometricorum Systematum Bases forent e-contra Unitate minores, evadent $C^x = C^{\infty} = 0$, atque consimili pacto $B^x = B^{\infty} = 0$. Series autem in abscondito manet Infinitum hunc Logarithmum positivum vel negativum quodammodo distinguens, ac instar symboli praebens: at isti etiam voto facillime satisfiet per Exponentialium evolutio-

nem.

nem. Esto namque $C^{-\infty} = 1 - \frac{\infty^1}{1} + \frac{\infty^2}{1.2} - \frac{\infty^3}{1.2.3} + \frac{\infty^4}{1.2.3.4} - \frac{\infty^5}{1.2.3.4.5}$
 $+ \frac{\infty^6}{1.2.3.4.5.6} - \frac{\infty^7}{1.2.3.4.5.6.7} + \&c: = 0 = \left(1 - \frac{\infty}{\infty}\right)^{\infty}$; et similiter
 $B^{-\infty} = 0 = \left(1 - \frac{\infty}{M.\infty}\right)^{\infty} = 1 - \frac{\left(\frac{\infty}{M}\right)^1}{1} + \frac{\left(\frac{\infty}{M}\right)^2}{1.2} - \frac{\left(\frac{\infty}{M}\right)^3}{1.2.3}$
 $+ \frac{\left(\frac{\infty}{M}\right)^4}{1.2.3.4} - \frac{\left(\frac{\infty}{M}\right)^5}{1.2.3.4.5} + \frac{\left(\frac{\infty}{M}\right)^6}{1.2.3.4.5.6} - \frac{\left(\frac{\infty}{M}\right)^7}{1.2.3.4.5.6.7} + \&c:.$ Ad hoc,

ut Series nuper descriptae sint Nullitati aequales, oportet vi Theorematum perpenforum in Capite I^o., ac praecipue num^{is}. 38. 39., esse $\tau\delta\infty$ Signum Infinitae Harmonicae Progressionis $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

$+ \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \&c:$, vel, quod eodem recidit, $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \&c:$, aut $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \&c:$, quia ∞ sive $\frac{\infty}{2}$ speciem

Infinitatis nec hilum immutant. Igitur huius ope artificii determinatur illud ignoti Infiniti genus ex nudo charactere Indicis Potestatum indeterminati $C^{-\infty}$ aut $B^{-\infty}$ deductum, simulque notio altera elegantissima acquiritur, in universalibus Formulis passim superius recensitis $\left(1 \pm \frac{\Delta x}{\infty}\right)^{\infty}$ Infinitatis

Notam Analytica necessitate Infinitum Harmonicum, quod nonnulli vocant *paradoxum*, denotare. Primum enim observari meretur, quod ex evolutione Binomii $\left(1 \pm \frac{\Delta x}{\infty}\right)^{\infty}$ constat tam Denominatoris, quam Indicis Infinita eiusdem esse speciei atque valoris. Praeterea nec posset, uti debet, supposito $x = \infty$, evadere $\left(1 - \frac{\infty}{\infty}\right)^{\infty}$, nec $\left(1 - \frac{\infty}{M.\infty}\right)^{\infty} = 0$ nisi $\frac{\infty}{\infty}$,

aut $\frac{\infty}{\infty} \cdot \frac{1}{M} = 1$, vel Fractioni finitae, scilicet nisi Infinitum, quod munere fungitur Denominatoris, ad eandem speciem pertineat Infiniti Numeratoris, nimirum ad illam Harmonicorum. Dum ergo ratiocinatio instituitur de aequipollentia expressionum C^x atque $\left(1 + \frac{x}{\infty}\right)^{\infty}$ subintelligi debet

Indicis

Indicis Infiniti Notam heic designare $\tau\delta r \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c: \right)$, quod etiam longius evolvere libet. Est

$$L_{\infty} = -L_0 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c:, \text{ et } C^{L_{\infty}} = \infty \text{ vi praecognitae Aequationis oecumenicae } C^{Lx} = x. \text{ In-}$$

$$\text{de profluit } C^{L_{\infty}} = \frac{1}{0}, \text{ vel } 0 = C^{-L_{\infty}} = \left(1 - \frac{L_{\infty}}{\infty} \right)^{\infty} =$$

$$\left(1 - \frac{L_{\infty}}{r.L_{\infty}} \right)^{r.L_{\infty}} \text{ dummodo } r \text{ sit Finitus, Unitate maior, ac Positivus.}$$

Eadem ratione $\left(1 + \frac{\Delta x}{\infty} \right) = C^{\Delta x}$ tractare licuisset. Timeas autem ve-

lim, si coniectando assumeretur $C^x = \left(1 + \frac{x}{L_{\infty}} \right)^{L_{\infty}}$, ideoque $C =$

$$\left(1 + \frac{1}{L_{\infty}} \right)^{L_{\infty}}, \text{ antilogiam } C^{L_{\infty}} = \left(1 + \frac{L_{\infty}}{L_{\infty}} \right)^{L_{\infty}} = 2^{L_{\infty}}, \text{ nimi-}$$

rum forte $C = 2$. Etenim nequidem casu, in quo sumus, Infiniti Expo-
nentis aequantur inter se $2^{L_{\infty}}$, et $(2,7182818284 \&c:)^{L_{\infty}}$, eoquod
0,7182818284 &c: Fractio exaltata ad unicum Infiniti Ordinis Potestatem
nullius est valoris, uti communis Arithmetica docet, at nequaquam idem
valet de Potestatum eius innumerarum congerie. Iniuria ergo conciperetur

$$C^{L_{\infty}} = 2^{L_{\infty}} = (2,7182818284 \&c:)^{L_{\infty}}, \text{ aut } 2 = 2,718 \&c:, \text{ vel etiam}$$

Binario cum qualibet alia subiuncta vera Fractione (*).

III. Altius promovetur haec quantulacumque sit disquisitio dum superveniat
examen rationis aut proportionis Harmonici illius Infiniti ad Infinitum Denomi-
natorem vel Indicem in Formula Binomiali $\left(1 \pm \frac{\Delta x}{\infty} \right)^{\infty}$, quae Exponentialia
universa eminenter complectitur respectu habito ad innumera Logarithmorum
Systemata. In Hyperbolico Ratio aequalitatis perquam facillime coniectatur

ut su-

(*) Vide Numerum 133., ubi quaedam alia de Infinitatis, prope dixeris, ludis ad-
notantur.

ut supra, si tamen liceret; in ceteris vero ea coniectaretur Proportio ab Aequa-

tione $1 - \frac{\infty'}{M \cdot \infty''} = 0$, videlicet $M = \frac{\infty'}{\infty''}$: quapropter fortassis duo inter

Infinita proposita $\infty' : \infty''$ esse videretur Geometrica Ratio $M : 1$, vel Systematis Moduli ad Unitatem. Sed huiusmodi effingens Infinitatis passiones vereor eximios tot Geometrarum errores, semperque horrens accedo quo tam saepe ii naufragia patiuntur aestuosae imaginationis vi in Infiniti pelagum ducti, tantisque territus male auspicatorum hominum exemplis manum cito retraho utcumque blandius ludentem, cum iamdiu senserim doloso sub cinere perpetuum ignem latere.

112. Varia Seriei Logarithmicae divergentis, aut non admodum convergentis medicamina qui rite intellecta possideat, mirabilia quaedam, nec amplius periculosa in Magnitudinum Finitarum supellestile perscrutabitur. Elegantiora subnecto. Erit primum ex num°. 104.

$$\begin{aligned}
 (1+z)^x = (1)^x & \left(1 + 2 \frac{\left(\frac{z}{z+2}\right)^1}{1} x^1 + 2 \frac{\left(\frac{z}{z+2}\right)^2}{1} x^2 + \frac{4}{3} \frac{\left(\frac{z}{z+2}\right)^3}{1} x^3 + \frac{2}{3} \frac{\left(\frac{z}{z+2}\right)^4}{1} x^4 + \frac{4}{15} \frac{\left(\frac{z}{z+2}\right)^5}{1} x^5 + \right. \\
 & + 2 \frac{\left(\frac{z}{z+2}\right)^3}{3} x^1 + 4 \frac{\left(\frac{z}{z+2}\right)^4}{3} x^2 + 4 \frac{\left(\frac{z}{z+2}\right)^5}{3} x^3 + \frac{8}{3} \frac{\left(\frac{z}{z+2}\right)^6}{3} x^4 + \frac{4}{3} \frac{\left(\frac{z}{z+2}\right)^7}{3} x^5 + \\
 & + 2 \frac{\left(\frac{z}{z+2}\right)^5}{5} x^1 + \frac{46}{9} \frac{\left(\frac{z}{z+2}\right)^6}{5} x^2 + \frac{56}{9} \frac{\left(\frac{z}{z+2}\right)^7}{5} x^3 + \frac{44}{9} \frac{\left(\frac{z}{z+2}\right)^8}{5} x^4 + \frac{76}{27} \frac{\left(\frac{z}{z+2}\right)^9}{5} x^5 + \\
 & + 2 \frac{\left(\frac{z}{z+2}\right)^7}{7} x^1 + \frac{88}{15} \frac{\left(\frac{z}{z+2}\right)^8}{7} x^2 + \frac{3272}{405} \frac{\left(\frac{z}{z+2}\right)^9}{7} x^3 + \frac{11888}{2025} \frac{\left(\frac{z}{z+2}\right)^{10}}{7} x^4 + \frac{2392}{675} \frac{\left(\frac{z}{z+2}\right)^{11}}{7} x^5 \\
 & + 2 \frac{\left(\frac{z}{z+2}\right)^9}{9} x^1 + \frac{1126}{175} \frac{\left(\frac{z}{z+2}\right)^{10}}{9} x^2 + \frac{1692}{175} \frac{\left(\frac{z}{z+2}\right)^{11}}{9} x^3 \quad \&c: \quad \&c: \\
 & + 2 \frac{\left(\frac{z}{z+2}\right)^{11}}{11} x^1 \quad \&c: \quad \&c: \\
 & \&c:
 \end{aligned}$$

Haec Series, quae universis valoribus $-z$, atque Indicis x positivi ac negativi extendi potest nulla difficultate prohibente, sed unius ope mutationis signorum, si concedit elegantiae, legisque nitore primaevae Seriei a directa Binomii Newtoniani resolutione genitae, nihilo tamen minus nec iniucunde progreditur, et valde ipsi praestat ob Coefficientium fidelem comitem Convergentiam, quam saepissime in priore nefas est obtinere. Utriusque vero Seriei, dum consulas terminorum singulorum progressum ac magnitudinem summo opere discrepantis, valor signanter identicus ex eo constabit, quod Coefficientis

$$\begin{aligned}
 \text{ordine prior } 2 \left(\frac{\left(\frac{z}{z+2}\right)^1}{1} + \frac{\left(\frac{z}{z+2}\right)^3}{3} + \frac{\left(\frac{z}{z+2}\right)^5}{5} + \frac{\left(\frac{z}{z+2}\right)^7}{7} + \frac{\left(\frac{z}{z+2}\right)^9}{9} \right. \\
 \left. + \frac{\left(\frac{z}{z+2}\right)^{11}}{11} + \&c: \right) \text{ fit aequalis } L(1+z), \text{ vel } \frac{1(1+z)}{M}, \text{ ceterique}
 \end{aligned}$$

Coeffi-

$$\frac{4}{45} \frac{\left(\frac{z}{z+2}\right)^6}{1} x^6 + \frac{8}{315} \frac{\left(\frac{z}{z+2}\right)^8}{1} x^7 + \frac{2}{315} \frac{\left(\frac{z}{z+2}\right)^8}{1} x^8 + \frac{4}{2835} \frac{\left(\frac{z}{z+2}\right)^9}{1} x^9 + \frac{4}{14175} \frac{\left(\frac{z}{z+2}\right)^{10}}{1} x^{10} + \&c.)$$

$$\frac{8}{15} \frac{\left(\frac{z}{z+2}\right)^8}{3} x^6 + \frac{8}{45} \frac{\left(\frac{z}{z+2}\right)^9}{3} x^7 + \frac{16}{315} \frac{\left(\frac{z}{z+2}\right)^{10}}{3} x^8 + \frac{4}{315} \frac{\left(\frac{z}{z+2}\right)^{11}}{3} x^9 + \&c: \frac{\left(\frac{z}{z+2}\right)^{12}}{3} x^{10}$$

$$\frac{172}{135} \frac{\left(\frac{z}{z+2}\right)^{10}}{5} x^6 + \frac{64}{135} \frac{\left(\frac{z}{z+2}\right)^{11}}{5} x^7 + \&c: \&c: \frac{\left(\frac{z}{z+2}\right)^{12}}{5} x^8 + \frac{\left(\frac{z}{z+2}\right)^{13}}{5} x^9 + \&c: \frac{\left(\frac{z}{z+2}\right)^{14}}{5} x^{10}$$

&c:

&c:

Haec

Coefficientes tali Lege iungantur, ut ordine eorum servato peraequent

$$\frac{(L(1+z))^2}{1 \cdot 2}, \frac{(L(1+z))^3}{1 \cdot 2 \cdot 3}, \frac{(L(1+z))^4}{1 \cdot 2 \cdot 3 \cdot 4}, \frac{(L(1+z))^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}, \text{ et sic deinceps in Infinitum.}$$

113. Nititur altera Series Aequalitati praeostensae inter $L(1+z)$ ac

$$\frac{\left(\frac{z}{1+z}\right)^1}{1} + \frac{\left(\frac{z}{1+z}\right)^2}{2} + \frac{\left(\frac{z}{1+z}\right)^3}{3} + \frac{\left(\frac{z}{1+z}\right)^4}{4} + \frac{\left(\frac{z}{1+z}\right)^5}{5} +$$

$$\frac{\left(\frac{z}{1+z}\right)^6}{6} + \frac{\left(\frac{z}{1+z}\right)^7}{7} + \frac{\left(\frac{z}{1+z}\right)^8}{8} + \frac{\left(\frac{z}{1+z}\right)^9}{9} + \frac{\left(\frac{z}{1+z}\right)^{10}}{10} + \&c:, ex-$$

hibetque, methodi toties repetitae iteratione facta, $(1+z)^x =$

 $(1)^x$

$$\begin{aligned}
(1)^x \times & \left(1 + \frac{\left(\frac{z}{1+z}\right)^1}{1} x^1 + \frac{\left(\frac{z}{1+z}\right)^2}{1.2} x^2 + \frac{\left(\frac{z}{1+z}\right)^3}{1.2.3} x^3 + \frac{\left(\frac{z}{1+z}\right)^4}{1.2.3.4} x^4 + \frac{\left(\frac{z}{1+z}\right)^5}{1.2.3.4.5} x^5 \right. \\
& + \frac{\left(\frac{z}{1+z}\right)^2}{1.2} x^1 + \frac{\left(\frac{z}{1+z}\right)^3}{1.2} x^2 + \frac{\left(\frac{z}{1+z}\right)^4}{1.4} x^3 + \frac{\left(\frac{z}{1+z}\right)^5}{1.3.4} x^4 + \frac{\left(\frac{z}{1+z}\right)^6}{1.2.4.6} x^5 \\
& + \frac{\left(\frac{z}{1+z}\right)^3}{1.3} x^1 + \frac{11\left(\frac{z}{1+z}\right)^4}{1.2.3.4} x^2 + \frac{7\left(\frac{z}{1+z}\right)^5}{1.2.3.4} x^3 + \frac{17\left(\frac{z}{1+z}\right)^6}{1.2.3.4.6} x^4 + \frac{5\left(\frac{z}{1+z}\right)^7}{1.2.3.4.6} x^5 \\
& + \frac{\left(\frac{z}{1+z}\right)^4}{1.4} x^1 + \frac{5\left(\frac{z}{1+z}\right)^5}{1.3.4} x^2 + \frac{5\left(\frac{z}{1+z}\right)^6}{1.4.4} x^3 + \frac{7\left(\frac{z}{1+z}\right)^7}{1.2.4.6} x^4 + \frac{7\left(\frac{z}{1+z}\right)^8}{1.3.6.8} x^5 \\
& + \frac{\left(\frac{z}{1+z}\right)^5}{1.5} x^1 + \frac{137\left(\frac{z}{1+z}\right)^6}{1.3.4.5.6} x^2 + \frac{29\left(\frac{z}{1+z}\right)^7}{1.3.5.6} x^3 + \frac{967\left(\frac{z}{1+z}\right)^8}{1.2.3.4.5.6.8} x^4 + \frac{1069\left(\frac{z}{1+z}\right)^9}{1.2.4.5.6.8.9} x^5 \\
& + \frac{\left(\frac{z}{1+z}\right)^6}{1.6} x^1 + \frac{7\left(\frac{z}{1+z}\right)^7}{1.4.5} x^2 + \frac{469\left(\frac{z}{1+z}\right)^8}{1.2.3.5.6.8} x^3 + \frac{89\left(\frac{z}{1+z}\right)^9}{1.3.4.5.8} x^4 + \frac{19\left(\frac{z}{1+z}\right)^{10}}{1.2.4.4.8} x^5 \\
& + \frac{\left(\frac{z}{1+z}\right)^7}{1.7} x^1 + \frac{363\left(\frac{z}{1+z}\right)^8}{1.4.5.7.8} x^2 + \frac{29531\left(\frac{z}{1+z}\right)^9}{1.2.3.5.6.7.8.9} x^3 + \frac{4523\left(\frac{z}{1+z}\right)^{10}}{1.3.4.5.6.7.9} x^4 & \&c: \\
& + \frac{\left(\frac{z}{1+z}\right)^8}{1.8} x^1 + \frac{761\left(\frac{z}{1+z}\right)^9}{1.5.7.8.9} x^2 + \frac{1303\left(\frac{z}{1+z}\right)^{10}}{1.3.4.6.7.8} x^3 & \&c: \\
& + \frac{\left(\frac{z}{1+z}\right)^9}{1.9} x^1 + \frac{7129\left(\frac{z}{1+z}\right)^{10}}{1.5.7.8.9.10} x^2 & \&c: \\
& + \frac{\left(\frac{z}{1+z}\right)^{10}}{1.10} x^1 & \&c: \\
& \&c:
\end{aligned}$$

quam novam Formulae ipsius Exponentialis evolutionem a divergentiae vizio perpolitam non ingratis arbitror futuram esse Geometris.

114. Consulto praetermissi pseudo-paradoxi imaginem quamdam, quae
tutio-

$$\begin{aligned}
& + \frac{\left(\frac{z}{1+z}\right)^6 x^6}{1.2.3.4.5.6} + \frac{\left(\frac{z}{1+z}\right)^7 x^7}{1.2.3.4.5.6.7} + \frac{\left(\frac{z}{1+z}\right)^8 x^8}{1.2.3.4.5.6.7.8} + \frac{\left(\frac{z}{1+z}\right)^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{\left(\frac{z}{1+z}\right)^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c: \\
& + \frac{\left(\frac{z}{1+z}\right)^7 x^6}{1.2.4.5.6} + \frac{\left(\frac{z}{1+z}\right)^8 x^7}{1.2.3.5.6.8} + \frac{\left(\frac{z}{1+z}\right)^9 x^8}{1.2.3.5.6.7.8} + \frac{\left(\frac{z}{1+z}\right)^{10} x^9}{1.2.3.4.6.7.8.10} \&c: \\
& + \frac{23\left(\frac{z}{1+z}\right)^8 x^6}{1.3.4.5.6.8} + \frac{13\left(\frac{z}{1+z}\right)^9 x^7}{1.4.5.6.8.9} + \frac{29\left(\frac{z}{1+z}\right)^{10} x^8}{1.4.6.7.8.9.10} \&c: \\
& + \frac{\left(\frac{z}{1+z}\right)^9 x^6}{1.2.5.8} + \frac{\left(\frac{z}{1+z}\right)^{10} x^7}{1.2.4.6.8} \&c: \\
& + \frac{3013\left(\frac{z}{1+z}\right)^{10} x^6}{1.2.4.5.6.8.9.10} \&c:
\end{aligned}$$

&c:

quam

tutioris ignara in Algebraicis facultatibus moderaminis laeta proficiensque mihi occurrebat versanti Logarithmicam Seriem primo retroactae Formulae Exponentialis Coefficienti aequalem, et adamussim consonam alteri supra de-

$$\text{scriptae } \frac{\left(\frac{v-1}{v}\right)^1}{1} + \frac{\left(\frac{v-1}{v}\right)^2}{2} + \frac{\left(\frac{v-1}{v}\right)^3}{3} + \frac{\left(\frac{v-1}{v}\right)^4}{4} + \frac{\left(\frac{v-1}{v}\right)^5}{5} \\ + \frac{\left(\frac{v-1}{v}\right)^6}{6} + \frac{\left(\frac{v-1}{v}\right)^7}{7} + \frac{\left(\frac{v-1}{v}\right)^8}{8} + \frac{\left(\frac{v-1}{v}\right)^9}{9} + \frac{\left(\frac{v-1}{v}\right)^{10}}{10}$$

+ &c: = $L_v = \frac{lv}{M}$. Accidit namque, quod videretur nonnullis mirabile

$$\text{dictu, Seriem ipsam abire in } -\frac{1}{0.1} + \frac{1}{0^2.2} - \frac{1}{0^3.3} + \frac{1}{0^4.4} - \frac{1}{0^5.5} + \frac{1}{0^6.6} \\ - \frac{1}{0^7.7} + \frac{1}{0^8.8} - \frac{1}{0^9.9} + \frac{1}{0^{10}.10} - \&c: \text{ tum cum } v = 0, \text{ sive } L_v = L_0.$$

Haec autem Series rite perpenſa plene conſentit expreſſioni $\frac{-L_2}{0} = -\infty$,

atque idcirco nihil abludit a tritiſſima Logometrica veritate. Quod ſi opta-
res plura de iſto Infinito cognoscere ne Algebram putes ſua dona avaram mi-

$$\text{ferrimamque nolle impertiri. Series etenim illa vertitur in } -\left(\frac{\infty^1}{1} - \frac{\infty^2}{2} \right. \\ \left. + \frac{\infty^3}{3} - \frac{\infty^4}{4} + \frac{\infty^5}{5} - \frac{\infty^6}{6} + \frac{\infty^7}{7} - \frac{\infty^8}{8} + \frac{\infty^9}{9} - \frac{\infty^{10}}{10} + \&c:\right),$$

quam eſſe aequalem $-L(\infty + 1)$, ſive $-L\infty$ neminem poſt praecognita latet.

$$\text{Erit itaque } L_0 = -L\infty = -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \right.$$

$$\left.\frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c:\right), \text{ quemadmodum ſuppoſito } v = \infty \text{ Coefficiens ipſe ſup-}$$

peditat, nimirum L_0 aequae ac $L\infty$ Harmonicam ſimul induent Infinita-
tem. Nec ſilentio praetereundum a Serie in num.^o 102. enucleata oriri eam-
dem Logarithmi ad Infinitum Numerum pertinentis expoſitionem, ſcilicet

$$2 \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \&c:\right), \text{ aut vi num. } 109. \text{ po-}$$

$$\text{tius } 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c: . \text{ Prae-}$$

$$\text{terea habetur exinde } L_2 = 2 \left(\frac{1}{3} + \frac{1}{81} + \frac{1}{1215} + \frac{1}{15309} + \frac{1}{177147} \right.$$

+ +

$\rightarrow \frac{1}{1948617} \rightarrow \&c: \rightarrow$), quae Series, Convergentiae quasi miraculum offerens.

in ordinem elegantiores digesta se prodit denuo, qualem in num.^o 103.,

notam formam recipiens $\frac{2}{1.3^1} + \frac{2}{3.3^3} + \frac{2}{5.3^5} + \frac{2}{7.3^7} + \frac{2}{9.3^9} + \frac{2}{11.3^{11}} +$
 $\frac{2}{13.3^{13}} \&c:$, ac aequalitatem, utut diverso charactere, servans perpetuam tam

respectu Series in num.^o 90. prolatae $\frac{1}{1.2^1} + \frac{1}{2.2^2} + \frac{1}{3.2^3} + \frac{1}{4.2^4} + \frac{1}{5.2^5}$

$\rightarrow \frac{1}{6.2^6} + \frac{1}{7.2^7} \&c:$, quam $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9}$

$- \frac{1}{10} \&c:$, cuius contemplationis exemplum locupletissimum Analyseos in

re ipsa dicenda atque ornanda varietatem demonstrat.

115. Recensitis Logarithmorum passionibus praecipuis investigandum proponatur, quid sibi velint, quaeve artificia desiderent Transcendentes expressiones, quas raro, tamen aliquando, Algebra colit $(Lx)^z$, et LLv , $LLLz \&c:$, $(LLx)^v$, $(LLLz)^x \&c:$, $(lx)^z$, $l'l''v$, Llv , llv , $l'l'l''z$, $l'l'l''z$, $ll'l''z$, $l'l'l''z$, ipsarumque Potestates, nec non $z^{\frac{Lx}{Lx}}$, $z^{\frac{LLx}{Lx}}$, $v^{\frac{Llz}{Lx}}$, $x^{\frac{LLv}{Lx}}$ &c:, $(Lx)^{Lz}$, $(lx)^{lz}$, $(lx)^{Lz}$, $(Lx)^{lz}$ &c: &c:, pluribusve Logarithmicis tessellis cumulatae. Parcius vero de iis differendum erit, ne videar plus moli verborum inanique fucati Calculi pompae, quam doctrinae promovendae consulere.

116. Antequam ultra procedam, quo argumenti ratio me vocat, absolum fortasse non erit obiter animadvertere, omnem Siglarum symbolicarum L , l' , $l'' \&c:$ combinationem, vel alternam vel successivam, in eandem Algorithmi formam facillime redigi. Et re quidem vera $l'x = m'Lx = L(x^{m'})$, et pari pacto $l''x = m''Lx = L(x^{m''})$ Systemate in quolibet altero Logarithmorum. Idcirco $l'l''v = m'L(m''Lv) = m'LL(v^{m''}) = L(L(v^{m''}))^{m'}$, quemadmodum contingeret in Systemate ipso Hyperbolico

si pro Lv in LLv scriberetur Potestas a respectivis Systematum Modulis definita $(L(v^{M''}))^{M'}$, quod nihil Calculum perturbat, cum sub speciebus v, x, z &c: universas possibiles variabilium Functiones comprehendi posse non sit qui non videat. Eo facilius Lx transformabitur in $LL(x^M)$, et LLx in $L(Lx)^M$. Consimili ritu $LL'l''x = Lm' Lm'' Lx = Lm' LL(x^{M''}) = LL(Lx^{M''})^{M'}$, ac $l'l''z = m' Lm'' Lz = m' LLL(z^{M''}) = L(LL(z^{M''}))^{M'}$, et denique $l'l''l'''v = m' Lm'' Lm''' Lv = m' Lm'' \times LL(v^{M'''}) = m' LL(L(v^{M'''}))^{M''}$, atque si velis $= L(L(L(v^{M'''}))^{M''})^{M'}$. Vanum igitur foret exempla in posterum mutuari a Logarithmis innumerorum Systematum, quum ad eundem typum Logarithmorum Naturalium sive Hyperbolicorum cetera omnia cudantur.

117. Eiusmodi foedere adeo simul iunguntur Logarithmorum-Logarithmo-Logarithmorum &c: sine limite Logarithmi, ut, quod per se evidens est, ab iis inferiorum graduum resolutiones superiorum servato ordine emergant. Clarius autem hoc patet in Potestatibus C^{LLx} , C^{LLLx} , B^{LLx} , B^{LLLx} , z^{LLx} , z^{LLLx} . Incoepti moliminis specimen primum esto in Basi super omnes singulari Systematis Hyperbolici, habebiturque $C^{LLx} = Lx$, atque insuper $C^{LLLx} = LLx$, quo in ordine infinities etiam continuato Numerus Siglarum L unitate semper minuitur. Eodem progressu $B^{LLx} = C^{(LLx)LB} = (Lx)^{LB} = (Lx)^{\frac{1}{M}}$, $B^{LLLx} = (LLx)^{LB} = (LLx)^{\frac{1}{M}}$, $z^{LLx} = (Lx)^{Lz}$, $z^{LLLx} = (LLx)^{Lz}$, quas Aequationes, istisque simillimas ex propria penu eruendas post praeostensa quisque demonstraturus tempus tereret frustra, decorumque etiam pulvisculum tyronibus invideret.

118. Magnitudini Logarithmico-Exponentiali $(Lx)^z$ in Seriem numero ter-

ro terminorum Infinitam evolvendae studens votum solvere perquam facilli-

me poterit ope Formulae $(1)^z \times \left(1 + \frac{(LLx)^1 z^1}{1} + \frac{(LLx)^2 z^2}{1.2} + \frac{(LLx)^3 z^3}{1.2.3} + \frac{(LLx)^4 z^4}{1.2.3.4} + \frac{(LLx)^5 z^5}{1.2.3.4.5} + \frac{(LLx)^6 z^6}{1.2.3.4.5.6} + \&c. \right)$, vel eloquio sane maximo $1^z \left(1 + \frac{(LLx)z}{\infty} \right)^\infty$. Valor autem $\tau\tilde{u}$ LLx per Lx exhibetur; namque

adaequat $\frac{(Lx)-1^1}{1} - \frac{(Lx)-1^2}{2} + \frac{(Lx)-1^3}{3} - \frac{(Lx)-1^4}{4} + \frac{(Lx)-1^5}{5} - \frac{(Lx)-1^6}{6} + \frac{(Lx)-1^7}{7} - \frac{(Lx)-1^8}{8} + \&c.$, sive commodiore convergentia,

nullaque Seriei divergentia evocata, $2 \left(\frac{((Lx)-1)^1}{((Lx)+1)^1} + \frac{((Lx)-1)^3}{((Lx)+1)^3} + \frac{((Lx)-1)^5}{((Lx)+1)^5} + \frac{((Lx)-1)^7}{((Lx)+1)^7} + \frac{((Lx)-1)^9}{((Lx)+1)^9} + \&c. \right)$, aut $\frac{((Lx)-1)^1}{Lx} + \frac{((Lx)-1)^2}{Lx} + \frac{((Lx)-1)^3}{Lx} + \frac{((Lx)-1)^4}{Lx} + \frac{((Lx)-1)^5}{Lx} + \frac{((Lx)-1)^6}{Lx} + \frac{((Lx)-1)^7}{Lx} + \&c.$ in Infinitum. Liquet has Analyticas expressiones a

prius reperto valore Lx, nimirum ex Logarithmo inferioris gradus pendere, de quo inveniundo iam Canones prostant tanta praediti subtilitate, sagacique Calculi compendio; ut, si prius novissent optime de Logometria meriti Tabularum Authores, immanem et improbum non parum sublevassent laborem, nec iniuria adseruerit universae Mathefeos Historiographus in II°. Volumine Numerorum omnium Logarithmos a Logarithmis solummodo obtineri potuisse priorum decem Seriei naturalis Arithmeticae Numerorum, praesertim Sharpiana methodo computatis (*).

119. Dum

(*) Utinam emendatiores Logarithmorum Gardineri Tabulas aliasve novorum Canonum auxilio supputandas Tabulis aeneis incidi aliquando curae habeant divites Philomathematici, prouti Iacobus Fergussonius optavit in *Selectis Mathefeos Exercitationibus* Londini editis anno MDCCLXXIII!

119. Dum Index Potestatis $(Lx)^z$ fuerit integer Numerus positivus enascitur more Neutoni simplicior expressio $(Lx)^z = (1 + (Lx) - 1)^z =$

$$1 + \frac{z}{1} \cdot (Lx) - 1 + \frac{z \cdot z - 1}{1 \cdot 2} \cdot (Lx)^2 - 1 + \frac{z \cdot z - 1 \cdot z - 2}{1 \cdot 2 \cdot 3} \cdot (Lx)^3 - 1 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot (Lx)^4 + \&c., \text{ utpote quae Progressionis forma ne-}$$

cessario abrumpetur. Nihilo tamen minus idem ea significabit ac superius proposita Series Limitem nesciens, et suo finu complectens sub Infinitiformis Functionis specie iuxta numerum 47. singulares etiam Potestatum perfectarum valores. Ipsamet Formula Logarithmico-Exponentialis $(Lx)^z$ evolvi insuper posset in Seriem numero terminorum Infinitam absque eo quod

$$\text{locum habeat } Lx. \text{ Revera } (Lx)^z = (L(1 + v))^z = \left(\frac{v^1}{1} - \frac{v^2}{2} + \frac{v^3}{3} - \frac{v^4}{4} + \frac{v^5}{5} - \frac{v^6}{6} + \frac{v^7}{7} - \frac{v^8}{8} + \frac{v^9}{9} - \frac{v^{10}}{10} + \&c. \right)^z; \text{ quamobrem, si Theo-}$$

rematum praecognitorum adiumento *exponentialiter* hoc Infinitinomium tractetur, parique ratione unum alterumve ex aliis Infinitinomiis, quae diversimode composita eundem tamen adaequant Lx , profluat Series artificio tali digesta, ut eliminato penitus Lx nil aliud praeter Potestates $\approx z$, et v seu $x - 1$, instar tritissimae Formulae in num^o. 44. prolatae, comprehendere va-

leat. Interim non videtur supervacaneum monere, quod de $(Lx)^z$ ratiocinationem ineuntes simul quoque de LLx nobis etiam invitis egisse contigerit; ita ut prorsus inutile foret prolixiora de hisce repetere potius quam commentari. Mirabilis equidem Analyseos, sibi que semper constans natura rursus elucet dum posito $x = 1$, $x = 0$, ac deinceps $= \infty$ evadit $LLx = Lo$, vel $= L(-\infty)$, aut denique $= L\infty$. Nam prima ex hypothese oritur triplici in Serie, de quibus praecedens numerus nonnulla perhibet,

$$\text{vel } \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \&c. \right), \text{ five } 2 \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \&c. \right), \text{ aut } \left(\frac{\infty^1}{1} - \frac{\infty^2}{2} + \frac{\infty^3}{3} - \frac{\infty^4}{4} + \&c. \right)$$

$\frac{\infty^4}{4} + \frac{\infty^5}{5} - \frac{\infty^6}{6} + \frac{\infty^7}{7} - \&c:) = -L\infty$, quemadmodum numerus
 habet 114.; praetereaue hypothesis tertia parit $(\frac{\infty^1}{1} - \frac{\infty^2}{2} + \frac{\infty^3}{3} -$
 $\frac{\infty^4}{4} + \frac{\infty^5}{5} - \frac{\infty^6}{6} + \frac{\infty^7}{7} - \frac{\infty^8}{8} + \&c:)$, vel $2(1 + \frac{1}{3} + \frac{1}{5} +$
 $\frac{1}{7} + \frac{1}{9} + \&c:)$, five tandem $(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} +$
 $\frac{1}{8} + \&c:)$, cuius aequae ac Seriei praecedentis veritas a superius exposi-
 tis positivo Signo acceptis lucem quammaximam mutuatur. Quod vero evo-
 lutionem respicit $\approx L(-\infty)$ obiter sciant Analystae a nostris Formulis derivari
 $-\frac{\infty^1}{1} - \frac{\infty^2}{2} - \frac{\infty^3}{3} - \frac{\infty^4}{4} - \frac{\infty^5}{5} - \frac{\infty^6}{6} - \frac{\infty^7}{7} - \frac{\infty^8}{8} - \&c: =$
 $L(1 - \infty)$, Seriem scilicet admodum falsam propter universos terminos di-
 vergentes, sed symbolum residuarum, quae eliciuntur, Serierum $2(1 + \frac{1}{3}$
 $+ \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \&c:)$, et $(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$
 $+ \frac{1}{8} + \&c:)$, quod equidem altius, quam poterit, animo reponendum.

120. Usque a num°. 92. elegantem conversionem ostendimus Exponen-
 tialis Magnitudinis z^{Lx} in alteram ipsi aequalem x^{Lz} : attamen proderit huic
 spartae promovendae nunc novam curam paullisper impendere. Quum enim
 $z^{Lx} = (1)^{Lx} \cdot (1 + \frac{(Lz)^1 (Lx)^1}{1} + \frac{(Lz)^2 (Lx)^2}{1.2} + \frac{(Lz)^3 (Lx)^3}{1.2.3} + \frac{(Lz)^4 (Lx)^4}{1.2.3.4}$
 $+ \frac{(Lz)^5 (Lx)^5}{1.2.3.4.5} + \frac{(Lz)^6 (Lx)^6}{1.2.3.4.5.6} + \frac{(Lz)^7 (Lx)^7}{1.2.3.4.5.6.7} + \&c:)$, simulque $x^{Lz} = (1)^{Lz} \cdot$
 $(1 + \frac{(Lx)^1 (Lz)^1}{1} + \frac{(Lx)^2 (Lz)^2}{1.2} + \frac{(Lx)^3 (Lz)^3}{1.2.3} + \frac{(Lx)^4 (Lz)^4}{1.2.3.4} + \frac{(Lx)^5 (Lz)^5}{1.2.3.4.5}$
 $+ \frac{(Lx)^6 (Lz)^6}{1.2.3.4.5.6} + \frac{(Lx)^7 (Lz)^7}{1.2.3.4.5.6.7} + \&c:)$, idcirco, ob unicum saltem valorem
 $z^{Lx} = x^{Lz}$, nil mirum si z^{Lx} variabilibus ordine inversis peraequet x^{Lz} ;
 quod

quod etiam comprobant notissima Binomia $\left(1 + \frac{Lz \cdot Lx}{\infty}\right)^{\infty} = z^{Lx}$, nec non $\left(1 + \frac{Lx \cdot Lz}{\infty}\right)^{\infty} = x^{Lz}$. Quilibet vel non monitus summa facilitate intuebitur, Series illas Infinitas a Siglis Logarithmicis liberari dummodo vice Lx et Lz homologa subrogentur Infinitinomia. Versis autem Formulis Exponentialibus z^{Lx} , ac x^{Lz} in ipsis aequales $C^{Lx \cdot Lz}$, $C^{Lz \cdot Lx}$, sive potius $(C^{Lx})^{Lz}$, $(C^{Lz})^{Lx}$ nulla Serierum novitas oriretur.

121. Eodem pacto Transcendens Potestas $(Lx)^{Lz}$ abit in $(1)^{Lz}$.

$$\left(1 + \frac{(LLx)^1(Lz)^1}{1} + \frac{(LLx)^2(Lz)^2}{1.2} + \frac{(LLx)^3(Lz)^3}{1.2.3} + \frac{(LLx)^4(Lz)^4}{1.2.3.4} + \frac{(LLx)^5(Lz)^5}{1.2.3.4.5} + \frac{(LLx)^6(Lz)^6}{1.2.3.4.5.6} + \frac{(LLx)^7(Lz)^7}{1.2.3.4.5.6.7} + \&c.\right)$$
, scilicet, neglecto Factore, in aequipollentem $\left(1 + \frac{LLx \cdot Lz}{\infty}\right)^{\infty}$, quarum in alterutra expressione fas erit, si libeat, Infinitinomia singillatim aequalia his Lz et LLx sufficere. Nec facilius res voto cessisset immutando Potestatem propositam $(Lx)^{Lz}$, ipsamque diversa sub specie $C^{Lz \cdot LLx}$, vel $(C^{Lz})^{LLx}$, sive $(C^{LLx})^{Lz}$ evolvendo, quam Formulae datae simplicissimam transformationem universaliorem effici posse in ceteris eo magis compositis Logarithmico-Exponentialibus quisque per se contemplatur.

122. Exempla desumpta a Formulis altioris ordinis $(LL\Delta)^{\Phi}$, $(LLL\Delta)^{\Phi}$ &c.; ubi Δ , ac Φ Functiones quaslibet variabilium designare velim, nullam quoque difficultatem praeseferent; eae namque vertuntur in $C^{\Phi \cdot LLL\Delta}$, et $C^{\Phi \cdot LLLL\Delta}$ &c.; vel $(C^{\Phi})^{LLL\Delta}$, ac $(C^{\Phi})^{LLLL\Delta}$ &c.; in Infinitum simillimo continuato progressu. Inde fit $(LL\Delta)^{\Phi} = (1)^{\Phi} \times \left(1 + \frac{(LLL\Delta)^1 \Phi^1}{1} + \frac{(LLL\Delta)^2 \Phi^2}{1.2} + \frac{(LLL\Delta)^3 \Phi^3}{1.2.3} + \frac{(LLL\Delta)^4 \Phi^4}{1.2.3.4} + \frac{(LLL\Delta)^5 \Phi^5}{1.2.3.4.5} + \frac{(LLL\Delta)^6 \Phi^6}{1.2.3.4.5.6} + \frac{(LLL\Delta)^7 \Phi^7}{1.2.3.4.5.6.7} + \&c.\right)$

$\frac{(LLL\Delta)^7\Phi^7}{1.2.3.4.5.6.7} + \&c:) = \left(1 + \frac{\Phi(LL\Delta)}{\infty}\right)^\infty$. Nec ab eadem explicandi ratio-

ne divergit altera expressio $(LLL\Delta)^\Phi$, cum abeat in $(1)^\Phi \times$

$$\left(1 + \frac{(LLLL\Delta)^1\Phi^1}{1} + \frac{(LLLL\Delta)^2\Phi^2}{1.2} + \frac{(LLLL\Delta)^3\Phi^3}{1.2.3} + \frac{(LLLL\Delta)^4\Phi^4}{1.2.3.4} + \frac{(LLLL\Delta)^5\Phi^5}{1.2.3.4.5} + \frac{(LLLL\Delta)^6\Phi^6}{1.2.3.4.5.6} + \frac{(LLLL\Delta)^7\Phi^7}{1.2.3.4.5.6.7} + \&c: \right), \text{ aut}$$

$(1)^\Phi \left(1 + \frac{(LLLL\Delta)\Phi}{\infty}\right)^\infty$ si ne Factorem quidem 1^Φ praetermittere libeat.

123. Ad hoc autem, ut Logarithmo-Logarithmicis Formulis lux quaedam maior non defuit, facile dictum est $\tau\delta$ LLLv peraequare Seriem numero terminorum Infinitam

$$\frac{(LLv)-1^1}{1} - \frac{(LLv)-1^2}{2} + \frac{(LLv)-1^3}{3} - \frac{(LLv)-1^4}{4} + \frac{(LLv)-1^5}{5} - \frac{(LLv)-1^6}{6} + \frac{(LLv)-1^7}{7} - \frac{(LLv)-1^8}{8} + \&c: . \text{ Quinim-}$$

mo ipsummet LLLv ex praecostensis redigitur in $2 \left(\frac{(LLv)-1^1}{(LLv)+1} + \frac{(LLv)-1^3}{(LLv)+1} + \frac{(LLv)-1^5}{(LLv)+1} + \frac{(LLv)-1^7}{(LLv)+1} + \&c: \right)$, aut, dum in ditif-

$$\frac{(LLv)-1^1}{(LLv)+1} + \frac{(LLv)-1^3}{(LLv)+1} + \frac{(LLv)-1^5}{(LLv)+1} + \frac{(LLv)-1^7}{(LLv)+1} + \&c: , \text{ aut, dum in ditif-}$$

fima Analyseos penu feligatur altera non minus elegans Series superius inter

$$\text{inventas, obviam veniet } LLLv = \frac{(LLv)-1^1}{LLv} + \frac{(LLv)-1^2}{LLv} +$$

$$\frac{(LLv)-1^3}{LLv} + \frac{(LLv)-1^4}{LLv} + \frac{(LLv)-1^5}{LLv} + \frac{(LLv)-1^6}{LLv} +$$

$$\frac{(LLv)-1^7}{LLv} + \&c: ; \text{ nec singularem meretur mentionem toties confecta ob-}$$

servatio Logarithmos ordinis superioris ab universis inferiorum graduum Logarithmis indissolubili necessitate quaesitam evolutionem recipere. Idem de LLLLz dicendum; idem de quacumque Functione pro z aut v suffecta, et

Logarithmica quotlibuerit Signa consequente; idem si variabilibus constantes magnitudines promiscue substituerentur; idem denique generatim asserendum quando omnia Logarithmorum Symbola instar numeri 119. eliminare prolixo, sed neutiquam difficili, calculo nobis animus foret.

124. Iterum prospectanda redit in LLLv, iteratis hypothesibus $v = 1$, $v = 0$, $v = \infty$, Serierum indoles exinde genitarum $\frac{\infty^1}{1} - \frac{\infty^2}{2} + \frac{\infty^3}{3} - \frac{\infty^4}{4} + \frac{\infty^5}{5} - \frac{\infty^6}{6} + \frac{\infty^7}{7} - \&c.$, aut $\frac{(L(-\infty))^1}{1} - \frac{(L(-\infty))^2}{2} + \frac{(L(-\infty))^3}{3} - \frac{(L(-\infty))^4}{4} + \frac{(L(-\infty))^5}{5} - \frac{(L(-\infty))^6}{6} + \frac{(L(-\infty))^7}{7} - \&c.$, vel $\frac{\infty^1}{1} - \frac{\infty^2}{2} + \frac{\infty^3}{3} - \frac{\infty^4}{4} + \frac{\infty^5}{5} - \frac{\infty^6}{6} + \frac{\infty^7}{7} - \&c.$

Ceteris autem Seriebus usitato Algorithmo rursus subactis nanciscemur hypothesium ordinem ipsum servantes $2 \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \&c. \right)$,

atque $\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \&c. \right)$, quod oculi ictu

experiri operis est nec periculosi nec summo ingenio vincendi. Quibus ingeminatorum Logarithmorum symptomata, Infinitatisque consequentes ludos meliori omine curae sit introspicere, iis haec paucula voveo non ingloriae originis additamenta. Esto in Seriem Logarithmicis carentem Signis evol-

venda Functio Transcendens $LL(1+z)$ vel $L\left(\frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \&c.\right)$. Rite confectis in Infinitinomio eisdem,

quae toties Magnitudinem Binomiale[m] versantes inivimus, emerget Analyticum Tetragonum elegantia ac Legis simplicitate praestantissimum. Fiet namque

$$LL(1+z)$$

$$\begin{aligned}
 LL(1+z) = & -1 + \frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \&c: \\
 & - \frac{1}{2} + \frac{z^1}{1} - \frac{z^2}{1} + \frac{5z^3}{6} - \frac{17z^4}{24} + \frac{37z^5}{60} - \frac{197z^6}{360} + \&c: \\
 & - \frac{1}{3} + \frac{z^1}{1} - \frac{3z^2}{2} + \frac{5z^3}{3} - \frac{5z^4}{3} + \frac{97z^5}{60} - \frac{559z^6}{360} + \&c: \\
 & - \frac{1}{4} + \frac{z^1}{1} - \frac{2z^2}{1} + \frac{17z^3}{6} - \frac{27z^4}{8} + \frac{37z^5}{10} - \frac{467z^6}{120} + \&c: \\
 & - \frac{1}{5} + \frac{z^1}{1} - \frac{5z^2}{2} + \frac{13z^3}{3} - \frac{73z^4}{12} + \frac{227z^5}{30} - \frac{1579z^6}{180} + \&c: \\
 & - \frac{1}{6} + \frac{z^1}{1} - \frac{3z^2}{1} + \frac{37z^3}{6} - \frac{241z^4}{24} + \frac{847z^5}{60} - \frac{1301z^6}{72} + \&c: \\
 & \&c: \quad \&c: \quad \&c: \quad \&c: \quad \&c: \quad \&c: - \frac{2071z^6}{120} + \&c: \\
 & \&c: .
 \end{aligned}$$

Huiusce Seriei prior Linea verticalis est Infinitum Harmonicum negativum, five Lo; secunda est Series Numerorum aequalium z , quorum Summa Magnitudinis pariter Infinitae, scilicet $\infty \cdot z$; verumtamen sequentium Lex Linearum haecenus iacet obscura (*). Nihilo tamen minus latens Lex ista detegi facile poterit dummodo retrogredi non pigeat, atque animadvertere priorem

Lineam huiusmodi Seriem esse, quae gaudeat Termino Generali $\frac{1}{x}$ vel

$\frac{1}{1} \cdot x^{-1}$; secundam ex Lineis verticaliter dispositis Coefficientes habere numericos

Y 2

1 1
0

(*) Primus, qui docuit in Trigonis, Tetragonis &c: Numerorum passionem memoriae commendare, fuit percelebris suo *Triangulo Arithmetico* summus ille germanae Veritatis venator Blaius Pascalius. Consule initium Voluminis V.ⁱ eius Operum nuperrimae Editionis Hagae-Comitum MDCCLXXIX. editorum.

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \quad \&c: \quad \left| \quad \text{Terminus Generalis } \frac{1}{1} \cdot x^0 \quad , \right.$$

quorum prima differentia nullefcit. Tertiae autem Lineae Coefficientes Seriem conftituunt Inaequalium Arithmeticam

$$\begin{array}{cccccc} \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \quad \&c: \quad \left| \quad \text{Term. Gen: } \frac{1}{1.2} \cdot x^1 \quad , \right.$$

cuius terminorum idcirco primas omnes norunt peraequari, aut fecundas evanefcere differentias. In quarta Linea contingit, inftar Numerorum Triangularium integrorum, tertiae nullitas differentiae, veluti fequens Typus offendit

$$\begin{array}{cccccc} \frac{1}{3} & \frac{5}{6} & \frac{5}{3} & \frac{17}{6} & \frac{13}{3} & \frac{37}{6} \\ \frac{3}{6} & \frac{5}{6} & \frac{7}{6} & \frac{9}{6} & \frac{11}{6} & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & \\ 0 & 0 & 0 & 0 & & \end{array} \quad \&c: \quad \left| \quad \text{Term. Gen: } \frac{1}{1.2.3} x^2 + \frac{1}{1.2.3} \quad . \right.$$

Denique quintae, sextae, feptimae &c: Linearum verticalium quoad Coefficientes affectio ita fe habet, ut aequales fiant, veluti in Numeris integris Pyramidalibus, Triangulo-Triangularibus &c:, differentiae ordinum tertii, quarti, quinti &c:, aut, quod eodem redit, quartae, quintae, sextae &c: differentiae in nihilum abeant. Et re quidem vera unico quafi profpectu univerfa, quae diximus, confpici poffunt.

$\frac{1}{4}$	$\frac{17}{24}$	$\frac{5}{3}$	$\frac{27}{8}$	$\frac{73}{12}$	$\frac{241}{24}$	&c:	Term. Gen: $\frac{1}{1.2.3.4} \cdot x^3 +$
	$\frac{11}{24}$	$\frac{23}{24}$	$\frac{41}{24}$	$\frac{65}{24}$	$\frac{95}{24}$		$\frac{1}{1.2.3} \cdot x^1 + \frac{1}{1.2.3.4}$
	$\frac{12}{24}$	$\frac{18}{24}$	$\frac{24}{24}$	$\frac{30}{24}$			
		$\frac{6}{24}$	$\frac{6}{24}$	$\frac{6}{24}$			
			0	0			

$\frac{1}{5}$	$\frac{37}{60}$	$\frac{97}{60}$	$\frac{37}{10}$	$\frac{227}{60}$	$\frac{847}{60}$	&c:	Term. Gen: $\frac{1}{1.2.3.4.5} \cdot x^4 +$
	$\frac{25}{60}$	$\frac{60}{60}$	$\frac{125}{60}$	$\frac{232}{60}$	$\frac{393}{60}$		$\frac{1}{1.2^2.3} \cdot x^2 + \frac{1}{1.2.3.4} \cdot x^1 +$
	$\frac{35}{60}$	$\frac{65}{60}$	$\frac{107}{60}$	$\frac{161}{60}$			$\frac{1}{1.3.5}$
		$\frac{30}{60}$	$\frac{42}{60}$	$\frac{54}{60}$			
			$\frac{12}{60}$	$\frac{12}{60}$			
				0			

$\frac{1}{6}$	$\frac{197}{360}$	$\frac{559}{360}$	$\frac{467}{120}$	$\frac{1579}{180}$	$\frac{1301}{72}$	$\frac{2071}{120}$	&c:	Term. Gen: $\frac{1}{1.2.3.4.5.6} \cdot x^5 +$
	$\frac{137}{360}$	$\frac{362}{360}$	$\frac{842}{360}$	$\frac{1757}{360}$	$\frac{3347}{360}$	$\frac{4912}{360}$		$+ \frac{1}{1.2^2.3^2} \cdot x^3 +$
	$\frac{225}{360}$	$\frac{480}{360}$	$\frac{915}{360}$	$\frac{1590}{360}$	$\frac{2565}{360}$			$\frac{1}{1.2^4.3} \cdot x^2 + \frac{29}{1.3.4.5.6} \cdot x^1 +$
	$\frac{255}{360}$	$\frac{435}{360}$	$\frac{675}{360}$	$\frac{975}{360}$				$+ \frac{13}{1.3.4.5.6}$
		$\frac{180}{360}$	$\frac{240}{360}$	$\frac{300}{360}$				
			$\frac{60}{360}$	$\frac{60}{360}$				
				0				

Iuvat

Iuvat ulterius ad exercendam Geometrarum candidatorum mentem subiungere, eas Series, ut aiunt, Algebraicas five Progressiones Numerorum Figuratorum super expositas non solummodo a notissimae Seriei Harmonicae termino respectivo singulas inchoari, sed etiam termino eidem constanter aequales fieri postremas, inter se pares, uniuscuiusque Seriei Differentias. Nul-
lus itaque dubito quin luculenter pateat universorum, de quibus sermo,

sive numero Coefficientium eo magis praecedentibus $\sum \frac{x}{x}$ ac $\sum x^\infty$ Infinitos

esse valores. Summis Serierum verticalium inventis cogetur Analyticum Te-

tragonum ipsum in formam $Lo + \frac{\infty \cdot z^1}{1^2} - \frac{\infty^2 \cdot z^2}{2^2} + \frac{\infty^3 \cdot z^3}{3^2} - \frac{\infty^4 \cdot z^4}{4^2} +$
 $\frac{\infty^5 \cdot z^5}{5^2} - \frac{\infty^6 \cdot z^6}{6^2} \&c: = LL(1 + z)$, cuius Progressionis elegantia opitulan-

tibus Terminis Generalibus non difficiliter adinvenitur. Formulae etenim

Summarum respective ita se habent „ $\sum x^{-1} \cdot z^0, \frac{x^1 \cdot z^1}{1}, \frac{(x^2 + x)}{4} z^2,$

$\frac{(x^3 - \frac{3}{4} x^2 + \frac{11}{4} x)}{9} z^3, \frac{(x^4 - \frac{14}{3} x^3 + \frac{32}{3} x^2 - 3x)}{16} z^4$
9 16 &c:

&c: „. Nec fileam quod, dum de LL2 reperiendo ratiocinationem inire liberet, tam primam verticalem, quam horizontalem priorem Tetragoni Lineam praeter — 1, in Harmonicam consimilem Progressionem satis iucunde tunc verfas apparere, alternis unam, successivis aliam terminorum Signis distinctam.

125. Quaenam igitur in Matheseos penetralibus arcana absconditur ratio, vi cuius primus tantum terminus Algebraici illius Tetragoni a sociorum communi regula veluti exlex abludat? En Problema ab Analystis solvendum. Solvat autem quisquis ab inventa Lege recedere Seriem ipsam Harmonicam putet, non ego qui rite, recteque, et summo iure coniungi ceteris

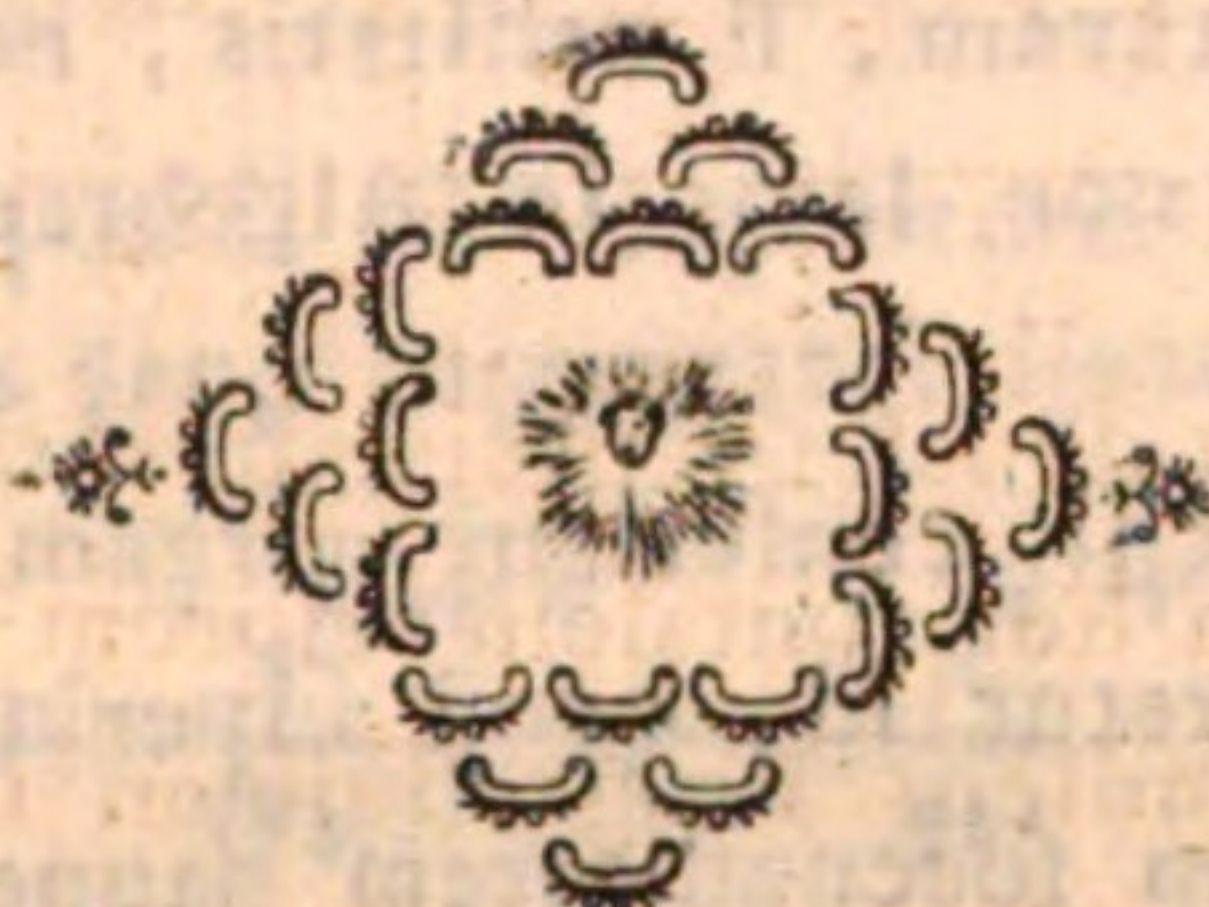
Progressionibus Figuratis, quarum vel primae, vel n^{simae} , vel ∞^{simae} Differentiae sint nullae, eam arbitrer, cuius Differentiae nusquam evanescant, quinimmo eo magis discrepent in Infinitum, unamque ita praebeant Coefficientium congeriem quasi Serierum Algebraicarum Limitem, omnibus evacuatis possibilium casuum homologis combinationibus cumulativum refertum.

Hoc

Hoc luculenter constabit si periculum facere Differentiarum progressus in

Harmonica Serie $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \&c.$, quae unica Summa-

tionem respuit Algebraicam, sed analogiam Termini Generalis praesefert, tyrones non horreant, dum nos properantes meliora sequemur, ne tot nugis quisquiliisque defatigata Lectorum patientia hi tergum vertant, et ad ea, quae fortasse maiori usui fuerint, longius ire recusent.



C A P U T IV.

THEOREMATA COMMUNIA UTRISQUE QUANTITATIBUS
EXPONENTIALIBUS ET LOGARITHMICIS.

126. **M**agnitudines Exponentiales, ac Logarithmos multa habere communiter ad rem ipsam facientia, atque ex eodem hausta veluti fonte non erit qui serio ambigere possit. Universa etenim argumentorum copia, in quibus hucusque sumus versati, fidem facit luculentissimam omne in Logometria punctum tulisse eum, qui vel nusquam, vel rarius a Logarithmis Exponentialia seiunxerit. Sicuti autem, ne dicendorum ordo perturbaretur rerumque conglomerata diversitas obscuritatem pareret, nonnulla aut vix delibavi de Logarithmis et Exponentialibus agens, aut nominatim postmodum illustranda reieci, ea hic in unum colligere tum decreveram, quae Theoriae, ac Praxi Logometricae magis intelligendae conducere, spe fretus nova quaedam inserturum, si utilitatis, non certe iucunditatis expertia. Accedit etiam ratio non levis in Algebrae Finitorum decus sequentia adiiciendi, ne, dum silentii sacramento quae adhuc recensenda maneant cohibuerim, aliquis Aristarchi censoriam virgam assumptus me deliberato animo praeteriisse interpretaretur id, quod superius confessam Formulae Binomii Newtoniani omnimodam foecunditatem pauperiorem factam, seu minus revera habentem actu detexisset.

127. Aequatio illa Exponentialis, quam obiter perspeximus in num^o. 37^{mo}. novo aspectu consideranda nunc redit. Alterutram primi ipsius Aequationis membri ibidem traditam formam $x^x - \left(\frac{x^2 + x}{1 \cdot 2}\right) x^{x-1} +$

$$\left(\frac{3x^4 + 2x^3 - 3x^2 - 2x}{1 \cdot 4 \cdot 6}\right) x^{x-2} - \left(\frac{x^6 - x^5 - 3x^4 + x^3 + 2x^2}{1 \cdot 2 \cdot 4 \cdot 6}\right) x^{x-3} +$$

$$\left(\frac{15x^8 - 60x^7 - 10x^6 + 192x^5 - 25x^4 - 180x^3 + 20x^2 + 48x}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 8}\right) x^{x-4} \dots\dots\dots$$

$$= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \dots\dots\dots x - 3 \cdot x - 2 \cdot x - 1 \cdot x = 0,$$

vel x^x

$$\text{vel } x^x = \left(\frac{x \cdot x + 1}{1 \cdot 2} \right)^{x-1} x + \left(\frac{x-1 \cdot x \cdot x + 1 \cdot 3x + 2}{1 \cdot 4 \cdot 6} \right)^{x-2} x^2 +$$

$$\left(\frac{x-2 \cdot x-1 \cdot x \cdot x + 1 \cdot x + 1}{1 \cdot 2 \cdot 4 \cdot 6} \right)^{x-3} x^3 +$$

$$\left(\frac{x-3 \cdot x-2 \cdot x-1 \cdot x \cdot x + 1 (15x^3 + 15x^2 - 10x - 8)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 8} \right)^{x-4} x^4 \dots \mp (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots$$

$x-4 \cdot x-3 \cdot x-2 \cdot x-1 \cdot x) = 0$, laboriosi quidem, sed non involuti Calculi fructum, eximia atque iam recensita comitatur proprietas, quod toties reapse evanescat Aequatio, quoties variabilis x adaequet Numerum integrum positivum. Ista ne mysterium sapiat singularis affectio auctor sum, namque a notissimis Combinationum canonibus simplicem ducit originem; verumtamen eo magis mirifica videretur, si primum libuisset Aequationem illam Logometrico stylo proponere sub forma difficilioris indaginis

$$+ \frac{x^{x+4}}{384} - \frac{x^{x+3}}{32} + \frac{83x^{x+2}}{576} - \frac{77x^{x+1}}{240} + \frac{403x^x}{1152} - \frac{5x^{x-1}}{32} +$$

$$\frac{x^{x-2}}{288} - \frac{x^{x-3}}{120} \dots \mp (x \cdot x - 1 \cdot x - 2 \dots$$

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 0$, sive potius ab Exponentialibus ad Logarithmos regrediendo $L2 + L3 + L4 + L5 \dots + L(x-2) + L(x-1)$

$$+ Lx = L \pm \left(\dots \frac{x^{x+4}}{384} - \frac{x^{x+3}}{32} + \frac{83x^{x+2}}{576} - \frac{77x^{x+1}}{240} + \frac{403x^x}{1152} - \frac{5x^{x-1}}{32} + \frac{x^{x-2}}{288} - \frac{x^{x-3}}{120} \dots \right).$$

Hunc in modum proposita Aequatio Logarithmico-Exponentialis, nisi prior praecognita fuerit, a qua deducatur, geometrica ingenia etiam sagaciora fortasse torqueret (*).

Z

128.

(*) Nititur haec Theoria inventis de Summa Potestatum x^n Numerorum naturaliter progredientium, cuius argumenti elegantissima prodiderunt Eulerus in Capite decimo Voluminis I. *Introductionis*, et in Parte secunda *Institutionum Calculi Differentialis*, nec non nuperrime Bordoisius in *Commentariis Academiae Bruxellensis* relatis ad annum proxime elapsam MDCCLXXX.

128. Numero infinitas Systematum Logarithmicorum Bases variis in hypothesibus perscrutaturi initio fingamus Exponentialis Magnitudinis $(1+z)^x$ Basin $1+z$ nulescere, quod idem est atque affirmare $z = -1$. Dum si-

mul Index x evanescat, Series Exponentialis vertetur in $(1)^0 \cdot \left(1 + \frac{(Lo)^1 o^1}{1} + \frac{(Lo)^2 o^2}{1.2} + \frac{(Lo)^3 o^3}{1.2.3} + \frac{(Lo)^4 o^4}{1.2.3.4} + \frac{(Lo)^5 o^5}{1.2.3.4.5} + \frac{(Lo)^6 o^6}{1.2.3.4.5.6} + \frac{(Lo)^7 o^7}{1.2.3.4.5.6.7} + \&c.\right)$, nempe $(1-1)^0 = (0)^0 = 1 - \frac{\infty^1 o^1}{1} + \frac{\infty^2 o^2}{1.2} - \frac{\infty^3 o^3}{1.2.3} + \frac{\infty^4 o^4}{1.2.3.4} - \frac{\infty^5 o^5}{1.2.3.4.5} + \frac{\infty^6 o^6}{1.2.3.4.5.6} - \frac{\infty^7 o^7}{1.2.3.4.5.6.7} + \&c.$, Signo ∞ Infinitum

Harmonicum denotante. Erit igitur $(0)^0 = (1-0)^\infty = 1^\infty = 1$, quod et Algebrae Finitorum doctrinam, et superiorum omnium Theorematum comprobat veritatem. Non autem dedignentur qui nimium in Infinito confidunt moneri, futile quondam fuisse nonnullorum Geometrarum Effatum, semper Magnitudinem definitam a Nihilo Infinites repetito veluti procreari:

Aequatio etenim $(0)^0 = 1$ alteram gignit $oLo = L1$, vel $\mp o.\infty = o$, quae non repugnat conclusio indeterminatae Aequalitati $o.\infty = n$, penes emunctae naris Analystas omnium possibilium valorum, ideoque et Nullitatis, capaci. Digna equidem observatione est ac prope cedro linenda singularis affectio Infinitinomii $1 - \frac{x^1.\infty^1}{1} + \frac{x^2.\infty^2}{1.2} - \frac{x^3.\infty^3}{1.2.3} + \frac{x^4.\infty^4}{1.2.3.4} - \frac{x^5.\infty^5}{1.2.3.4.5} + \frac{x^6.\infty^6}{1.2.3.4.5.6} - \&c. = 0$ ex num^o. 38., quemcumque habeat x va-

lorem finitum positivum; namque, retenta sub Signo ∞ Harmonicae Infinitatis significatione, repente Infinitinomium ipsum fit 1 vixdum x in Nihilum abeat. Proprietas ostensa $o.Lo = o$. — ∞ Harm: $= o = o.\infty$ Har: $= oLo$ initio subsequenteris numeri confirmabitur; sed interim proderit ad Theoriae huius amplificationem ita quoque ratiocinari. Esto $o.Lo$, vel $-o.\infty =$

$-n$, evadetque $o^0 = 1 - \frac{n}{1} + \frac{n^2}{1.2} - \frac{n^3}{1.2.3} + \frac{n^4}{1.2.3.4} - \frac{n^5}{1.2.3.4.5} + \frac{n^6}{1.2.3.4.5.6} - \&c. = C^{-n} = 1$, cui Aequationi satisfieri ullo modo

nequit

nequit nisi $n = 0$. Valde magis haec multiplex demonstratio Theorematis elegantissimi $0 \cdot \infty$ Harm: $= 0$ aestimanda videtur, quia deficit isto in casu Regulae Bernoullianae applicatio; quod infortunium in eccellente *Institutionum Calculi Differentialis* Capite XV. non adnotavit Eulerus. Profecto

$0 \cdot Lo = \frac{1+x}{1:L(1+x)}$ quando $x = -1$. Igitur illius Regulae iussu erit $0 \cdot Lo$

$$= \frac{0}{0} = \frac{d(1+x)}{d(1:L(1+x))} = \frac{1}{(1+x)(L(1+x))^2} =$$

$$= \frac{1}{0 \cdot \infty^2}; \text{ rursusque } \frac{1}{(1+x)(L(1+x))^2} \text{ verso in } \frac{-(1+x)}{1:(L(1+x))^2} \text{ habebitur}$$

$$\frac{d(-(1+x))}{d(1:(L(1+x))^2)} = \frac{(1+x)(L(1+x))^3}{2} = \frac{0 \cdot \infty^3}{2}, \text{ et sic deinceps}$$

in Infinitum; nimirum semper semperque valor aderit Indeterminatus. Commodi id accidit, ut sibi mutuo robur lucemque praestent Exponentialium Formularum ope ambae expressiones $(1-1)^0 = 0^0 = 1$, et $(1+\infty)^0 = \infty^0 = 1$, determinantque unum et idem esse Infinitum illud Lo ac $L\infty$, quod secus incognitum iaceret in Binomio ad morem Newtoni evoluto. Hinc

$$\text{paradoxon explicatur non modo Aequalitatis } 1 - \frac{n}{1} + \frac{n(n-1)}{1 \cdot 2} -$$

$$\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} - \&c: = 0 \text{ dum } n \text{ sit finitus}$$

$$\text{positivus, at } = 1 \text{ dum } n \text{ evanescat, verum etiam alterum } 1 + \frac{n \cdot \infty}{1} +$$

$$\frac{n(n-1)\infty^2}{1 \cdot 2} + \frac{n(n-1)(n-2)\infty^3}{1 \cdot 2 \cdot 3} + \frac{n(n-1)(n-2)(n-3)\infty^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c: =$$

∞ quoad n fuerit finitus positivus, sed statim atque nullefcit $= 1$. Quemadmodum enim primum Infinitinomialium aequipollet, immutato solum, uti

in num°. 46., terminorum ordine, Magnitudini Exponentiali $C^{n \cdot Lo}$, ita se-

cundum aequivalet $C^{n \cdot L\infty}$, vel $C^{-n \cdot Lo}$ pari arguendi ratione.

129. Dum vero Basis Magnitudinem induat Infinitam emerget eodem

$$\text{ritu } \left(\frac{1}{0}\right)^0, \text{ vel } (\infty)^0 = (1+\infty)^0 = \left(1+\frac{1}{0}\right)^0 = 1 + \frac{(L\infty)'0^1}{1} +$$

$$+ \frac{(L\infty)^2 0^2}{1 \cdot 2} + \&c:$$

$$\frac{(L\infty)^2 0^2}{1.2} + \frac{(L\infty)^3 0^3}{1.2.3} + \frac{(L\infty)^4 0^4}{1.2.3.4} + \frac{(L\infty)^5 0^5}{1.2.3.4.5} + \frac{(L\infty)^6 0^6}{1.2.3.4.5.6} + \frac{(L\infty)^7 0^7}{1.2.3.4.5.6.7} \\ + \&c: = 1 + \frac{\infty^1 0^1}{1} + \frac{\infty^2 0^2}{1.2} + \frac{\infty^3 0^3}{1.2.3} + \frac{\infty^4 0^4}{1.2.3.4} + \frac{\infty^5 0^5}{1.2.3.4.5} +$$

$$\frac{\infty^6 0^6}{1.2.3.4.5.6} + \frac{\infty^7 0^7}{1.2.3.4.5.6.7} + \&c: = (1 + 0)^\infty = 1. \text{ Necesse itaque est, ut}$$

praetensa Infiniti potentia hoc in casu non valeat subsequentes post Unitatem terminos Seriei Exponentialis Nihilo affectos foecundare; quod aliter contingere nequit, quippe quum $L\infty = -Lo$ sit Infinitum Harmonicum, cuius inter dudum ostensa symptomata adesse illud quoque rarissimum $0.\infty = 0$, et inde $0^n . \infty^n = 0^n = 0$ fuit observatum.

130. Hisce praemissis qui pro Basi Logarithmorum ludicri instar experimenti Nihilum eligere peramaverit non parum difficultatis reperiet in paradoxo aliquantisper moderando curiosi equidem Logarithmici Systematis ista Basi suffulti, cuius Systematis Modulus idcirco congruet cum Lo , sive $-\infty$. Ac imprimis Progressio Numerorum Geometrica ab Exponentiali Magnitudine $(0)^x$ orta componitur, Unitate intermedia, Nihilis sine numero ex una parte, et Infinitis ex altera. Revera $(0)^0 = 1$; sed, dum Index x positivus quomodocumque parvulus concipiatur, evadet $(0)^x = (1 - 1)^x =$

$$(1)^x \times \left(1 - \frac{\infty^1 . x^1}{1} + \frac{\infty^2 . x^2}{1.2} - \frac{\infty^3 . x^3}{1.2.3} + \frac{\infty^4 . x^4}{1.2.3.4} - \frac{\infty^5 . x^5}{1.2.3.4.5} + \frac{\infty^6 . x^6}{1.2.3.4.5.6} - \right.$$

$$\text{GEOM}^A. \&c: \infty, \infty, \infty, \infty, \infty, \infty, \infty, \infty, \infty,$$

$$\text{ARITH}^A. \&c: -11x, -10x, -9x, -8x, -7x, -6x, -5x, -4x,$$

quarum postremam Progressionem nihil a communibus differre quisque sibi suadet; altera autem principibus etiam de re Analytica scriptoribus videtur pene divinitus cultae Continuitatis Legi refragari. Sed pace tantorum virorum dicere ausim, nullum in Geometrica Progressione, nec in Formula eius

$$\text{genitrice } 1 \pm \frac{\infty^1 . x^1}{1} + \frac{\infty^2 . x^2}{1.2} \pm \frac{\infty^3 . x^3}{1.2.3} + \frac{\infty^4 . x^4}{1.2.3.4} \pm \frac{\infty^5 . x^5}{1.2.3.4.5} + \frac{\infty^6 . x^6}{1.2.3.4.5.6} \pm$$

$$\frac{\infty^6 \cdot x^6}{1.2.3.4.5.6} - \frac{\infty^7 \cdot x^7}{1.2.3.4.5.6.7} + \frac{\infty^8 \cdot x^8}{1.2.3.4.5.6.7.8} - \frac{\infty^9 \cdot x^9}{1.2.3.4.5.6.7.8.9} + \frac{\infty^{10} \cdot x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.) = 0 \text{ vi praecedentis numeri, propter Infinitum Har-}$$

monicum Symbolo ∞ comprehensum, atque x non ut prius nallescentem. Ex opposito, etiamsi minimus foret in $(o)^{-x} = \frac{1}{o^x} = (1-1)^{-x}$

negativus Exponens x , habebit Poteſtas infinite magnum valorem, qui ſatis clare demonſtratur ex forma $(1)^{-x} \cdot \left(1 + \frac{\infty^1 \cdot x^1}{1} + \frac{\infty^2 \cdot x^2}{1.2} + \frac{\infty^3 \cdot x^3}{1.2.3} + \dots\right)$

$$+ \frac{\infty^4 \cdot x^4}{1.2.3.4} + \frac{\infty^5 \cdot x^5}{1.2.3.4.5} + \frac{\infty^6 \cdot x^6}{1.2.3.4.5.6} + \frac{\infty^7 \cdot x^7}{1.2.3.4.5.6.7} + \frac{\infty^8 \cdot x^8}{1.2.3.4.5.6.7.8} +$$

$$\frac{\infty^9 \cdot x^9}{1.2.3.4.5.6.7.8.9} + \frac{\infty^{10} \cdot x^{10}}{1.2.3.4.5.6.7.8.9.10} \&c.) \text{ Arithmetica vero respondentium}$$

Logarithmorum Progressio eodem pacto, quo in ceteris Systematum Basibus, omnes Numeros positivos, ac negativos complectitur intermedia pariter Unitate, cui convenit Nullitas pro Logarithmo. Peculiare ergo Logarithmicum Systema, cuius nunc mentio facta, utrisque Progressionibus infra concinnatis coalescit

GEOM^A.

$\infty^+, \infty, (\infty, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \&c:$

$-3x, -2x, -x, 0, x, 2x, 3x, 4x, 5x, 6x, 7x, 8x, 9x, 10x, 11x \&c;$

quarum

$$\frac{\infty^6 \cdot x^6}{1.2.3.4.5.6} \pm \frac{\infty^7 \cdot x^7}{1.2.3.4.5.6.7} + \frac{\infty^8 \cdot x^8}{1.2.3.4.5.6.7.8} \&c: = (1 \pm x)^\infty \text{ ex legem factum}$$

inesse; quod evidenter patebit, dummodo mente volvamus Formulam ipsam Infinitinomiam non singillatim considerari debere, at potius respectu habito ad aliam continuo variabilem, quae originem ducat a Basi z semper semperque

semperque sine ullo Limite minuenda. Ita latissimo sensu acceptum Expo-

mentiale $(0)^{\pm x}$ extremus erit veluti terminus $\tau_z(z)^{\pm x} = (1)^{\pm x}$.

$$\left(1 \pm \frac{(Lz)^1 x^1}{1} \pm \frac{(Lz)^2 x^2}{1.2} \pm \frac{(Lz)^3 x^3}{1.2.3} \pm \frac{(Lz)^4 x^4}{1.2.3.4} \pm \frac{(Lz)^5 x^5}{1.2.3.4.5} \pm \frac{(Lz)^6 x^6}{1.2.3.4.5.6} \pm \frac{(Lz)^7 x^7}{1.2.3.4.5.6.7} \pm \frac{(Lz)^8 x^8}{1.2.3.4.5.6.7.8} \pm \&c. \right),$$

ubi infra Unitatem gradatim decre-

sciente Basi z eo magis Logarithmis $+x$ respondentes Numeri z^x in immen-
sum minuuntur, atque ex adverso Numeri z^{-x} Logarithmis homologi $-x$
augentur, usque dum, rigore Continuitatis servato, tandem Basis nulle-
scat, ac priores Numeri in Nihilum, alteri in Infinitum iuxta Limitum naturam
vertantur. Qui alacriore polleant in Mathematicis disciplinis imaginandi fa-
cultate aequae Continuum esse, ac ceterae finitis omnibus Numeris coale-
scentes, Progressionem Geometricam superius descriptam, et praeter 1 Nul-
litatibus, atque Infinitis conflatam, causamque specialis huiusmodi Progref-
sionis ipsa in Infinitate, et Rationum Limitibus elucescere sentient. Alii vero
lentiolem concipiendi vim adiuuabunt si diversas Curvas Logisticas Aequa-
tione $z^x = y$, sed Basi variabili praeditas Unitate minori, quemadmodum
feci in num°. 133., sibi depinxerint, spectabuntque Curvas illas Curvedinem
paullatim aperientes laxantesque in Angulum vergere, et tandem abire ma-
nente Continuitatis Lege in Rectarum Systema, instar Parabolae posita
Parametro Infinites magna in Tangentem, et Hyperbolae Apollonianae dum
eius Potentia evanescat in Asymptotas transeuntium. (*)

131. Eaedem, sed conversim, habentur mendosae Incontinuitatis species
cum fuerit Logarithmorum Basis Infinita, veluti $(\infty)^{\pm x} = \left(\frac{1}{0}\right)^{\pm x} = (1)^{\pm x}$

$$\times \left(1 \pm \frac{\infty^1 x^1}{1} \pm \frac{\infty^2 x^2}{1.2} \pm \frac{\infty^3 x^3}{1.2.3} \pm \frac{\infty^4 x^4}{1.2.3.4} \pm \frac{\infty^5 x^5}{1.2.3.4.5} \pm \frac{\infty^6 x^6}{1.2.3.4.5.6} \pm \frac{\infty^7 x^7}{1.2.3.4.5.6.7} \pm \frac{\infty^8 x^8}{1.2.3.4.5.6.7.8} \pm \&c. \right)$$

(*) Errasse mihi videtur Vallisus dum in Tractatu de Sectionibus Conicis Analogiam
Hyperbolae, et Ellipsis nimio oestro contemplatus asseruit in eadem unius Ellipsis Peri-
metro existere Ramos oppositos Ramosque Coniugatos, quemadmodum in Hyperbolarum
Systemate.

$$\left(\frac{\infty^7 \cdot x^7}{1.2.3.4.5.6.7} + \frac{\infty^8 \cdot x^8}{1.2.3.4.5.6.7.8} \&c: \right) = (0)^{\frac{1}{x}}, \text{ hocque satis aperte demon-}$$

strant opposita Signa Formulae Infinitinomialis, ceteroquin quoad superius notatam identicae. Nec Theoriae claritatem infirmat consideratio Aequationis $(0)^0 = (1)^0$, quam nonnulli consecutione non saturi $0.L0 = 0.L1$, vel $0 = 0$, ita deinde divisionem praetexentes per 0 communem Factorem corrumpunt, $L0 = L1$, sive $\mp \infty = 0$, quod Confectarii monstrum universae Algebrae adversaretur. Quinimmo, cum generatim $z^0 = 1 = 1^0$, iure confirmili inferri posset $Lz = 0$, et ideo in maius serperet error labefactaretque primitiva Logometriae fundamenta. Verum Analyseos etiam non Logarithmicae universa corruerent hucusque inconcussa principia si forte etiam fieret, eliminatis Indicibus, $0 = 1 = z$; quae credat Iudaeus Apella. Vitium argumentationis facili didascalico examine detegitur latens in hypothesi facta $\pi \approx 0$ veri Coefficientis numerici tam $0.L0$, quam $0.L1$, aut $0.Lz$, nec non $z^0, 0^0, 1^0, \&c:$ perinde ac si forent illae verae $z, 0, 1$ Potestates, non ut Algebra docet fictitius tantum expressionis Analyticae modus ad vacuum Potestatum Progressionis loco Unitatis implendum. Repugnat itaque Dialecticae Legibus ad modos vel potius *privaciones* aptare Canones ipsos, qui substantialem Potestatum existentiam supponunt, et eo magis adversatur quia in spuriis sive *modalibus* Aequationibus ipsis $0^0 = z^0 = 1^0 \&c:$ tacite iam contineatur hypothesis, cui *Circuli vitiosi* nomen Logici imponderent, Aequalitatis postea iniuria suppositae $0 = 1$, vel $L0 = L1$, similiterque $z = 1$, et $Lz = L1$; nam omnes sciunt Potestates cuiusque ordinis veras, aequalibus Indicibus praeditas, tum solum inter se peraequari, cum Bases Radicesve habeant aequales. Nec praecedentia simul animo volutans unumque cum al-

$$\text{tero conferens censeas invicem adversari Series } 1 - \frac{\infty^1}{1} + \frac{\infty^2}{1.2} - \frac{\infty^3}{1.2.3}$$

$$+ \frac{\infty^4}{1.2.3.4} - \frac{\infty^5}{1.2.3.4.5} + \frac{\infty^6}{1.2.3.4.5.6} - \frac{\infty^7}{1.2.3.4.5.6.7} + \&c:, \text{ et } 1 - \infty^1$$

$$+ \infty^2 - \infty^3 + \infty^4 - \infty^5 + \infty^6 - \infty^7 + \&c:, \text{ quarum utralibet ex numeris } 38. \text{ ac } 60. \text{ Nullitatem significat. Nihil enim latet absurdi in posteriore expressione, quae, utpote Series divergens, Residui eget nullatenus negligendi, et hoc addito formam acquirit identicae Aequalitatis } 1 - \infty^1$$

$$+ \infty^2$$

id pariter confirmante utraque Progressione in num°. 130. evoluta. En itaque unicos casus, quibus m. Halleii stylo, vel k more Euleri Infinitum Harmonicum negativum, aut positivum, ipsorumque reciprocae Magnitudi-

nes $\frac{1}{m}$, vel $\frac{1}{k}$, aut Moduli, Nihilum repraesentabunt. Unico Schemate

XVIII. non tantummodo affectionis huiusce ratio intelligitur, verum etiam geometricum spectaculum aperitur omnium quibuslibet Basibus gaudentium Systematum Logarithmicorum in Curvis Logisticis, quae, cum in unico extremo puncto B Ordinatum-applicatae OB Unitatem designantis concurrant, Logarithmos dant positivos Numeris Unitate minoribus et viceversa, eoque magis accedunt ex parte Z Asymptoto SOT, nec non ex parte L indefinitae OBR prouti Bases IC, ID, IE, IF, IK &c: infra Unitatem magis decrefcant, donec Basi tandem evanescente Curva coincidat cum lateribus indefinitis Anguli TOR. Vicissim crescentibus supra Unitatem Basibus IA, IM, IN, IH, IQ, &c: eadem ex latere X, vel G, sed adverso situ, contingunt symptomata; adeo ut evadente Basi Infinita rursus Curva vertatur in Rectas Lineas SO, OR angulariter positas, mediamque teneat sedem inter Logisticarum cohortem sine fine descriptibilium Recta YB Φ Asymptoto ST parallela, in quam definit Curva casu Baseos IP = OB peraequanti Unitatem. Eam quasi Logisticarum filiationem instar praelocutionis levi ductu pingere statui, ne minoris evidentiae notam illa, quae sequuntur, praeseferant. Dum in Infinito sumus pauca contemplanda manent de Formulae B $^\infty$ mirabilibus sane affectionibus. Proponatur valoris $\tau \approx 0$. B $^\infty$ Analytica investigatio. Oritur istud

Exponentiale ab expressione oecumenica $xB^{\frac{1}{x}}$, cuius Limes est, dum x in

Nihilum abeat. Porro $xB^{\frac{1}{x}} = x \cdot 1^{\frac{1}{x}} \left(1 + \frac{\left(\frac{LB}{x}\right)^1}{1} + \frac{\left(\frac{LB}{x}\right)^2}{1.2} + \frac{\left(\frac{LB}{x}\right)^3}{1.2.3} \right.$

$\left. + \frac{\left(\frac{LB}{x}\right)^4}{1.2.3.4} + \frac{\left(\frac{LB}{x}\right)^5}{1.2.3.4.5} + \&c: \right) = 1^{\frac{1}{x}} \left(x + \frac{(LB)}{1} + \frac{(LB)^2}{1.2.x} + \frac{(LB)^3}{1.2.3.x^2} \right.$

$\left. + \frac{(LB)^4}{1.2.3.4.x^3} + \frac{(LB)^5}{1.2.3.4.5.x^4} + \&c: \right)$, atque idcirco $0.B^{\frac{1}{0}} = 0.B^\infty =$

$$0 + LB + \frac{(LB)^2}{1.2.0} + \frac{(LB)^3}{1.2.3.0^2} + \frac{(LB)^4}{1.2.3.4.0^3} + \frac{(LB)^5}{1.2.3.4.5.0^4} + \&c: = LB +$$

∞ , et huiusce Infiniti genus ex natura Problematis facillime definitur. Iucundissimum autem hac in exornanda Infinitorum provincia mihi videtur

$$\text{Exponentiale } 0.C^{\infty \text{ Harm:}}, \text{ sive } 0.C^{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \&c:},$$

utpote quod $= 1$ iuxta subiectam demonstrationem. Vel

$$\text{me non monente patet } xC^{L\left(\frac{1}{x}\right)} = x \cdot \frac{1}{x} = dx \cdot \frac{1}{dx} = 1, \text{ quicum-}$$

que valor variabili x tribuatur. Igitur non tam $0.C^{L\infty}$, prouti nunciavi, quam $\infty.C^{L0}$, Anti-Logarithmi $\tau\tilde{e}$ Indicis, et Logarithmi $\tau\tilde{e}$ Coeffi-

tis conversione facta, $= 1$, interea dum $0.C^{1+1+1+1+1+1+1+1+1\&c:}$

$= \infty$. Sed etiam universalius $0.B^{\infty \text{ Harm:}} = C^{(LB-1)\infty \text{ Harm:}}$, quod ex

primitiva fuit Logometriae natura. Hinc abunde intelligitur esse $0.B^{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \&c:}$

$$+ \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \&c: = \infty \text{ donec } LB > 1, \text{ atque e-contra}$$

$= 0$ si $LB < 1$; scilicet ∞ dum $B > 2.7182818 \&c:$, ac 0 dum $B < 2.7182818 \&c:$, quos inter valores medium tenet hypothesis $LB = 1$, seu

$B = C = 2.7182818 \&c:$, tunc abeunte $0.C^{\infty \text{ Harm:}}$ in $C^{(1-1)\infty \text{ Harm:}}$

$= C^{0.\infty \text{ Harm:}} = 1$; hocque strenue confirmat Aequalitatem superius ac toties demonstratam $0.\infty \text{ Harm:} = 0$.

134. Esto nunc in Exponentiali z^x Basis $z > 1$, habebiturque Hyperbolicus ipsius Logarithmus ex praeostensis Numerus Positivus, videlicet positi-

vus quoque $\frac{1}{Lz}$, aut Systematis Modulus. Quae cum ita sint, comparata

erit, prouti in num°. 99., tali modo omnigena, sed Unitate maior, Logarithmo-

rum

rum Basis, ut Numeri $(1)^{+x} \times \left(1 + \frac{\left(\frac{x}{M}\right)^1}{1} + \frac{\left(\frac{x}{M}\right)^2}{1.2} + \frac{\left(\frac{x}{M}\right)^3}{1.2.3} + \right.$

$\frac{\left(\frac{x}{M}\right)^4}{1.2.3.4} + \frac{\left(\frac{x}{M}\right)^5}{1.2.3.4.5} + \frac{\left(\frac{x}{M}\right)^6}{1.2.3.4.5.6} + \frac{\left(\frac{x}{M}\right)^7}{1.2.3.4.5.6.7} + \frac{\left(\frac{x}{M}\right)^8}{1.2.3.4.5.6.7.8} + \&c.)$ po-

sitivi Unitate maiores habeant Logarithmos $+x$, et vicissim Numeri quo-

que positivi Unitate minores seu fracti $(1)^{-x} \times \left(1 - \frac{\left(\frac{x}{M}\right)^1}{1} + \frac{\left(\frac{x}{M}\right)^2}{1.2} - \frac{\left(\frac{x}{M}\right)^3}{1.2.3} + \frac{\left(\frac{x}{M}\right)^4}{1.2.3.4} - \frac{\left(\frac{x}{M}\right)^5}{1.2.3.4.5} + \frac{\left(\frac{x}{M}\right)^6}{1.2.3.4.5.6} - \frac{\left(\frac{x}{M}\right)^7}{1.2.3.4.5.6.7} + \&c.)$ Lo-

garithmis $-x$ distinguantur. Id autem commodi accidit, quod ope Infini-

tinomiorum Exponentialium omnis Multiformitas Numerorum ab Indice $\pm x$

pendens fracto vel transcendente in Multiformitatem Factoris $(1)^{\pm x}$ unice reiciatur. Nec dubium suboriri potest, postremum Infinitinomialium, hypothefi facta negativi Logarithmi $-x$, esse Numerum Unitate minorem, propterea quod vi numeri 76. idem Infinitinomialium adaequet Unitatem per primum divisam, ideoque Numerum fractum latissimo sensu, etiam *transcendentaliter*, sumptum.

135. Basis e-converso Systematis Logarithmici vel fundamentalis Numerus z Magnitudinis Exponentialis z^x esto infra Unitatem. Nanciscemur eo casu Lz , et $\frac{1}{Lz}$ five $-M$ Modulum negativum. Hoc praesupposito

Logarithmis $+x$ respondebunt Numeri positivi Unitate minores $(1)^{+x}$.

$\left(1 - \frac{\left(\frac{x}{M}\right)^1}{1} + \frac{\left(\frac{x}{M}\right)^2}{1.2} - \frac{\left(\frac{x}{M}\right)^3}{1.2.3} + \frac{\left(\frac{x}{M}\right)^4}{1.2.3.4} - \frac{\left(\frac{x}{M}\right)^5}{1.2.3.4.5} + \frac{\left(\frac{x}{M}\right)^6}{1.2.3.4.5.6} - \frac{\left(\frac{x}{M}\right)^7}{1.2.3.4.5.6.7} + \frac{\left(\frac{x}{M}\right)^8}{1.2.3.4.5.6.7.8} - \&c.)$, Logarithmisque $-x$ Numeri

Unitatem superantes, videlicet, ut omnes iam norunt, $(1)^{-x} \times$

$$\left(1 + \frac{\left(\frac{x}{M}\right)^1}{1} + \frac{\left(\frac{x}{M}\right)^2}{1.2} + \frac{\left(\frac{x}{M}\right)^3}{1.2.3} + \frac{\left(\frac{x}{M}\right)^4}{1.2.3.4} + \frac{\left(\frac{x}{M}\right)^5}{1.2.3.4.5} + \frac{\left(\frac{x}{M}\right)^6}{1.2.3.4.5.6} + \frac{\left(\frac{x}{M}\right)^7}{1.2.3.4.5.6.7} + \&c. \right),$$

observationum cumulum repetere nec iuvat, nec decet. Uno verbo asseram universa, quae valeant in Logarithmorum Systemate, cui Basis conveniat $z > 1$, sine novis computationibus, Tabulisve condendis, perquam facillime transferri Systema ad alterum Logarithmicum Basis $\frac{1}{z}$ subnixum, et vicissim: nam huiusmodi foedere, atque harmonia coniunguntur utrique Systematum modi, ut omnibus differentiis sublati Signa tantum eiusdem Numeri dati Logarithmorum sibi invicem, salva aequalitate, opponantur; quod ipsamet Infinitinomia demonstrant.

136. Universi itaque Numeri positivi, immo et ambo ipsorum Limites Nihilum atque Infinitum, cuiuslibetcumque Systematis varios Logarithmos regentium Basium munere functi totas exhaustiunt possibiles Numerorum positivorum Geometricas Progressiones, nullumque commonstrant aditum ad Series continuas Logarithmorum negativis Numeris respondentium. Vidimus quoque, omnes reales Systematum Modulos $\pm M$, videlicet vel positivos vel negativos cuiusvis magnitudinis, quod idem est ac affirmare universas ipsorum Systematum Subtangentes, sive $\frac{y}{k}$ incipiendo a Nihilo, vel Limite

positivas ac negativas seiungente, in Systemate Basis evanescentis aut Infinitae, generatim evacuari a Geometricis Progressionibus, quae, saltem continuos, Numeros negativos excludant. Summo igitur iure Leibnitius contra Ioannem Bernoullium scribens in Epistola CCII. nonnullisque sequentibus eas inter, quae exstant secundo Volumine *Commercii Philosophici et Mathematici* contendebat pro aris focisque nusquam fieri posse quantitatem Exponentialem Basis positiva affectam n^y continui-negativi valoris capacem, quam Analyticam impossibilitatem, dum Index y sit realis quicumque iuxta hypothesein

$$\text{Leibnitianam Infinitinomialium } (1)^y \cdot \left(1 + \frac{(\text{Ln})^1 y^1}{1} + \frac{(\text{Ln})^2 y^2}{1.2} + \frac{(\text{Ln})^3 y^3}{1.2.3} + \dots \right)$$

$$\begin{aligned}
 & + \frac{(\text{Ln})^4 y^4}{1.2.3.4} + \frac{(\text{Ln})^5 y^5}{1.2.3.4.5} + \frac{(\text{Ln})^6 y^6}{1.2.3.4.5.6} + \frac{(\text{Ln})^7 y^7}{1.2.3.4.5.6.7} + \frac{(\text{Ln})^8 y^8}{1.2.3.4.5.6.7.8} + \\
 & \frac{(\text{Ln})^9 y^9}{1.2.3.4.5.6.7.8.9} + \frac{(\text{Ln})^{10} y^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c:) \text{ ex sui ipsius indole luculenter}
 \end{aligned}$$

concludit. Modulus namque, vel etiam eius inversa Magnitudo $\frac{1}{M} = \text{Ln}$,

quae positivos ac negativos explet, nullo vacuo remanente, universos valores in tota Systematum Logarithmicorum familia, praebet casu Ln positivi Infinitinomialium illud absque dubio positivum, et casu $-\text{Ln}$ aequale positivum,

$$\begin{aligned}
 \text{propterea quod } 1 &= \frac{(\text{Ln})^1 y^1}{1} + \frac{(\text{Ln})^2 y^2}{1.2} + \frac{(\text{Ln})^3 y^3}{1.2.3} + \frac{(\text{Ln})^4 y^4}{1.2.3.4} + \frac{(\text{Ln})^5 y^5}{1.2.3.4.5} \\
 &+ \frac{(\text{Ln})^6 y^6}{1.2.3.4.5.6} + \frac{(\text{Ln})^7 y^7}{1.2.3.4.5.6.7} + \frac{(\text{Ln})^8 y^8}{1.2.3.4.5.6.7.8} + \&c: =
 \end{aligned}$$

$$\begin{aligned}
 & 1 \\
 & + \frac{(\text{Ln})^1 y^1}{1} + \frac{(\text{Ln})^2 y^2}{1.2} + \frac{(\text{Ln})^3 y^3}{1.2.3} + \frac{(\text{Ln})^4 y^4}{1.2.3.4} + \frac{(\text{Ln})^5 y^5}{1.2.3.4.5} + \frac{(\text{Ln})^6 y^6}{1.2.3.4.5.6} + \frac{(\text{Ln})^7 y^7}{1.2.3.4.5.6.7} + \frac{(\text{Ln})^8 y^8}{1.2.3.4.5.6.7.8} + \&c:
 \end{aligned}$$

Negligi attamen non meretur Factor ille omnium Infinitinomialium $(1)^y$,

qui Multiformitatem in Exponentiali n^y superius reconditam Calculo subiicit, incomparabilique commodo, quando y fuerit Functio Uniformis, fecernit ab Serie numero terminorum Infinita: vi etenim superius memorati Factoris, quoties y Fractionarius sit irreducibilis, denominatorem habens Numerum parem, Irrationalis, vel etiam Transcendens eadem praeditus conditione, tunc

$$\begin{aligned}
 (1)^y, \text{ ideoque et } n^y \text{ geminum exhibet aequalem valorem, unum nempe} \\
 \text{positivum, alterum negativum; quamobrem reiectis omnibus Imaginariis bi-} \\
 \text{formis fit } n^y, \text{ quippe } = \pm \left(1 + \frac{(\text{Ln})^1 y^1}{1} + \frac{(\text{Ln})^2 y^2}{1.2} + \frac{(\text{Ln})^3 y^3}{1.2.3} + \right. \\
 \left. \frac{(\text{Ln})^4 y^4}{1.2.3.4} + \frac{(\text{Ln})^5 y^5}{1.2.3.4.5} + \frac{(\text{Ln})^6 y^6}{1.2.3.4.5.6} + \frac{(\text{Ln})^7 y^7}{1.2.3.4.5.6.7} + \frac{(\text{Ln})^8 y^8}{1.2.3.4.5.6.7.8} + \&c: \right),
 \end{aligned}$$

ac Locus Exponentialis Aequatione distinctus $n^y = v$ Ordinatum applicatas negativas, numero infinitas, invicem per intervalla positas, vel ad Curvae continuam perimetrum non pertinentes, sed ad successionem indefinitam

Puncto-

Punctorum, quae vocant Coniugata, minimis tamen, et inassignabilibus mutuis distantis vacuisve interspersa (*).

137. Nisi Curvae, de qua nunc agimus, Exponentialis singulare symptoma nativo veluti ductu ab evolutione directa emanasset finitae Binomii Formulae Neutonianae, suspicio fortasse aliqua animo obversaretur obliquum Calculi artificium ab Infiniti obscuritate excitatum quaedam peregrina, et rei Logometricae non affinia attulisse, ac potissimum, praeter Logisticae verum Ramum in regione positivorum, alterum Crus ex parte negativorum continuum, aut melius dixeris coalescentem Punctis pseudo-Perimetrum. Huius quasi pleonasmii Algebraici, aut fastidiosae saepius foecunditatis exempla non infrequentia in Problematibus resolvendis occurrunt, quae aliquando a nimis contorto et artificii plenissimo Calculi gyro derivata quaestionis propositae confiniis circumscribi non patiuntur, verum ultra pererrant redduntque abstractas, ut Analystae ferunt, resolutiones concretis admodum involutiores. Quod infortunium nullo modo perturbare, atque inficere potuisset conclusionem superius expositam, cum nihil Algorithmi mere Cartesiano exoticum supervenerit metuendum, nihil in Calculi progressu oculos fugerit, omniaque praeter spem exactissime fuerint veluti commensurata, et relligioso mentis iudicio subacta. Nec inter Analyseos monstra reponendam, si profecto adforet Curvae alicui Transcendenti, arbitrarer seriem illam Punctorum Coniugatorum Θ, Ω, Ψ &c: in Schemate pictam XVIII^o., utcumque mirificam, cuius mentio pluries invaluit. Libenter enim amplector clarissimi Boscovichii sententiam in calce Operis praecitati Iesuitae Luini *De Progressionibus ac Seriebus* prolatam multaque in Transcendentibus Curvis consimilia Incontinuitatis, vel peculiaris Continuitatis paradoxa decernentem, tametsi Dalembertio mentis amaritudinem quamdam ea parere videantur. Et tanto magis haec mihi arridet opinio, quia Curvae quandoque Geometricae Punctis istis Coniugatis, sed finito, ut par est, numero orrientur, aut in illa denique abeant. Has inter, absque eo quod Magni illius Galilaei in *Dialogis de nova Mo-*

(*) Reales esse posse Numerorum Negativorum Logarithmos, etsi non omnium, Editores *Dictionary Encyclopedici* in Voce *Logarithmus* hoc etiam argumento $a^{\frac{1}{2}} = \pm \sqrt{a}$, et generalius $C^{\frac{1}{2}} = \pm \sqrt{C}$ ostendere satagunt.

va Motus Scientia circa Linearum compositionem ex Punctis vacuisque innumeris sententiam inire nobis sit animus, percelebris est Ovalis Cassiniana in duo Punctorum Coniugatorum Systema transformabilis demonstrante id primum Davide Gregorio, ac valde plura ex iis Punctis praeseferunt Lineae quartum Ordinem superantes. Vires autem ratiocinationis huiusce aliquantisper cumulatibus aliis observationibus intendi, et ad meliora firmanda revocari poterunt tum, quum in Capite VIII. Logisticam Curvam non tantummodo Limitem esse ostendemus omnium Paraboloeidon, uti facillime patefacit Infinitinomialium, in quod evolvitur Formula Exponentialis, verum etiam Parabolarum ab Apolloniana initio desumpto; cuius maximopere felicitis ideae usus uberrimus elucescet in Logisticae proprietatibus una prope Calculi linea eruendis, ac demonstrandis.

138. Universa Logarithmorum Systemata compendio sane eleganti complectitur Formula canonica $L(x^M) = lx$, quam non forte putes consentire cum Aequatione a Dalembertio iam prodita $l(nx) = lx$ in VI°. primi *Opusculorum Mathematicorum* Voluminis Schediasmate. Postrema namque unum ac idem Systema Logarithmicum praesupponit, interea dum prior Hyperbolicum comparat non Hyperbolico, cui Modulus M characteristicam quasi mensuram definiens conveniat. Toto itaque coelo distat ab altera memoratarum Aequationum prima, quae etiam universalius, sublato Systemate singulari Hyperbolico, ita exprimi posset $l'(x^{M''}) = l''(x^{M'})$, vel $l'\left(\frac{M''}{x^{M'}}\right) = l''(x)$, aut potius $l'(x) = l''\left(\frac{M'}{x^{M''}}\right)$. Verumtamen nobis operam dantibus Aequationi Dalembertianae illustrandae rem altius repetenda erit; libentiori equidem animo, propterea quod nonnulla ad praxin accommodata Systemata Logarithmica, nec satis adhuc theorices luce perfusa commentari huc illuc feret occasio, et praesertim legitimum omnibusque numeris absolutum Systema Neperianum, ac deinde Magnum, atque Logisticum.

139. Logarithmos a Nepero primum recensitos, Tabulisque conscriptos ferme omnes cum Naturalium, sive Hyperbolicorum Systemate commiscant, cum tamen, etsi quantitate non discrepent, genesi et qualitate inter se non leviter differant. Theoretice enim loquendo Basis Neperianorum

est Ba-

est Baseos Naturalium Reciproca, nimirum $\frac{1}{C}$, vel Decimalis Fractio 0,367879441 &c: in praecedenti num.^o 97. adumbrata, illudque exornans Systema, quod venia a Geometris petita nuncupavimus Anti-Hyperbolicum. Inde consequitur Modulum veri Systematis Neperiani esse — 1, dum + 1 Modulum perhibet Hyperbolici. Subtangens ergo, et characteristicus Numerus k Logisticae Neperianae Unitatem negativam, et Logisticae Systematis Hyperbolici positivam peraequant; unde ambae Logisticae eadem sunt Curva, sed, ut in Schemate XIX. paullisper deformato, inverse disposita. Hae Systematum differentiae num.^o etiam 105. praenunciatae cuivis constabunt perspicienti *Mirificum Neperi Canonem*, ubi Sinubus - rectis, aut quod idem est Lineis Radio vel Unitate Trigonometrica minoribus Logarithmi respondent positivi, et vicissim negativi Secantibus, Tangentibusve Radio ipso maioribus, adeo ut vi Moduli negativi Infinitum positivum Harmonicum fiat Nihili, ac negativum Infinitum pariter Harmonicum Infinitatis absolutae $\frac{1}{1-x}$

Anti-Hyperbolicus Logarithmus. Id confirmat Definitio celeberrima Logarithmorum Auctoris verbis heic repetita, et Fluxionum, Fluentiumque methodi fortassis mater: statuit enim ipse Logarithmum generatim esse *Numerum illum, qui quamproxime definit Lineam, quae aequaliter crevit, interea dum Sinus - totius Linea proportionaliter in Sinum - rectum decrevit*. Obiter tamen monere iuvabit, ne vocum disparitas aliquando confusione ansum praebeat, nomine Anti-Logarithmorum Neperum ab stylo nuperrimo discrepantem significasse non Numeros Logarithmis in *Canone* oppositos, sed Logarithmos Cosinum - rectorum, Cotangentium, Cosecantium &c:, uno verbo Linearum Trigonometricarum Arcubus - Complementaribus correspondentium.

140. De Logisticis-Logarithmis acturus pauca quaedam perlustrare oportebit, quia communi Briggiano vel Tabulari Systemate sustinentur. Si enim constantis Numeri 3600., cui Logarithmus Tabularis convenit 3,556302500767287265017533595959216671936637491305608812280586&c:, valorem specie compendiosiore A indigitare libeat habebuntur Logarithmus-Logisticus llog., ad eumque pertinens Numerus N expressi Aequatione 1A —

$$1N = \text{llog. } N, \text{ sive } l\left(\frac{A}{N}\right) = \text{llog. } N, \text{ denotante } l \text{ usitatos Logarithmos}$$

Briggianos, seu Tabulares. Nec maiorem praesferret difficultatem translatio

tio Aequationis ipsius in Logarithmorum Systema Hyperbolicum. Ea quip-

pe foret $mL\left(\frac{A}{N}\right) = llog. N$, vel $llog. N = L\left(\left(\frac{A}{N}\right)^m\right)$, subintellecto

Modulo m Briggiani Systematis, de quo fusior mentio facta in praeterito Num.^o 96. adinvenitur. Eodem pacto facillime emerget aut in Systemate Tabulari, aut in simpliciore Hyperbolico Exponentialis Aequatio Numeros N ac Logarithmos-Logisticos x invicem nectens, quam Aequalitatem de-

$$\text{finit vel expressio } B^{\frac{x}{lB}} = \frac{A}{N}, \text{ scilicet } N = \frac{A}{B^x} = \frac{B^{lA}}{B^x} = B^{lA-x} \\ = B^{3,556302 \&c: - x} = (10)^{3,556302 \&c: - llog. N}, \text{ vel altera } C^x = \left(\frac{A}{N}\right)^m$$

transiens in $N = \frac{A}{C^{\frac{x}{m}}} = \frac{A}{C^{xLB}}$. Numeri itaque Logisticis respondentes

Logarithmis inversam habent Rationem ad Numeros Tabulares communibus Logarithmis appositos; adeo ut, dato Logarithmo-Logistico, si pro Tabulari sumatur, erit quaesitus Numerus aequalis Quotienti a partitione orto illius fundamentalis 3600 per Numerum Tabularem, atque vicissim. Elegantiore autem prospectu nexus Logarithmorum-Logisticorum, atque communium patebit statim ac adseruerim universaliter Logisticos, et Briggianos veluti Coordinatim-applicatas ad Hyperbolen Apollonianam pertinentes Potentia A . i praeditam rite recteque coniungi. Dum vero Logarithmo Briggiano constantis Numeri 3600 subrogare placuisset Hyperbolicum, vel Naturalem, tunc

nacti fuissetus vice $N = \frac{A}{B^x} = \frac{A}{C^{xLB}}$ alteram identicam Aequationem,

ut quisque experietur, $N = \frac{C^{lA}}{C^{xLB}} = C^{lA-xLB} =$

$(2,7182818 \&c:)^{8,188689 \&c: - 2,302585 \&c: \cdot x}$

$= (2,7182818 \&c:)^{8,188689 \&c: - 2,302585 \&c: \cdot llog. N}$; nam $L(3600)$ pro-

xime Calculo inito aequivalet $8,188689124444201369660937 \&c:$, nimirum

B b

rum

rum conficit in multiplicatione per Modulum M five 0,4342944819031518
276511289 &c: praedefinitum Logarithmum Tabularem Numeri A. Qui
spicilegium facilius huiusmodi Logarithmorum in Logistica Curva desi-
deret Schema XX. iisdem, de quibus supra, Siglis Numerisque consulto
exornatum, at quoad Linearum veras longitudines neglectum, inspiciat, et
protinus sentiet Numero 3600 Nihilum pro Logarithmo-Logistico contin-
gere, negativos autem Logarithmos ex parte R Numeris N', vel 3600 supe-
rantibus, ac ex latere P positivos iis N respectu 3600 aut A minoribus. Ne
sublime quiddam censeatur consimilia sine Numero Logarithmorum Systema-
ta mentis acumine contemplari addam Aequationem universalissimam $y =$
 m^{a-x} , quae tali artificio comparatur, ut y Numeros, m Basin quamcum-
que, a Constantem Geometrae arbitrio seligendam, m^a Protonumerum, qui
in Systematibus haecenus explicatis Unitas erat, et x Logarithmos Logistico-
rum instar computatos generatim exhibeant.

141. Quemadmodum universa Logisticorum possibilium Logarithmorum Sy-
stemata in Aequatione continentur $y = m^{a-x}$, ita Logarithmi, quos Magnos vo-
care solent nonnulli Analyistarum, Aequatio analoga praefert $y = m^{x-a}$. Logi-
sticos igitur, atque Magnos Logarithmos unica etiam Aequalitas $y = m^{\pm(a-x)}$
habens omnia praeter Signum communia comprehendere valet. Percelebre est
autem indefinita inter Systemata Magnorum Logarithmorum illud actuali
Praxi devotum, in quo, firmis manentibus Basi, Modulo &c: Briggiani Sy-
stematis, constans Numerus a Denarium adaequet. His positis Aequatio ge-
neralis in peculiarem vertetur $y = (10)^{x-10}$, five $N = (10)^{1. \text{Mag. } N-10}$
quae abit in $1. \text{Mag. } (N) = 1N - 10 = 1 (10000000000 \cdot N)$, pro-
uti penes Geometras usuvenit. Liqueat ex dictis dato cuiuslibet Numeri Lo-
garithmo-Magno hunc Logarithmo Tabulari semper aequalem fieri Numeri
Multipli ab 10000000000^{cuplo} denominati, interea dum ex adverso Loga-
rithmus-Logisticus Numeri dati aequat Tabularem Logarithmum sub-Mul-
tipli 3600 denominati ab ipsomet Numero. *Characteristica* Logarithmorum
Magnorum decem ideo Unitatibus illam Tabularium excedit, ipsa tamen
salva

salva Decimali Fractione post memoratam Characteristicam consequente, si-
ve, ut aiunt, *Mantissa*. In hoc ulterius conveniunt Logarithmi-Logistici,
et Magni, quod initium supputationis vel Nihilum Logarithmorum ab
Unitate recedat, sed in priorum Systemate Numerus 3600, et universalis
 m^a , in aliorum vero Numerus fractus 0,0000000001, ac generatim m^{-a}
vel $\frac{1}{m^a}$ loco Logarithmi Nihilum habeant. Exinde fit Logarithmos - Magnos

summo rei Numerariae, et Trigonometricae commodo nusquam abire in
negativos nisi, quod rarissimum est, Numerus fractus infra Centenorum-
Millesimorum Quadratum descenderit. Commodum istud sane maximum
Logarithmos negativos in Astronomia Calculatoria, et in Trigonorum re-
solutione vitandi rationem praebet laudabilem usitatae praxeos Systematis
Logarithmorum, de quibus est sermo; ac singulariter memoria dignum vi-
detur Logarithmos - Magnos perfecte congruere Tabularibus Linearum Tri-
gonometricarum Logarithmis, cum ii sint tali artificio digesti, ut Radio
sive 10000000000 Logarithmus 10 adnumeretur. Immo harmonice conso-
nant cum Logarithmis iisdem Magnis Fractorum Numerorum Logarithmi,
quos in Calculo vocant Complementa Arithmetica. Solemne autem pal-
mariumque est, ut omnis fileat dubitatio, in Logometria nedum $\frac{(z-1)^1}{1} -$

$$\frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \frac{(z-1)^5}{5} - \&c., \text{ aut } M \left(\frac{(z-1)^1}{1} - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \frac{(z-1)^5}{5} - \&c. \right), \text{ verum etiam}$$

$$\frac{(z-1)^1}{1} - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \frac{(z-1)^5}{5} - \&c. \pm A,$$

$$\text{feu } M \left(\frac{(z-1)^1}{1} - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \frac{(z-1)^5}{5} - \&c. \right)$$

$\pm B$ Logarithmum exhibere posse Numeri z . Profecto, quam ex num.^o 91.,
ac ex toto prope Capite III. evidentissimum sit priorem Infinitinomii for-
mam progredi Arithmetice interea dum z Geometricam sensim sequitur
Progressionem, et ab communi Algebra erudiamur ceteras formas instar
primae Progressionem Arithmeticam servare quemcumque induant valorem,

etiam Imaginarium, Constantes M, A, B ad typum excusae cuiusvis Systematis, patebit assumpta propositio sine ullius suspensionis discrimine.

142. Logarithmos, si mavis, Hyperbolicos Briggianorum loco substitue-
re in promptu erit. Porro servatis iam familiaribus Signis conficietur Ae-

$$\text{quatio l.Mag. } N = \frac{L(10000000000 \cdot N)}{L 10}, \text{ scilicet } = 0,434294481903251827$$

6511289 &c.: $L(10000000000 \cdot N)$; quamobrem transitu facto ad Expo-

ponentialia emerget altera Aequalitas $N = C^{\frac{L(10)(\text{l.Mag. } N - 10)}{L 10}}$, nimirum
respectivis valoribus substitutis $N =$

$$(2,7182818 \&c:)^{2,302585 \&c: \cdot \text{l.Mag. } N - 23,0258509 \&c:}. \text{ Supposita vero Ae-}$$

quatione universalissima $y = m^{x-a}$, haec ad Basin C Naturalium Loga-
rithmorum summa transferetur simplicitate, nam eo in casu oritur Formu-

$$\text{la } y = C^{\frac{Lm(x-a)}{Lm}}.$$

143. Ope variorum Systematum, ex quorum Theorice nuper explicita
Logarithmi Magni, ac Logistici cuiusvis speciei originem ducunt, Lex illa
ad nutum Geometrae perturbatur, quae prius visa fuerat in Mathesi in-
violabilis, Nihilumque Unitati respondentem Logarithmum decreverat. For-

mula etenim Exponentialis m^{x+a} , Indicibus positivis x , vel a , positi-
vis, aut negativis, meridiana luce clarius nos edocet Systematum Lo-
garithmicorum inexhaustum fontem, quae Basin habeant seu Numerum-
rectorem m , Modulum Subtangenteve $\frac{1}{Lm}$, et characteristicam Mensuram
 $k = Lm$, absque eo quod Logarithmus x nulescens ad Unitatem pertineat,

sed potius ad Numerum m^a , vel C^{aLm} , intereadum Unitas gaudeat pe-
culiari Logarithmo, quem determinat $-a$, scilicet positivo si a negati-
vum valorem praeferat, et viceversa. Numeri ac Logarithmi horum sine

sine Systematum in Aequatione oecumenica continentur $m^{\frac{x+a}{Lm}} = (1)$

$$\times \left(1 + \frac{(Lm)^1 \overline{x+a}^1}{1} + \frac{(Lm)^2 \overline{x+a}^2}{1.2} + \frac{(Lm)^3 \overline{x+a}^3}{1.2.3} + \frac{(Lm)^4 \overline{x+a}^4}{1.2.3.4} + \dots \right)$$

$$+ \frac{(Lm)^5 \overline{x+a^5}}{1.2.3.4.5} + \frac{(Lm)^6 \overline{x+a^6}}{1.2.3.4.5.6} + \&c: \Big), \text{ five Trigonum Analyticum}$$

$$\text{aemulando } m^{\overline{x+a}} = (1) \times$$

$$\begin{aligned} & \left(1 + \frac{(Lm)^1 x^1}{1} + \frac{(Lm)^2 x^2}{1.2} + \frac{(Lm)^3 x^3}{1.2.3} + \frac{(Lm)^4 x^4}{1.2.3.4} + \frac{(Lm)^5 x^5}{1.2.3.4.5} + \frac{(Lm)^6 x^6}{1.2.3.4.5.6} + \&c: \right) \cdot \\ & + \frac{(Lm)^1 a^1}{1} + \frac{(Lm)^2 a^1 x^1}{1} + \frac{(Lm)^3 a^1 x^2}{1.2} + \frac{(Lm)^4 a^1 x^3}{1.2.3} + \frac{(Lm)^5 a^1 x^4}{1.2.3.4} + \frac{(Lm)^6 a^1 x^5}{1.2.3.4.5} \\ & + \frac{(Lm)^2 a^2}{1.2} + \frac{(Lm)^3 a^2 x^1}{1.2} + \frac{(Lm)^4 a^2 x^2}{1.4} + \frac{(Lm)^5 a^2 x^3}{1.3.4} + \frac{(Lm)^6 a^2 x^4}{1.2.4.6} \\ & + \frac{(Lm)^3 a^3}{1.2.3} + \frac{(Lm)^4 a^3 x^1}{1.2.3} + \frac{(Lm)^5 a^3 x^2}{1.3.4} + \frac{(Lm)^6 a^3 x^3}{1.2.3.6} \\ & + \frac{(Lm)^4 a^4}{1.2.3.4} + \frac{(Lm)^5 a^4 x^1}{1.2.3.4} + \frac{(Lm)^6 a^4 x^2}{1.2.4.6} \\ & + \frac{(Lm)^5 a^5}{1.2.3.4.5} + \frac{(Lm)^6 a^5 x^1}{1.2.3.4.5} \\ & + \frac{(Lm)^6 a^6}{1.2.3.4.5.6} \end{aligned}$$

quo pacto melius pater Aequalitas cum Algebrae regulis mirum in mo-

dum consentiens, sistensque $m^{\overline{x+a}} = m^{\overline{x}} \cdot m^{\overline{a}} = 1 \cdot \left(1 + \frac{(Lm)^1 x^1}{1} + \right.$

$$\left. \frac{(Lm)^2 x^2}{1.2} + \frac{(Lm)^3 x^3}{1.2.3} + \frac{(Lm)^4 x^4}{1.2.3.4} + \frac{(Lm)^5 x^5}{1.2.3.4.5} + \frac{(Lm)^6 x^6}{1.2.3.4.5.6} + \&c: \right) \times$$

$$1^a \cdot \left(1 + \frac{(Lm)^1 a^1}{1} + \frac{(Lm)^2 a^2}{1.2} + \frac{(Lm)^3 a^3}{1.2.3} + \frac{(Lm)^4 a^4}{1.2.3.4} + \frac{(Lm)^5 a^5}{1.2.3.4.5} + \right.$$

$$\left. \frac{(Lm)^6 a^6}{1.2.3.4.5.6} + \&c: \right) (*). \text{ Quin et ex hac Formula luculentissime erui-}$$

tur Unitatem non esse Baseos Systematis m Logarithmum, verumtamen Nu-
meri

(*) Perspiciatur numerus 18., ac 93. Qui in Basi antonomastica *C Proto-Numerum* optaret ab Unitate diversum m , voti compos fiet ope unius Exponentialis mC^x .

meri m^{a+1}, vel C^{Lm(a+1)} = (1)^{a+1} × (1 + $\frac{(Lm)^1 \overline{a+1}^1}{1}$ + $\frac{(Lm)^2 \overline{a+1}^2}{1.2}$ + $\frac{(Lm)^3 \overline{a+1}^3}{1.2.3}$ + $\frac{(Lm)^4 \overline{a+1}^4}{1.2.3.4}$ + $\frac{(Lm)^5 \overline{a+1}^5}{1.2.3.4.5}$ + $\frac{(Lm)^6 \overline{a+1}^6}{1.2.3.4.5.6}$ + &c:), quae Aequatio, si de Basi Hyperbolica C agatur, transfit in simpliciore (1)^{a+1} × (1 + $\frac{\overline{a+1}^1}{1}$ + $\frac{\overline{a+1}^2}{1.2}$ + $\frac{\overline{a+1}^3}{1.2.3}$ + $\frac{\overline{a+1}^4}{1.2.3.4}$ + $\frac{\overline{a+1}^5}{1.2.3.4.5}$ + $\frac{\overline{a+1}^6}{1.2.3.4.5.6}$ + &c:), interea dum eadem hypothesi facta Numero (1)^a . (1 + $\frac{a^1}{1}$ + $\frac{a^2}{1.2}$ + $\frac{a^3}{1.2.3}$ + $\frac{a^4}{1.2.3.4}$ + $\frac{a^5}{1.2.3.4.5}$ + $\frac{a^6}{1.2.3.4.5.6}$ + $\frac{a^7}{1.2.3.4.5.6.7}$ + &c:) Logarithmus o hoc in Systemate adscriberetur. (*)

144. Infinites igitur multiplicia Logarithmorum Systemata concipi, ac existere possunt, quae non Unitati, sed cuilibet Numero dato Nihilum praestent ceu Logarithmum. Huius affectionis evidentiam continent Series in calce numeri 141. prolatae, quae in unica hypothesi $\tilde{x} A$, seu $B=0$ Logarithmum tribuunt evanescentem Numero $z=1$. Exinde est, quod cum Riccato vocantibus Systematis Logarithmici Proto-Numera eum Numerum habentem Nihilum pro Logarithmo, ac Basin ipsius Systematis Numerum alterum, cui pro Logarithmo stet Proto - Numerus, satisfaciant

hae Formulae universales: $m^a = \text{Proto-Num:}$; $m^{(a+m)} = \text{Bas. in Systemate oecumenico}$; $m^{a+1 \text{ Num:}} = \text{Num:}$. Revera quidnam vetat Progressionem Arithmetica exhibere tali obvio artificio Geometricae respondentem, ut Nullitas in Arithmetica Numero cuivis x in Geometrica subscrubatur? Nonne ita efficiendo Logometriae ius omne suum tribueretur, quod nil aliud exigit nisi mutuum comparationem terminorum quomodocumque dispositorum Progressionis Geometricae, et Arithmeticae? Accidit sane in

Logistica

(*) De simpliciore Proto - Numero aliqua habet numerus 94.

Logistica harumce Progressionum Typo veluti Lineari quod universis Curvis Algebraicis, vel Transcendentibus communiter conceditur, nimirum Abscissarum initium huc illuc in Axe vel Asymptoto transferri ad libitum posse, salva Parametro, aut Modulo Aequationem afficiente. Hinc resolutio obtinetur Aequalitatis prima fronte paradoxicae $ly = l(ny)$, quam loco citato Dalembertius proposuit, et minime contradictoriam esse omni iure affirmavit. Esto etenim M Modulus sive Subtangens, B Basis Systematis, vel Rector-numerus, et Aequatio illa vertetur in $B^x = B^x + \frac{ln}{lB} = B^{x+ln}$ ob $lB = 1$ eidem Logisticae Subtangente M praeditae convenientem, quae satis ostendit Logarithmos x ab diverso tantum in Axe puncto numeratos invicem peraequari. Nec moror in Exponentiali Aequatione diutius adstruenda, cum omnes iam noverint Proportionem $B^x : B^{x+ln} :: 1 : B^{ln} :: 1 : n :: y : ny$. Breviter, ac perfunctorie potius adseram Aequationem istam $ly = l(ny)$ redigi in oecumenicam superius enucleatam $z = m^{v+a}$, videlicet in Logarithmos, qui constante numerica Magnitudine excedant sive deficient ab Logarithmis v , nec nova indigeant idcirco Tabularum computatione.

145. Praecipua nunc habenda ratio de Logarithmorum Systematibus illis, quae ab negativa Basi suppeditentur. Inusitata equidem, nec bonis avibus in Praxin transferenda haec nobis obversantur Systemata; attamen theoreticam saltem excitat curiositatem argumentum hucusque, quod sciam, incompletum. Ostendimus passim in num°. 71., quod, ne Serierum fallacia, dum in divergentes abeunt, Exponentialia perturbare atque inficere valeat, opus sit, ut Potestas indefinita Basi negativa suffulta $(-z)^x$ in Factum $(-1)^x (z)^x$ sedulo vertatur. Hac methodo, quae universis certe antecellit, tractationis symptomatumque diversitas, si aliqua adsit, reiicietur in meras Potestates negativae Unitatis.

146. Sit primum $(-z)^x = (-1)^x$, vel Basis Systematis Unitas negativa. Nullus dubito, quominus per se pateat $(-1)^0 = 1$, et universalis $(-z)^0 = 1$. Qui affectionem istam rigide demonstrare in votis haberet his satisfaciet dummodo meminerit esse $(-1)^0 = (-1)^1 (-1)^{-1} = -1 \cdot -1 = 1$, eademque ratione ex Potestatum Algorithmis deducta $(-z)^0 = (-z)^1 (-z)^{-1} = \frac{-z}{-z} = 1$; nec mirum, ex eo quod evo-

lutio

Intio $\tau z (-z)^0$ pendeat ab altera $(-1)^0$ vi Aequalitatis $(-z)^0 = (-1)^0 z^0 = (-1)^0 \cdot 1 = (-1)^0$ iuxta Elementorum doctrinam. Series numero terminorum Infinita, in quam vertitur $(1-z)^x = (-1)^x$ casu $x=0$ nullam equidem lucem perfunderet, quinimmo, prouti decernunt antecedentes numeri 72. ac 73., Infinitate superveniente quaestio omnis illustranda.

&c: $D^{-1^0}, D^{-2}, D^{-3}, D^{-4}, D^{-5}, D^{-6}, D^{-7}, D^{-8}, D^{-9}, D^{-10},$

&c: $-10x, -9x, -8x, -7x, -6x, -5x, -4x, -3x, -2x, -x,$

Cave vero ne censeas Numeros Geometricè progredientes univèrse futuros Reales, quippe cum Baseos negativae necessitate Imaginariis quoque commisceantur. Index etenim x , quum quoslibet significare Numeros valeat atque idcirco etiam fractos vel fractionarios irreductibiles, quorum Numerator impar, Denominator autem par, Magnitudines in Geometrica Progressione emergent

&c: $-1, \frac{1}{\sqrt[4]{-1}}, -\sqrt{-1}, \frac{1}{\sqrt[4]{-1}}, +1, \frac{1}{\sqrt[4]{-1}}, -\sqrt{-1}, \frac{1}{\sqrt[4]{-1}}, -\sqrt{-1}, \frac{1}{\sqrt[4]{-1}}, -\sqrt{-1}, \frac{1}{\sqrt[4]{-1}},$

&c: $-3, -\frac{11}{4}, -\frac{5}{2}, -\frac{9}{4}, -2, -\frac{7}{4}, -\frac{3}{2}, -\frac{5}{4}, -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4},$

observandumque monemus Quadriformes esse Radices $\sqrt[4]{-1}$, et Biformes $\sqrt{-1}$, quod fufius in Capite VII. dilucidabitur. Multo magis Numeri Imaginariis exorientur statim atque $\tau d x$ valores acquirat Irrationales, et Transcendentes.

147. Eam, quam Dalembertius perhibet Arithmeticam Progressionem Nihilis absolutis existentem $0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0$ &c: in infinitum, Seriei Geometricae Imaginario-reali superius expositae logometricè respondentem, ratum firmumque fit ab nullius Exponentialis evolutione ortum ducere, nec Basin ullam assignari posse Realem, positivam, vel negativam, cuius ope hoc singulare Systema *exponentialiter* enascatur. Nec interim ausim infirmare principium, cui iamdudum voverunt sublimiores Geometrae, duas quaslibet Progressiones, Arithmeticam unam, Geometricam alteram, Numeris Realibus, Imaginariis, vel partim Realibus, partim Imaginariis compositas, Systema

stranda tenebris obrueretur. Ambae igitur Progreſſiones Arithmetica, nec non Geometrica, quae curiositate praestans Logometricum Systema componunt, cuius Basis vel Numerus-rector fit -1 , ita se habent, brevitatis causa praesupposito $(-1)^x = D^x$,

&c: D^{-1} .

$1, D^1, D^2, D^3, D^4, D^5, D^6, D^7, D^8, D^9, D^{10}$ &c:

$0, x, 2x, 3x, 4x, 5x, 6x, 7x, 8x, 9x, 10x$ &c:.

emergent Imaginariae, adeo ut Progreſſio ipsa coalescat Imaginariis, et Rea- libus Numeris, positivis, ac negativis. Exemplo sit $x = \frac{1}{4}$, eaque hypo- thesi facta Logometrici evadet Systematis pars ea, quam nunc compendio maximo concinnamus,

$-1,$

$+1, \sqrt[4]{-1}, \sqrt{-1}, \sqrt[4]{-1}, -1, \sqrt[4]{-1}, \sqrt{-1}, \sqrt[4]{-1}, +1, \sqrt[4]{-1}, \sqrt{-1}, \sqrt[4]{-1}, -1$

$0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, \frac{9}{4}, \frac{5}{2}, \frac{11}{4}, 3$ &c:

Systema Numerorum, ac Logarithmorum, dum simul quocumque ordine iun- gantur, efformare; sed ea tantum hic contemplor Systemata, quae ab omni- modis Logarithmicis Basibus originem suam *exponentialiter* ducant.

148. Modulum Systematis Logarithmici ab Aequatione universali $(-1)^x = y$ comprehensi qui investigare vellet oleum operamque perderet existi- mans ab Formulis numerorum 72. et 95. promi posse, analogia veluti facem ferente. Fit enim $(1-z)^x = (-1)^x$ tum cum $z = -\frac{1}{2}$; quamobrem Coefficientis secundi termini praecitatae Seriei peraequaret eo casu Seriem Geometrico-Harmonicam negativam $-\left(\frac{2^1}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \frac{2^4}{4} + \frac{2^5}{5} + \frac{2^6}{6} + \frac{2^7}{7} + \frac{2^8}{8} + \frac{2^9}{9} + \frac{2^{10}}{10} + \&c\right)$, vel Infinitam Ma- gnitudinem nihil determinantem, utpote ab mendosa Serie deductam. Exclu-

fo seu potius ob eius indeterminationem neglecto pseudo-Modulo, qui si analogiae fides in

$$\frac{2^1}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \frac{2^4}{4} + \frac{2^5}{5} + \frac{2^6}{6} + \frac{2^7}{7} + \frac{2^8}{8} + \frac{2^9}{9} + \frac{2^{10}}{10} \text{ \&c:} \quad \text{abiret,}$$

frustra verus Systematis Basi -1 distincti Modulus quaereretur in Nihilo, in Infinito, in Numero quodam reali integro, fracto &c., vel positivi, vel negativi valoris. Docent consulto praemissa, quae num°. 136. leguntur, Numeros universos reales, quin et ipsum Nihilum, atque Infinitum, Moduli sive characteristicae Mensurae munere fungi ad omnigena distinguenda Logarithmorum Systemata possibilia Basibus innixa positivis, integris, aut fractis &c., Nihiloque vel Infinito etiam, si libeat, aequalibus. Dummodo itaque in confesso sit, quod post numerum 146. demonstratione non eget, Systema Logometricum ortum ex Aequatione Exponentiali $(-1)^x = y$ sum-mopere differre ab Systemate, cuius Aequatio genitrix foret $(m)^x$, subintel-lecta magnitudine m positiva, nemo inficiari unquam poterit primo Syste-mati nullam Subtangente, Modulum, vel Numerum characteristicum k realem convenire, verumtamen impossibilem.

149. Idem concludit aliud quodcumque Logometriae Systema Basi ne-gativa, sed ab Unitatis pariter negativae valore diversa praeditum. Data namque Aequatione $(-z)^x = y$ Logarithmos, ac Numeros amice coniu-ngente vertitur haec in $(-1)^x (z)^x = y$, ne Serierum fallacia, ut in num°. 71. iam vidimus, Exponentialium perturbet utilem evolutionem; quo facto omnia pendent ab Systemate superius perpenso Aequationis Genitricis $(-1)^x = y$, communiaque sunt, partita eadem sorte, symptomata. Et re quidem vera Logarithmis x respondebunt Numeri aequales producto ex multiplicatione Numerorum realium positivorum in valores $(-1)^x$, qui quandoque positivi, negativi, reales, imaginarii per numerum 146. geome-trice tamen progrediuntur. Modulus ergo, Subtangens, aut Numerus k im-possibilis universaliter erit dum de Systemate agatur Logarithmico ad Ba-sin quomodolibet negativam relato, confirmante hanc tantae molis Analy-ticam veritatem argumento alias ducto ab Coefficiente secundi termini Seriei Exponentialis, in quam $(-z)^x$ evolvatur. Evadit namque $(-z)^x = (1 - (z+1))^x$, et idcirco pseudo-Modulus indeterminatus, mutusque fiet

$$\frac{(z+1)^1}{1} + \frac{(z+1)^2}{2} + \frac{(z+1)^3}{3} + \frac{(z+1)^4}{4} + \frac{(z+1)^5}{5} + \frac{(z+1)^6}{6} + \frac{(z+1)^7}{7} + \frac{(z+1)^8}{8} \&c;$$

vel $\frac{-1}{\infty \text{ Plusquam-Harm:}}$ (*), seu $-\infty$, admodum absurde instar moniti praecedentis.

150. Comprobat denique Modulum, aut Subtangenter esse Impossibile quoties sermo habeatur de Systematum Basibus generatim negativis Curva ipsa Logistica, quippe cum propter praestensa hisce in casibus nec ex parte positivarum, nec ex parte negativarum Coordinatim-applicatarum ullam Curvilineae perimetri speciem illa praeseferat, sed in medio stantibus Imaginariis Ordinatis innumeris Puncta tantummodo Coniugata huc illuc interspersa designet; quo facto nullius erit capax realis Tangentis, seu Subtangenter. Nullefcente enim Perimetro, prouti pictum in Schemate effingitur XXI., ridiculum quidem foret Lineas petere Tangentiales, quae necessario Curvarum Perimetrorum existentia negata negantur. Aequae a proposita abluderet meta ille, qui Systematis Punctorum Coniugatorum investigandis Tangentibus operam daret, quemadmodum si Curvis in Puncta abeuntibus illarum euthusein, sive tetragonismum somnianti vel aegro non impar quaerere moliretur (**). Haec dicta sint fortasse immaturius, non ut singulas arbitrer obscuritates sublatas, quas Logometricae disciplinae recondunt statim ac transitus fiat a veris Basibus positivis ad negativas, quin etiam imaginarias, et ab Indicibus Baseos, seu Logarithmis realibus ad impossibiles, vel imaginarietate notatos. Manent adhuc in Capitibus V. ac IX. difficiliora recensenda idem caute nendo Binomii Neutoniani filum, quod felicibus, ni fallor, auspiciis usque nunc patefecit non paucos rei Logarithmico-Exponentialis labyrinthos. Fundamenta iam posui, nec mutilae, quam haecenus fateor, theoreticae translationi plurima forte deerunt postmodum superaddenda, quae praecedentium adiumento non resolvantur. Interea universa fere, quae Logometricam respiciebant eruditionem, turpeque fuisset silentio praetermittere, aut incomptis strictim perscribere verbis, partim in Prolegomenis, partim Capite isto, in quo nuper versati, sedem, nisi eloquii cultu nobilitatam, saltem aliquam obtinere.

CAPUT

(*) Vide numerum 390.

(**) Adde dicenda in numero 394.

C A P U T V.

EVOLUTIO SERIERUM TRIGONOMETRICARUM.

151. **I**N recentiorum Algebrae ortu Magnitudines omnes non positivae, quas Problematum etiam numericorum resolutio suppeditabat, non modo veluti inutiles censebantur Calculi a quaestione abludentis superfoetationes, quin et mendosae, cauteque reiiciendae. Ita de negativis tunc temporis animadversis Aequationum Radicibus sensere Cardanus, Harriotus, Vieta, cunctique, qui annum MDCXXXVII. in Litteratorum Fastis percelebrem, utpote editione *Geometriae* Cartesianae condecoratum, antecesserunt. Nonnulli fictitios, privativos alii, defectivosve nuncuparunt valores subtractionis charactere distinctos, et, perinde ac si Algebra fallendo ludere vellet, aut nihil aut monstra quaedam illos significare sunt arbitrati. Ipse, nec dicere metuum, summus ipse Cartesius dubius haesisset fortassis in natura, usuque Radicum negativarum detegendis, quum *falsas* minus apte appellaverit, nisi Curvarum duce Geometria qualitatem Magnitudinum Continuarum, vel diversum existendi modum praeter illarum quantitatem contemplari, novoque uberri-
mo principio Analysin locupletare ei feliciter contigisset. Quae, cum in Algebrae incunabulis circa Magnitudines negativas, olim iacturae proximas, nunc utilitatis refertas, accidisse Matheseos historia nos doceat, veniam dabimus libentissime primis Aequationum cultoribus, si imaginarias expressiones, negativis equidem perplexiores, in Calculo vix salutaverint. Hoc unum tantummodo mirari oporteret, quod, tametsi anno MDCXXIX. Albertus Girardus, Aequationis Cubicae ante omnes fidus interpret, formulas Cardanicas Aequationum tertii Gradus imaginarietate turbatas triplicem realem Magnitudinem nihilominus praebere demonstraverit, longiore postmodum, quam par fuisset, mora Analystae Imaginariorum usum in Algebra horrere fuerint. Experientia etenim rerum universarum magistra regiam viam ope illius exempli aperuerat utilitati, quin etiam Imaginariorum necessitati in Magnitudinum Calculo pernoscentiae.

152. Fictitiam habere existentiam Quantitates Imaginarias vel nomen ipsum

ipsum luculenter ostendit. Qui autem Matheseos doctrinae navarunt operam ignorare sine turpitudine nequeunt, saepenumero Magnitudines impossibiles symbolis, notisve in Algorithmo prouti existerent rite tractatas maxima ad rem Mathematicam amplificandam commoda asserre. Elementa siquidem Magnitudinum Infinitesima aequae imaginaria sunt, et impossibilis existentiae: his tamen, et Nihilis ipsis, atque Infinitis nullo reali modo, sed entium potius quadam contradictione posita existentibus dum conventionales ac repraesentativae tesserae tribuantur, recentissimae Geometriae miracula prodierunt. Quandoque enim ob mutuam Imaginariorum elisionem Realia extricantur summo Formularum universalitatis iure, quod sine Imaginarietatis specie commixta turbatum esset, commodoque manentibus. Nec raro accidit, insertis Calculo Imaginariis perinde ac si forent Reales, mirificum in modum Analyticas quasdam expressiones eloquentiores redditas contrahi, et innocui huius mendivi referari difficiliorum Theorematum penetralia. Ista dum argumenti occasione obiter memoro, ne quis me nimio fictitii, aut symbolici Calculi amore raptum putet, nimiaeve insimulet credulitatis. Publicam namque litteratorum societatem hic obtestari non timeo improbandas esse Nihilorum et Infinitorum veras, quas nonnulli vindicant, proportionales, adnumerandas aegri veluti somniis Magnitudinum negativarum quasi reapse Nihilo minorum contemplationes, et universa de minusquam-Nihilis, ac de plusquam-Infinitis, quae Analysin deturpavere, delenda. Fateor, quum dicam Imaginarias Radices, semper tacito mentis conceptu me subintelligere nullas, non quia evanescentes, immo potius uti sensus expertes, atque impossibiles. Omnia fateor Polynomia, ideoque prima Aequationum Membra, si paris Gradus fuerint in Factores Trinomiales reales quacumque hypothese facta resolvi posse, si disparis in unum Binomiale Factorem ac Trinomiales ceteros, nec praeter Trinomiales ipsos veram ulteriorem decompositionem haec admittere Polynomia nisi conditionibus Coefficientium ita ferentibus, ut omnis Imaginarietatis species removeatur. Verumtamen, quum Imaginariarum praesentia Magnitudinum aliquando Formulas Realem intimo sensu complectentes valorem inficiat, et artificio non levi opus sit ad contrariorum Impossibilium praecisam destruitilitatem diiudicandam, liceat *Postulatorum* more, non secus ac in asymmetris Quantitatibus iamdiu usuvenit, Signis quibusdam haec Imaginaria, vel Impossibilia Algorithmi subiicere, et de mendosis Magni-

tudinibus non mendosum Calculum instituere. Rei prima fronte sterilis, pueriliterque iocosae non modicos fructus innumera evincunt exempla; at eo magis hoc confirmabit tota pene sublimior Trigonometria Symbolis Imaginariorum innixa, ac inde sine Differentialium, vel Infinitorum subsidio ultra spem exornata.

153. Binomii ope Neutroniani integrum Caput II.^{um} decernit haberi z^x

$$= (1)^x \times \left(1 + \frac{(Lz)^1 x^1}{1} + \frac{(Lz)^2 x^2}{1.2} + \frac{(Lz)^3 x^3}{1.2.3} + \frac{(Lz)^4 x^4}{1.2.3.4} + \frac{(Lz)^5 x^5}{1.2.3.4.5} + \frac{(Lz)^6 x^6}{1.2.3.4.5.6} \right. \\ \left. + \frac{(Lz)^7 x^7}{1.2.3.4.5.6.7} + \frac{(Lz)^8 x^8}{1.2.3.4.5.6.7.8} + \frac{(Lz)^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{(Lz)^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c. \right). \text{ In}$$

hac Aequatione universalissima, tam Basi z , quam Indice x quomodocumque variabilibus praedita, fingamus nova quaerentes Indicem x Imaginarium, prius simplicem $\sqrt{-1}$, posteaque, sed admodum serius, compositum $y + \sqrt{-1}$. Basis tamen z hoc interim maneat Realis, ne graviore fatiscant mole, fordeantque in obscuris quaesita Theoremata. Ut vero intempestivis repetitionibus Lector ab incoepta semita non deflectatur, curae nobis sit insuper

$$\text{analogae Aequalitas } z^{-x} = (1)^{-x} \times \left(1 - \frac{(Lz)^1 x^1}{1} + \frac{(Lz)^2 x^2}{1.2} - \frac{(Lz)^3 x^3}{1.2.3} \right. \\ \left. + \frac{(Lz)^4 x^4}{1.2.3.4} - \frac{(Lz)^5 x^5}{1.2.3.4.5} + \frac{(Lz)^6 x^6}{1.2.3.4.5.6} - \frac{(Lz)^7 x^7}{1.2.3.4.5.6.7} + \frac{(Lz)^8 x^8}{1.2.3.4.5.6.7.8} \right. \\ \left. - \frac{(Lz)^9 x^9}{1.2.3.4.5.6.7.8.9} + \frac{(Lz)^{10} x^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c. \right), \text{ nec oculos fugiat Factum } \sqrt{-1}.$$

$y\sqrt{-1}$ peraequare $-y$, ideoque Uniformem esse Functionem, non Biformem, quam fortassis absurditatem aliquis non satis cautus amplecteretur multiplicationem iniens $\sqrt{-v^2} \cdot \sqrt{-y^2}$ non parum periculosam. Erit itaque Functio Uniformis etiam Factum sive Quadratum $\sqrt{-v} \cdot \sqrt{-v}$, aequabitque tantummodo $-v$, quod evidenter satis aequipollentia alterius Facti demonstrat $\sqrt{v} \cdot \sqrt{-1} \times \sqrt{v} \cdot \sqrt{-1}$, aut $(\sqrt{v}(\sqrt{-1}))^2$. Nullam praeterea patitur

difficultatem Aequationum veritas prope intuitiva $\frac{+1}{\pm\sqrt{-1}} = \mp\sqrt{-1}$, et e-

converso $\frac{-1}{\pm\sqrt{-1}} = \pm\sqrt{-1}$, quibus expressionibus dum maxima accedat

univer-

universalitas, $\frac{+x}{\pm\sqrt{-x}} = \mp\sqrt{-x}$, nec non $\frac{-x}{\pm\sqrt{-x}} = \pm\sqrt{-x}$, vel alia in-

ducta forma $\frac{+x}{\pm x\sqrt{-1}} = \mp\sqrt{-1}$, $\frac{-x}{\pm x\sqrt{-1}} = \pm\sqrt{-1}$ singulae eva-

dent; nam de gemino $\sqrt{-1}$ valore, seu valoris absurdi Symbolo, ne minimae quidem dubitationi fit locus.

154. His in antecessum recensitis nemo non videt $z^v\sqrt{-1}$ abire in subiectam numero terminorum Seriem Infinitam $(1)^{v\sqrt{-1}} \times$

$$\left(1 + \frac{(Lz)^1 v \sqrt{-1}}{1} - \frac{(Lz)^2 v^2}{1.2} - \frac{(Lz)^3 v^3 \sqrt{-1}}{1.2.3} + \frac{(Lz)^4 v^4}{1.2.3.4} - \right.$$

$$\frac{(Lz)^5 v^5 \sqrt{-1}}{1.2.3.4.5} - \frac{(Lz)^6 v^6}{1.2.3.4.5.6} - \frac{(Lz)^7 v^7 \sqrt{-1}}{1.2.3.4.5.6.7} + \frac{(Lz)^8 v^8}{1.2.3.4.5.6.7.8} -$$

$$\frac{(Lz)^9 v^9 \sqrt{-1}}{1.2.3.4.5.6.7.8.9} - \frac{(Lz)^{10} v^{10}}{1.2.3.4.5.6.7.8.9.10} - \frac{(Lz)^{11} v^{11} \sqrt{-1}}{1.2.3.4.5.6.7.8.9.10.11} + \&c: \left. \right),$$
 eadem-

que ratione ob geminum $\sqrt{-1}$ Signum, positivi ac negativi valoris Imagi-

narii, haberi $z^{-v\sqrt{-1}} = (1)^{-v\sqrt{-1}} \times \left(1 - \frac{(Lz)^1 v \sqrt{-1}}{1} - \right.$

$$\frac{(Lz)^2 v^2}{1.2} + \frac{(Lz)^3 v^3 \sqrt{-1}}{1.2.3} + \frac{(Lz)^4 v^4}{1.2.3.4} - \frac{(Lz)^5 v^5 \sqrt{-1}}{1.2.3.4.5} - \frac{(Lz)^6 v^6}{1.2.3.4.5.6}$$

$$+ \frac{(Lz)^7 v^7 \sqrt{-1}}{1.2.3.4.5.6.7} + \frac{(Lz)^8 v^8}{1.2.3.4.5.6.7.8} - \frac{(Lz)^9 v^9 \sqrt{-1}}{1.2.3.4.5.6.7.8.9} - \frac{(Lz)^{10} v^{10}}{1.2.3.4.5.6.7.8.9.10}$$

$$+ \frac{(Lz)^{11} v^{11} \sqrt{-1}}{1.2.3.4.5.6.7.8.9.10.11} + \&c: \left. \right).$$
 In utraque expressione nefas esset inobser-

vatam relinquere elegantissimam Signorum dispositionem, quae fert, ut non termini singuli, sed successivi terminorum Binarii alternatim sint positivi,

vel negativi. Morem autem ipsum sequuti reticentiae Factoris $(1)^{\pm v\sqrt{-1}}$, cui ius omne in integrum restituemus Capite VII. ad fartam testam Multiformitatem Functionis servandam, secernamus in praecedentibus compendii, nitorisque maioris causa Realia ab Imaginariis, et facile novus ordo retroa-

starum

Starum Aequationum elicietur $z^v \sqrt{-1} = 1 - \frac{(Lz)^2 v^2}{1.2} + \frac{(Lz)^4 v^4}{1.2.3.4} -$

$$\frac{(Lz)^6 v^6}{1.2.3.4.5.6} + \frac{(Lz)^8 v^8}{1.2.3.4.5.6.7.8} - \frac{(Lz)^{10} v^{10}}{1.2.3.4.5.6.7.8.9.10} + \frac{(Lz)^{12} v^{12}}{1.2.3.4.5.6.7.8.9.10.11.12} -$$

$$\frac{(Lz)^{14} v^{14}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14} + \frac{(Lz)^{16} v^{16}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16} -$$

$$\frac{(Lz)^{18} v^{18}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16.17.18} + \&c: + \left(\frac{(Lz)^1 v^1}{1} - \frac{(Lz)^3 v^3}{1.2.3} + \right.$$

$$\frac{(Lz)^5 v^5}{1.2.3.4.5} - \frac{(Lz)^7 v^7}{1.2.3.4.5.6.7} + \frac{(Lz)^9 v^9}{1.2.3.4.5.6.7.8.9} - \frac{(Lz)^{11} v^{11}}{1.2.3.4.5.6.7.8.9.10.11} +$$

$$\frac{(Lz)^{13} v^{13}}{1.2.3.4.5.6.7.8.9.10.11.12.13} - \frac{(Lz)^{15} v^{15}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15} +$$

$$\frac{(Lz)^{17} v^{17}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16.17} - \frac{(Lz)^{19} v^{19}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16.17.18.19}$$

$$+ \&c:) \sqrt{-1}, \text{ nec non } z^{-v} \sqrt{-1} = 1 - \frac{(Lz)^2 v^2}{1.2} + \frac{(Lz)^4 v^4}{1.2.3.4} -$$

$$\frac{(Lz)^6 v^6}{1.2.3.4.5.6} + \frac{(Lz)^8 v^8}{1.2.3.4.5.6.7.8} - \frac{(Lz)^{10} v^{10}}{1.2.3.4.5.6.7.8.9.10} +$$

$$\frac{(Lz)^{12} v^{12}}{1.2.3.4.5.6.7.8.9.10.11.12} - \frac{(Lz)^{14} v^{14}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14} +$$

$$+ \frac{(Lz)^{16} v^{16}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16} - \frac{(Lz)^{18} v^{18}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16.17.18}$$

$$+ \&c: - \left(\frac{(Lz)^1 v^1}{1} - \frac{(Lz)^3 v^3}{1.2.3} + \frac{(Lz)^5 v^5}{1.2.3.4.5} - \frac{(Lz)^7 v^7}{1.2.3.4.5.6.7} + \frac{(Lz)^9 v^9}{1.2.3.4.5.6.7.8.9} \right.$$

$$- \frac{(Lz)^{11} v^{11}}{1.2.3.4.5.6.7.8.9.10.11} + \frac{(Lz)^{13} v^{13}}{1.2.3.4.5.6.7.8.9.10.11.12.13} -$$

$$\frac{(Lz)^{15} v^{15}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15} + \frac{(Lz)^{17} v^{17}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16.17} -$$

$$\frac{(Lz)^{19} v^{19}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16.17.18.19} + \&c:) \sqrt{-1}. \text{ Verumtamen liceat}$$

priorem Serierum partem Realem, quae alternis itidem Signis praedita nul-

lo modo differt tam in evoluta Exponentiali Magnitudine $z^v \sqrt{-1}$, quam

in alia

in alia $z^{-v\sqrt{-1}}$, five $\frac{1}{z^{v\sqrt{-1}}}$, significare littera Δ , partemque alteram

Imaginario Factore distinctam, Signoque gemino duplicem redditam littera Φ ; adeo, ut unica obtineatur brevior Aequatio sub specie Imaginariorum, quae apte composita formam induat $z^{\pm v\sqrt{-1}} = \Delta \pm \Phi \sqrt{-1}$. Nec ut immoremur oportet in analogis, directeque fluentibus Aequationibus demon-

strandis $z^{v\sqrt{-1}} = \left(1 + \frac{(Lz)v\sqrt{-1}}{\infty}\right)^{\infty}$, et $z^{-v\sqrt{-1}} =$

$\left(1 - \frac{(Lz)v\sqrt{-1}}{\infty}\right)^{\infty}$, quippe cum comprobentur ab Formula Neutroniani Binomii, atque notissimis IIⁱ. Capitis Aequalitatibus generatim ostensis, dummodo tantum Indici $\pm x$ alter $\pm v\sqrt{-1}$ substituatur.

155. Age nunc Serierum Δ , et Φ naturam, praecipuaque symptomata investigemus. Statim vero geometricam excitat sagacitatem utrorumque Exponentialium comparatio $z^{v\sqrt{-1}}$ ac $z^{-v\sqrt{-1}}$; eorum etenim Summa, et

Differentia Aequationes progignunt $z^{v\sqrt{-1}} + z^{-v\sqrt{-1}} = 2\Delta$, ideoque

$$\frac{z^{v\sqrt{-1}} + z^{-v\sqrt{-1}}}{2} = \Delta = 1 - \frac{(Lz)^2 v^2}{1.2} + \frac{(Lz)^4 v^4}{1.2.3.4} - \frac{(Lz)^6 v^6}{1.2.3.4.5.6} + \frac{(Lz)^8 v^8}{1.2.3.4.5.6.7.8} - \frac{(Lz)^{10} v^{10}}{1.2.3.4.5.6.7.8.9.10} + \frac{(Lz)^{12} v^{12}}{1.2.3.4.5.6.7.8.9.10.11.12} - \frac{(Lz)^{14} v^{14}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14} + \frac{(Lz)^{16} v^{16}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16} - \frac{(Lz)^{18} v^{18}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16.17.18} + \&c: =$$

$$\left(1 + \frac{(Lz)v\sqrt{-1}}{\infty}\right)^{\infty} + \left(1 - \frac{(Lz)v\sqrt{-1}}{\infty}\right)^{\infty}, \text{ et vicissim } z^{v\sqrt{-1}} -$$

$$z^{-v\sqrt{-1}} = 2\Phi\sqrt{-1}, \text{ nimirum } \frac{z^{v\sqrt{-1}} - z^{-v\sqrt{-1}}}{2\sqrt{-1}} = \Phi = \frac{(Lz)^1 v^1}{1}$$

E e

$$= \frac{(Lz)^3 v^3}{1.2.3}$$

198 C A P U T V.

$$\begin{aligned}
 & \frac{(Lz)^3 v^3}{1.2.3} + \frac{(Lz)^5 v^5}{1.2.3.4.5} + \frac{(Lz)^7 v^7}{1.2.3.4.5.6.7} + \frac{(Lz)^9 v^9}{1.2.3.4.5.6.7.8.9} \\
 & + \frac{(Lz)^{11} v^{11}}{1.2.3.4.5.6.7.8.9.10.11} + \frac{(Lz)^{13} v^{13}}{1.2.3.4.5.6.7.8.9.10.11.12.13} \\
 & + \frac{(Lz)^{15} v^{15}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15} + \frac{(Lz)^{17} v^{17}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16.17} \text{ \&c.} \\
 & = \frac{\left(1 + \frac{(Lz) v \sqrt{-1}}{\infty}\right)^\infty}{2 \sqrt{-1}} - \frac{\left(1 - \frac{(Lz) v \sqrt{-1}}{\infty}\right)^\infty}{2 \sqrt{-1}}. \text{ Quae ditissima expref-}
 \end{aligned}$$

fiones, cum paullulam aberrent a confimilibus in num°. 78. prolatis, fecunditate etiam pari insigniuntur, Formulasque imitantur ab Simpsonio in *Traſtatibus Miſcellaneis*, aliſque ſexcentis prolixiore Algorithmi Infinitesimalis ductu conſcriptas

$$\frac{M^A + M^{-A}}{2}, \text{ nec non } \sqrt{-1} \left(\frac{M^A - M^{-A}}{2} \right), \text{ poſita Ma-}$$

gnitudine $M = N^{\sqrt{-1}}$. Namque ope ſubſtitutionis hae poſtremae ver-

$$\text{tuntur in } \frac{N^{A\sqrt{-1}} + N^{-A\sqrt{-1}}}{2}, \text{ ac } \sqrt{-1} \left(\frac{N^{A\sqrt{-1}} - N^{-A\sqrt{-1}}}{2} \right) \text{ vel}$$

potius $\frac{N^{A\sqrt{-1}} - N^{-A\sqrt{-1}}}{2\sqrt{-1}}$ propter antea notatam in num°. 153. reci-
procitatis $\tau \sqrt{-1}$ affectionem.

156. Cum itaque eadem, quae ſupra tanto facilitatis proſpectu fluxe-

$$\text{re in num°. 80., molimina repetens inveniam } \left(\frac{z^{v\sqrt{-1}} + z^{-v\sqrt{-1}}}{2} \right)^2 =$$

$$\frac{z^{2v\sqrt{-1}} + 2 + z^{-2v\sqrt{-1}}}{4}, \text{ ſimiliterque } \left(\frac{z^{v\sqrt{-1}} - z^{-v\sqrt{-1}}}{2\sqrt{-1}} \right)^2 =$$

$$\frac{-z^{2v\sqrt{-1}} + 2 - z^{-2v\sqrt{-1}}}{4}, \text{ hinc fiet ſemper conſtans, et aequalis } \frac{2+2}{4}$$

vel 1², Summa Quadratorum $\Delta^2 + \Phi^2$, quando ex adverſo in Exponentia-
libus

libus Reali Indice praeditis Differentiam Quadratorum peraequare 1^2 iam exhibitum fuit, mirifica certe, ac praestantissima analogia Imaginarios nunc sociante Realibus olim recensitis Indicibus. Conclusionem ipsam definiunt Infinitinomia Δ , et Φ : sunt etenim eorum Potestates secundae simul sumptae

$$\begin{aligned}
 & 1 - \frac{(Lz)^2 v^2}{1} + \frac{(Lz)^4 v^4}{1.4} - \frac{(Lz)^6 v^6}{1.2.3.4} + \frac{(Lz)^8 v^8}{1.4.9.16} - \frac{(Lz)^{10} v^{10}}{1.3.4.5.9.16} + \frac{(Lz)^{12} v^{12}}{1.4.9.16.25.36} - \&c: \\
 & + \frac{(Lz)^4 v^4}{1.3.4} - \frac{(Lz)^6 v^6}{1.3.4.5.6} + \frac{(Lz)^8 v^8}{1.2.3.4.5.6} - \frac{(Lz)^{10} v^{10}}{1.2.3.4.5.6.7.8} + \frac{(Lz)^{12} v^{12}}{1.2.5.6.7.8.9.16} - \&c: \\
 & + \frac{(Lz)^8 v^8}{1.3.4.5.6.7.8} - \frac{(Lz)^{10} v^{10}}{1.3.4.5.6.7.8.9.10} + \frac{(Lz)^{12} v^{12}}{1.2.3.4.5.6.7.8.9.10} - \&c: \\
 & + \frac{(Lz)^{12} v^{12}}{1.3.4.5.6.7.8.9.10.11.12} - \&c: \\
 & = 1^2 \\
 & + \frac{(Lz)^2 v^2}{1} - \frac{(Lz)^4 v^4}{1.3} + \frac{(Lz)^6 v^6}{1.4.9} - \frac{(Lz)^8 v^8}{1.2.4.5.9} + \frac{(Lz)^{10} v^{10}}{1.4.9.16.25} - \frac{(Lz)^{12} v^{12}}{1.3.4.5.6.7.8.9.10.11} + \&c: \\
 & + \frac{(Lz)^6 v^6}{1.3.4.5} - \frac{(Lz)^8 v^8}{1.3.4.5.6.7} + \frac{(Lz)^{10} v^{10}}{1.2.4.5.6.7.9} - \frac{(Lz)^{12} v^{12}}{1.2.4.5.6.7.8.9.9} + \&c: \\
 & + \frac{(Lz)^{10} v^{10}}{1.3.4.5.6.7.8.9} - \frac{(Lz)^{12} v^{12}}{1.3.4.7.9.16.25} + \&c:
 \end{aligned}$$

propter mutuam terminorum omnium elisionem excepta Unitate ipsiusve Quadrato, quilibet valor variabilibus z , atque v ab Geometrarum pendens placitis tribuatur.

157. Si ergo variabilium z , atque v Functiones Δ , et Φ ita ex praecostensis comparantur, ut $\Delta^2 + \Phi^2 = 1^2 = 1$, ad Circuli Circumferentiam veluti

Sinus ac Cosinus-recti Arcus eiusdem $(Lz)v$, aut $v(Lz)$, vel $L(z^v)$, cuius Radius sit Unitas, iure merito referri poterunt. Nec difficultate laborat alterutrius Functionis in Sinum, vel Cosinum distinctio, propterea quod evanescente Arcu $(Lz)v$ evadit $\Delta = 1$ vel Radius, nullefcitque Φ ; scilicet Δ est Cosinus, et Φ Sinus-rectus indefiniti Arcus $(Lz)v$ Circularis, ideoque

$z^{\pm v \sqrt{-1}} = \text{Cosin. Arc. C. } (Lz)v \pm (\text{Sin. Arc. C. } (Lz)v) \sqrt{-1}$. Ratum igitur,

tur,

cuius Hyperbolicus, vel Naturalis Logarithmus est Unitas, $C^{x\sqrt{-1}} = \Delta'$

$+ \Phi' \sqrt{-1} = \text{Cofin. Arc C. x} + (\text{Sin. Arc C. x}) \sqrt{-1}$, $C^{-x\sqrt{-1}} =$

$\frac{1}{C^{x\sqrt{-1}}} = \Delta' - \Phi' \sqrt{-1} = \frac{1}{\Delta' + \Phi' \sqrt{-1}} = \text{Cofin. Arc C. x} - (\text{Sin.}$

$\text{Arc C. x}) \sqrt{-1} = \frac{1}{\text{Cofin. Arc C. x} + (\text{Sin. Arc C. x}) \sqrt{-1}}$, $\frac{C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}}{2}$

$= \text{Cofin. Arc C. x}$, $\frac{C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}}{2\sqrt{-1}} = \text{Sin. Arc C. x}$, quos veluti pri-

mos fructus, non autem immaturos, sedulus colligo, ne opima futurae mes-

sis mole premente minutiores aristae forte pereant. 158. Sunt nonnulli, qui modo scribendi, non re differunt de Aequa-

tionibus agentes nuperrime concinnatis. Penes istos $\frac{C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}}{2}$ eva-

dit $\frac{C^{x\sqrt{-1}} + 1}{C^{x\sqrt{-1}}}$, vel $\frac{C^{2x\sqrt{-1}} + 1}{2C^{x\sqrt{-1}}}$, eodemque ritu altera Formula

$\frac{C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}}{2\sqrt{-1}}$ fit $\frac{C^{x\sqrt{-1}} - 1}{C^{x\sqrt{-1}}}$, sive, uti superius innui,

$\frac{C^{2x\sqrt{-1}} - 1}{2\sqrt{-1} (C^{x\sqrt{-1}})}$. Nec Eulerianas promere Formulas quiddam habet,

quod Geometram torqueat, quandoquidem num¹. 155. adiumento tritissimum

est fore ex iam dictis $C^{\pm x\sqrt{-1}} = \left(1 \pm \frac{x\sqrt{-1}}{\infty}\right)^{\infty} = \text{Cofin. Arc C. x} \pm$

F f

(Sin.

(Sin. Arc C. x) $\sqrt{-1}$, pariterque liquet esse $\frac{C^{x\sqrt{-1}} + C^{-x\sqrt{-1}}}{2} =$

$$\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} + \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{2} = \text{Cofin. Arc C. x, \&}$$

$$\frac{C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}}{2\sqrt{-1}} = \frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{2\sqrt{-1}} = \text{Sin.}$$

Arc C. x. Haftenus tradita Trigonometriae campum latissime aperiunt, in quo, si exspatiari liberet praeruptae imaginationis fraena laxando, nulla equidem prohiberent confinia.

159. Dum haec ipsa Theoremata ab Basi C ad illam Systematis Logarithmorum Decadici sive Briggsiani, aut ad aliam quamcumque transferre animus fuerit, id consulto paravimus usque a num°. 154. Exponentiali ma-

gnitudine universalissima $z^{\pm v\sqrt{-1}}$ incoeptam theoricen superstruentes. Esto etenim Basis assumpta quaelibet B, nimirum Numerus ille arbitrarii Systematis Logarithmici, cuius Hyperbolicus Logarithmus sit reciprocus Mo-

duli. Effecta hac hypothesi adipiscemur $B^{x\sqrt{-1}} = \text{Cofin. Arc C. (LB) x}$

$+ (\text{Sin. Arc C. (LB) x}) \sqrt{-1}$, praetereaue $B^{-x\sqrt{-1}} = \text{Cofin. Arc C. (LB) x}$
 $- (\text{Sin. Arc C. (LB) x}) \sqrt{-1}$. Insuper profluent secundariae Aequationes ex

primis compositae $B^{\pm x\sqrt{-1}} = \left(1 \pm \frac{x\sqrt{-1}(LB)}{\infty}\right)^{\infty}$, vel $= \left(1 \pm \frac{L(B^{x\sqrt{-1}})}{\infty}\right)^{\infty}$,

nec non $\frac{B^{x\sqrt{-1}} + B^{-x\sqrt{-1}}}{2} = \text{Cofin. Arc C. (LB) x} = \text{Cofin. Arc C. L(B^x)} =$

$\frac{\left(1 + \frac{x\sqrt{-1}(LB)}{\infty}\right)^{\infty} + \left(1 - \frac{x\sqrt{-1}(LB)}{\infty}\right)^{\infty}}{2}$, et denique $\frac{B^{x\sqrt{-1}} - B^{-x\sqrt{-1}}}{2\sqrt{-1}} =$

$\frac{(1 + \dots)}{2}$

$$\frac{\left(1 + \frac{x\sqrt{-1}(\text{LB})}{\infty}\right)^{\infty} - \left(1 - \frac{x\sqrt{-1}(\text{LB})}{\infty}\right)^{\infty}}{2\sqrt{-1}} \&c: . \text{Quae omnia, quanto sint usui}$$

in Trigonometricis disciplinis, ii sentiunt sublimiorum Mathematicum non ignari. Ceteris interim affectionibus usque nunc memoratis perpulchra supereminet proprietas perpetuae Formularum Trigonometricarum $\text{Cofin. Arc C. } x \rightarrow (\text{Sin. Arc C. } x) \sqrt{-1}$, atque $\text{Cofin. Arc C. } x \leftarrow (\text{Sin. Arc C. } x) \sqrt{-1}$ alternationis, qua evenit, ut $\text{Cofin. Arc C. } x \rightarrow (\text{Sin. Arc C. } x) \sqrt{-1} = \frac{1}{\text{Cofin. Arc C. } x \leftarrow (\text{Sin. Arc C. } x) \sqrt{-1}}$, five $= (\text{Cofin. Arc C. } x \leftarrow (\text{Sin. Arc C. } x) \sqrt{-1})^{-1}$, eodemque pacto vicissim $\text{Cofin. Arc C. } x \leftarrow (\text{Sin. Arc C. } x) \sqrt{-1} = \frac{1}{\text{Cofin. Arc C. } x \rightarrow (\text{Sin. Arc C. } x) \sqrt{-1}}$ $= (\text{Cofin. Arc C. } x \rightarrow (\text{Sin. Arc C. } x) \sqrt{-1})^{-1}$ iuxta demonstrationem in num^o. proditam 157. Haec affabra inversio coruscat facilius in primigenia Circuli natura, scilicet in Aequatione fundamentali $1 = (\text{Cofin. Arc C. } x \rightarrow (\text{Sin. Arc C. } x) \sqrt{-1}) (\text{Cofin. Arc C. } x \leftarrow (\text{Sin. Arc C. } x) \sqrt{-1}) = (\text{Cofin. Arc C. } x)^2 \rightarrow (\text{Sin. Arc C. } x)^2$, atque evidentem servat analogiam Theorematis ad calcem Capitis II., praesertim num^o. 75., et deinceps allatis. Quin etiam neminem fugere opinor universalioris Aequationis exinde genitae $(\text{Cofin. Arc C. } x \rightarrow (\text{Sin. Arc C. } x) \sqrt{-1})^{\pm v} = (\text{Cofin. Arc C. } x \leftarrow (\text{Sin. Arc C. } x) \sqrt{-1})^{\mp v}$, cum ea quippe ab Potentiatione prodeat perquam facillime.

160. Nostrae autem methodo exhibere se properat altera oecumenica, et prae omnibus, quae consequentur, insignior Aequalitas, cui penitus demonstrandae plerique inductione tantum satisfacere potuerunt Geometrarum, nimirum $(\text{Cofin. Arc C. } x \pm (\text{Sin. Arc C. } x) \sqrt{-1})^z = \text{Cofin. Arc C. } zx \pm (\text{Sin. Arc C. } zx) \sqrt{-1}$, ubi Indicem z te interpretari generaliter velim, uti in aliis Exponentialibus, quemcumque vel positivum Numerum vel negativum.

Pronum namque est vi anteaetae Aequationis $C^{\pm x\sqrt{-1}} = \text{Cofin. Arc C. } x \pm (\text{Sin. Arc C. } x) \sqrt{-1}$ alteram statim surgere sine ulla Indicis exceptione

$$C^{\pm zx\sqrt{-1}} = \text{Cofin. Arc C. } zx \pm (\text{Sin. Arc C. } zx) \sqrt{-1} = (C^{\pm x\sqrt{-1}})^z$$

[Faint, mostly illegible text consisting of several lines of mathematical derivations and commentary.]

[Faint text at the bottom of the page, possibly a concluding remark or a reference.]

$$\begin{aligned}
 x \sqrt{-1} &= v \sqrt{-1} - \frac{v^2}{2} * - \frac{v^4}{4} * - \frac{v^6}{16} * \\
 &+ \frac{v^2}{2} + \frac{v^3 \sqrt{-1}}{2} - \frac{v^4}{8} + \frac{v^5 \sqrt{-1}}{8} - \frac{v^6}{16} + \frac{v^7 \sqrt{-1}}{16} \\
 &- \frac{v^3 \sqrt{-1}}{3} + \frac{v^4}{2} + \frac{v^5 \sqrt{-1}}{4} + \frac{v^6}{12} + \frac{v^7 \sqrt{-1}}{8} \\
 &- \frac{v^4}{4} - \frac{v^5 \sqrt{-1}}{2} + \frac{3v^6}{8} * \\
 &+ \frac{v^5 \sqrt{-1}}{5} - \frac{v^6}{2} - \frac{v^7 \sqrt{-1}}{2} \\
 &+ \frac{v^6}{6} + \frac{v^7 \sqrt{-1}}{2} \\
 &- \frac{v^7 \sqrt{-1}}{7}
 \end{aligned}$$

Verum in hoc Analytico Trigono (*) Lineae omnes verticales Imaginarie-
tate

(*) Vacuum mirare Lector stellula maiore distinctum, ac in nucleo quasi positum
Trianguli ipsius Analytici, cuius passionis exemplum vel nullum vel rarissimum extat
in Algebra universa.

$$\begin{array}{rcl}
 -\frac{5v^8}{128} & * & -\frac{7v^{10}}{256} & * & -\frac{21v^{12}}{1024} & * & -\&c: \\
 -\frac{5v^8}{128} + \frac{5v^9\sqrt{-1}}{128} - \frac{7v^{10}}{256} + \frac{7v^{11}\sqrt{-1}}{256} - \frac{21v^{12}}{1024} & & & & & & \&c: \\
 -\frac{v^8}{32} + \frac{5v^9\sqrt{-1}}{64} + \frac{v^{10}}{64} + \frac{7v^{11}\sqrt{-1}}{128} + \frac{7v^{12}}{768} & & & & & & \&c: \\
 -\frac{11v^8}{64} + \frac{v^9\sqrt{-1}}{32} + \frac{13v^{10}}{128} + \frac{v^{11}\sqrt{-1}}{32} + \frac{35v^{12}}{512} & & & & & & \&c: \\
 -\frac{v^8}{8} - \frac{3v^9\sqrt{-1}}{16} + \frac{19v^{10}}{160} - \frac{3v^{11}\sqrt{-1}}{32} + \frac{3v^{12}}{32} & & & & & & \&c: \\
 -\frac{5v^8}{8} - \frac{7v^9\sqrt{-1}}{24} - \frac{5v^{10}}{32} - \frac{7v^{11}\sqrt{-1}}{32} - \frac{v^{12}}{24} & & & & & & \&c: \\
 -\frac{v^8}{2} + \frac{3v^9\sqrt{-1}}{4} - \frac{v^{10}}{2} + \frac{v^{11}\sqrt{-1}}{16} - \frac{5v^{12}}{16} & & & & & & \&c: \\
 -\frac{v^8}{8} - \frac{v^9\sqrt{-1}}{2} + \frac{7v^{10}}{8} + \frac{3v^{11}\sqrt{-1}}{4} - \frac{7v^{12}}{64} & & & & & & \&c: \\
 & + \frac{v^9\sqrt{-1}}{9} - \frac{v^{10}}{2} - v^{11}\sqrt{-1} + \frac{25v^{12}}{24} & & & & & \&c: \\
 & + \frac{v^{10}}{10} + \frac{v^{11}\sqrt{-1}}{2} - \frac{9v^{12}}{8} & & & & & \&c: \\
 & - \frac{v^{11}\sqrt{-1}}{11} + \frac{v^{12}}{2} & & & & & \&c: \\
 & & & & & & \frac{v^{12}}{12} & \&c: \\
 & & & & & & & \&c:
 \end{array}$$

Verum

tate non affectae fingillatim componuntur terminis se mutuo elidentibus; quam-

 obrem Aequatio simplicior redditur, evaditque $x\sqrt{-1} = \frac{v\sqrt{-1}}{1} + \frac{v^3\sqrt{-1}}{6}$

208 C A P I T U L U M V.

$$\frac{v^3\sqrt{-1}}{6} + \frac{3v^5\sqrt{-1}}{40} + \frac{5v^7\sqrt{-1}}{112} + \frac{35v^9\sqrt{-1}}{1152} + \frac{63v^{11}\sqrt{-1}}{2816} + \&c., \text{ et}$$

ope partitionis per $\sqrt{-1}$ eliminatis omnibus Imaginariis fit tandem, fortuna

$$\text{admodum favente, } x = \frac{v^1}{1} + \frac{v^3}{6} + \frac{3v^5}{40} + \frac{5v^7}{112} + \frac{35v^9}{1152} + \frac{63v^{11}}{2816} +$$

&c.: Uterius habetur propter facile detectam Coefficientium Legem $x =$

$$v^1 + \frac{1^2 \cdot v^3}{2 \cdot 3} + \frac{1^2 \cdot 3^2 \cdot v^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot v^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot v^9}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot v^{11}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11}$$

+ &c.; aut, quemadmodum placuit Ioanni Bernoullio in *Anecdosis*, $x = v$

$$+ \frac{1 \cdot v^3}{2 \cdot 3} + \frac{1 \cdot 3 \cdot v^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 \cdot v^7}{2 \cdot 4 \cdot 6 \cdot 7} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot v^9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot v^{11}}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 11} + \&c., \text{ et eapro-$$

pter alterutrius adiumento elegantissimi canonis in promptu erit huiusce Seriei indefinita protractio.

162. Haud secus ac itaque Series iam concinnavimus, quibus Numeros Logarithmis, et e-converso Logarithmos Numeris exprimere liceat, Sinus etiam recti Arcubus Circularibus, Arcusque vicissim rectis-Sinibus cognitis obtinebuntur. Prospectum utrarumque Serierum hic addere iuvat.

$$\text{Arc C.} = \frac{(\text{SinR})^1}{1} + \frac{1^2(\text{SinR})^3}{1 \cdot 2 \cdot 3} + \frac{1^2 \cdot 3^2(\text{SinR})^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{1^2 \cdot 3^2 \cdot 5^2(\text{SinR})^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2(\text{SinR})^9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2(\text{SinR})^{11}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \&c.$$

$$\text{Sin R.} = \frac{(\text{ArcC})^1}{1} - \frac{(\text{ArcC})^3}{1 \cdot 2 \cdot 3} + \frac{(\text{ArcC})^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{(\text{ArcC})^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{(\text{ArcC})^9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} - \frac{(\text{ArcC})^{11}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \&c.$$

Frustra immorarer in comparandis superius scriptis Seriebus, quum absque vocum necessitate mutuae earum eluceant differentiae, mirabilesque similitudines. Nec aegre ferant Analystae Aequationem posuisse primitivam $x\sqrt{-1} = L(\text{Cofin. Arc C. } x + (\text{Sin. Arc C. } x)\sqrt{-1})$, vel $= -L(\text{Cofin. Arc C. } x - (\text{Sin. Arc C. } x)\sqrt{-1})$ loco alterius usitatoris, et ab Eulero quoque adhibitae $x\sqrt{-1} =$

$$\frac{1}{2} L \left(\frac{\text{Cofin. Arc C. } x + (\text{Sin. Arc C. } x)\sqrt{-1}}{\text{Cofin. Arc C. } x - (\text{Sin. Arc C. } x)\sqrt{-1}} \right), \text{ propterea quod } \tau \delta$$

$$\frac{1}{2} L \left(\frac{\text{Cofin. Arc C. } x + (\text{Sin. Arc C. } x) \sqrt{-1}}{\text{Cofin. Arc C. } x - (\text{Sin. Arc C. } x) \sqrt{-1}} \right) \text{ peraequat}$$

$$\frac{1}{4} L \left(\frac{\text{Cofin. Arc C. } x + (\text{Sin. Arc C. } x) \sqrt{-1}}{1} \right)^2, \text{ et ideo par fit } L (\text{Cofin. Arc C. } x + (\text{Sin. Arc C. } x) \sqrt{-1}), \text{ quae Formula omnibus simplicitate praecellit.}$$

163. Series numero terminorum Infinitas ad Circulares Arcus, rectosque eorundem Sinus pertinentes Convergentium classi generaliter adscribi debere in confesso iam est apud omnes. Liceat vero in ipsarum usum tuendum nostrae quoque methodi vim experiri. Prior Series novis alphabeticis

$$\text{elementis exposita praebet } x = v + \frac{1^2 \cdot A v^2}{2 \cdot 3} + \frac{3^2 \cdot B v^2}{4 \cdot 5} + \frac{5^2 \cdot C v^2}{6 \cdot 7} + \frac{7^2 \cdot D v^2}{8 \cdot 9} + \frac{9^2 \cdot E v^2}{10 \cdot 11} + \frac{11^2 \cdot F v^2}{12 \cdot 13} + \&c: \text{ modo, admissa Signorum considera-}$$

tione, hic et in ceteris Seriebus maiores Litterae A, B, C, D, E, F &c: loco sint terminorum propius antecedentium. Quilibet itaque terminus

esimus
n. post primum non adnumerandum erit ad sequentem $n+1$ esimum

in Ratione Geometrica $1 : \frac{(2n-1)^2 v^2}{2n(2n+1)}$, quae Proportio ob v Sinum-rectum

Unitate, seu Radio minorem semper erit maioris inaequalitatis, excepto tantum casu $v = 1$: tunc etenim postremus veluti Series terminus, ad quem nusquam pervenire sit datum, fiet propter n Infinitum suo praecedenti termino aequalis. Altera Series prioris conversa, nimirum $v =$

$$\frac{x^1}{1} - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{x^9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} -$$

$$\frac{x^{11}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \frac{x^{13}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13} - \&c: \text{ quae formae con-}$$

$$\text{tractionis est capax } v = x - \frac{A'x^2}{2 \cdot 3} - \frac{B'x^2}{4 \cdot 5} - \frac{C'x^2}{6 \cdot 7} - \frac{D'x^2}{8 \cdot 9} -$$

$$\frac{E'x^2}{10 \cdot 11} - \frac{F'x^2}{12 \cdot 13} - \&c: \text{ eo magis evidenter convergit, quum terminus}$$

H h

primo

primus
 primo excluso n. sit ad statim subsequentem ut $1: \frac{x^2}{2n(2n+1)}$, in qua

Proportione observari meretur, Quadratum Arcus x^2 , quomodocumque Arcus idem praegrandis supponatur, et quantolibet etiam Peripheriarum numero coalescens, dempto, vel addito Arcu z , finitam habiturum magnitudinem, interea dum Denominator $2n(2n+1)$ sine Limite excrescere poterit, et tandem reddere Rationis praefatae antecedentem suo consequente maiorem.

164. Porro ultra imaginationis, conceptusve hominum vires in Infinitate se condit adpropinquatio praesertim Seriei, qua Sinus Arcu cognito datur; sed spicilegium aliquod, multiplicationibus actu confectis, non ingratum fortassis erit ob oculos ponere. En Denominatoribus evolutis Formula ipsa superius exposita

$$v = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \frac{x^{11}}{39916800} + \frac{x^{13}}{6227020800} - \frac{x^{15}}{1307674368000} + \frac{x^{17}}{355687428096000} - \frac{x^{19}}{121645100408832000} + \frac{x^{21}}{51090942171709440000} - \&c.;$$

nullumque Natura in stupendo Denominatorum augmento Limitem novit. Accedit etiam ad Series Convergentes illa, quae dato Arcu Cofinum exhibet, nimirum

$$z = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \frac{x^{12}}{479001600} - \frac{x^{14}}{87178291200} + \frac{x^{16}}{20922789888000} - \frac{x^{18}}{6402373705728000} + \frac{x^{20}}{2432902008176640000} - \&c.;$$

Et re quidem vera, dum Series eadem usitato compendio maximo

$$disponatur in formam $z = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} - \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \frac{x^{12}}{1.2.3.4.5.6.7.8.9.10.11.12} - \&c. = 1$

$$- \frac{A''x^2}{1.2} - \frac{B''x^2}{3.4} - \frac{C''x^2}{5.6} - \frac{D''x^2}{7.8} - \frac{E''x^2}{9.10} - \frac{F''x^2}{11.12} - \&c.;$$$$

hic typus ipse demonstrat esse terminum, excluso confimiliter primo, n.

ad alium

ad alium illico sequentem veluti 1 ad $\frac{x^2}{2n(2n-1)}$, videlicet in Rationis huiusmo-

di natura, quae continuata terminorum Seriesi successione, ideoque audio sine fine Numero n evadet aliquando maioris Ratio inaequalitatis, quomodo- libet Arcus x semper finitus, et assignabilis magnitudine excrescat. Quin etiam idem contingit alteri Seriesi, quae praecognito Arcu ducit ad Si- nus-versi valorem, eoquod vi numⁱ. 157. praeter Unitatem, Signorumque inversionem eadem sit ac nuperrime enucleata Series pro Cosinu. Tuto igitur consilio universae Series Trigonometricae hactenus ad examen revo- catae sublimiori Analyfi inservient.

165. Regressus a Serie dudum prodita Cosinus per Arcum ad eam Arcus per Cosinum iactis in num^o. 161. sustinetur principiis. Ipsa siqui- dem universalis Aequatio $x\sqrt{-1} = L(\text{Cosin. Arc } C. x \rightarrow (\text{Sin. Arc. } C. x) \sqrt{-1})$ suppeditat facilem novi Problematis resolutionem. Nullus etenim dubito, quominus cunctis pateat relationem dupli Arcus ad duplum eius Sinum-rectum, vel Arcus ad Chordam, sive Subtensam haberi adiu- mento illius praenotatae Aequationis, scilicet, subintellectis semper Pote- statibus Unitatis, vel Radii in quocumque ex Denominatoribus, $\frac{x}{2} = \frac{y^1}{2^1}$

$$+ \frac{y^3}{2^3.6} + \frac{3y^5}{2^5.40} + \frac{5y^7}{2^7.112} + \frac{35y^9}{2^9.1152} + \frac{63y^{11}}{2^{11}.2816} + \&c., \text{ aut } x = y$$

$$+ \frac{y^3}{2^2.6} + \frac{3y^5}{2^4.40} + \frac{5y^7}{2^6.112} + \frac{35y^9}{2^8.1152} + \frac{63y^{11}}{2^{10}.2816} + \&c., \text{ posita } y \text{ loco}$$

Subtensae Arcui x respondentis. Atqui pari evidentia gaudet, quod, dum Sagittam, seu Sinum-versum Arcus x denotet z , Radiusque iuxta morem

Unitate repraesentetur, substitutionibus rite confectis erit $x = (2z)^{\frac{1}{2}}$

$$+ \frac{(2z)^{\frac{3}{2}}}{4.6} + \frac{3(2z)^{\frac{5}{2}}}{16.40} + \frac{5(2z)^{\frac{7}{2}}}{64.112} + \frac{35(2z)^{\frac{9}{2}}}{256.1152} + \frac{63(2z)^{\frac{11}{2}}}{1024.2816} + \&c., \text{ et asy-}$$

metria in unicum Factorem reiecta, Legeque Coefficientium in ordinem

$$\text{proprium digesta } x = (2z)^{\frac{1}{2}} \left(1 + \frac{1^2.z^1}{2^2.3} + \frac{1^2.3^2.z^2}{2^3.3.4.5} + \frac{1^2.3^2.5^2.z^3}{2^4.3.4.5.6.7} + \right.$$

$$\left. \frac{1^2.3^2.5^2.7^2.z^4}{2^5.3.} \right)$$

$$\frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2}{2^5 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot 11^2}{2^6 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \&c: \Big), \text{ aut demum } x =$$

$$(2z)^{\frac{1}{2}} \left(1 + \frac{z^2}{12} + \frac{3z^4}{160} + \frac{5z^6}{896} + \frac{35z^8}{18432} + \frac{63z^{10}}{90112} + \&c: \right). \text{ Interim itine-}$$

ris occasione acquisivimus Seriem converſam, cuius ſubſidio Arcus invenia-
tur in antecellum eius Sinu-verſo adſignato. Ab una quippe Aequatione

$$\text{Sin. V.} = \frac{(\text{Arc C})^2}{1.2} - \frac{(\text{Arc C})^4}{1.2.3.4} + \frac{(\text{Arc C})^6}{1.2.3.4.5.6} - \frac{(\text{Arc C})^8}{1.2.3.4.5.6.7.8} + \frac{(\text{Arc C})^{10}}{1.2.3.4.5.6.7.8.9.10} - \frac{(\text{Arc C})^{12}}{1.2.3.4.5.6.7.8.9.10.11.12} + \&c:$$

retrocedentibus pervenire nunc licet ad alteram

$$\text{Arc. C.} = (2\text{Sin V.})^{\frac{1}{2}} \left(1 + \frac{1^2 \cdot (\text{Sin V.})^2}{2^2 \cdot 3} + \frac{1^2 \cdot 3^2 \cdot (\text{Sin V.})^4}{2^3 \cdot 3 \cdot 4 \cdot 5} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot (\text{Sin V.})^6}{2^4 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot (\text{Sin V.})^8}{2^5 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \&c: \right),$$

de qua perpetuum fere, nec ob minorem certe elegantiam, apud Analyſtas
Elementographos ſilentium. In hac Serie condenda erravit Montuclas ad
Caput IV. *Historiae tentaminum pro inveniendâ Circuli Quadratura.*

166. Occurrit etiam vel nobis invitis gemina Aequatio tribuens Sub-
tenſam ope Arcus, et viciffim Arcum ex Chorda data, nec immerito utrae-
que heic ſedem expoſcunt.

$$\text{Chord.} = (\text{Arc C})^1 - \frac{(\text{Arc C.})^3}{4^1 \cdot 1.2.3} + \frac{(\text{Arc C.})^5}{4^2 \cdot 1.2.3.4.5} - \frac{(\text{Arc C.})^7}{4^3 \cdot 1.2.3.4.5.6.7} + \frac{(\text{Arc C.})^9}{4^4 \cdot 1.2.3.4.5.6.7.8.9} - \frac{(\text{Arc C.})^{11}}{4^5 \cdot 1.2.3.4.5.6.7.8.9.10.11} + \&c:$$

ſive contractionibus ſymbolis, evolutisſque Denominatoribus

$$y = x^1 - \frac{x^3}{24} + \frac{x^5}{1920} - \frac{x^7}{322560} + \frac{x^9}{92897280} - \frac{x^{11}}{40874803200} + \frac{x^{13}}{25505877196800} - \&c:,$$

deindeque viciffim

$$\text{Arc. C.} = (\text{Chord.})^1 + \frac{1^2 \cdot (\text{Chord.})^3}{2^2 \cdot 2.3} + \frac{1^2 \cdot 3^2 \cdot (\text{Chord.})^5}{2^4 \cdot 2.3.4.5} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot (\text{Chord.})^7}{2^6 \cdot 2.3.4.5.6.7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot (\text{Chord.})^9}{2^8 \cdot 2.3.4.5.6.7.8.9} + \&c:,$$

$$\text{vel } x = y^1 + \frac{1^2 \cdot y^3}{4^1 \cdot 2.3} + \frac{1^2 \cdot 3^2 \cdot y^5}{4^2 \cdot 2.3.4.5} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot y^7}{4^3 \cdot 2.3.4.5.6.7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot y^9}{4^4 \cdot 2.3.4.5.6.7.8.9} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot y^{11}}{4^5 \cdot 2.3.4.5.6.7.8.9.10.11} + \&c:.$$

Haſce Aequationes qui rite, recteque Analyticis periculis ſubiecerit ſibi
facilius,

facilius, quam reperita superioris methodi applicatione, summopere adpropinquantem istarum Serierum valorem suadebit. Ab eisdem admodum tritis considerationibus renovandis in posterum quoque abstinemus.

167. Sin littera t nunc Cofinum Arcus x denotemus, fiet Sinus- versus $z = 1 - t$; quamobrem post congruas substitutiones in Formula nume-

ri 165. sine ulla Calculi perplexitate obtinebimus $x = (2 \cdot \sqrt{1-t})^{\frac{1}{2}}$

$$\left(1 + \frac{1-t}{12} + 3 \cdot \frac{1-t^2}{160} + 5 \cdot \frac{1-t^3}{896} + 35 \cdot \frac{1-t^4}{18432} + 63 \cdot \frac{1-t^5}{90112} + \&c. \right),$$

vel genesi Coefficientium numericorum melius explicata $x = (2 \cdot \sqrt{1-t})^{\frac{1}{2}}$

$$\left(1 + \frac{1^2 \cdot 1-t}{2^2 \cdot 3} + \frac{1^2 \cdot 3^2 \cdot 1-t^2}{2^3 \cdot 3 \cdot 4 \cdot 5} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 1-t^3}{2^4 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 1-t^4}{2^5 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \right.$$

$$\left. \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot 1-t^5}{2^6 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot 11^2 \cdot 1-t^6}{2^7 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13} + \&c. \right). \text{Quin etiam ditif-}$$

sima Analysis peculiari artificio asymmetriam omnem ab expressione Arcus per Cofinum eliminare valet ita se gerens. Esto v Sinus-rectus Circularis Arcus x , et habebitur simul t Sinus-rectus Complementi Arcus x , five Arcus $Q - x$, adhibita specie Q loco Quadrantis Peripheriae. Igitur $Q - x =$

$$\frac{t^1}{1} - \frac{t^3}{6} + \frac{3t^5}{40} - \frac{5t^7}{112} + \frac{35t^9}{1352} - \frac{63t^{11}}{2816} + \&c.; \text{ aut } x =$$

$$Q - \left(t^1 - \frac{1^2 \cdot t^3}{2 \cdot 3} + \frac{1^2 \cdot 3^2 \cdot t^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot t^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot t^9}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} - \right.$$

$$\left. \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot t^{11}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \&c. \right), \text{ vel, dempto Arcus Quadrantal symbolo,}$$

$$x = 1 - \frac{1^2}{2 \cdot 3} + \frac{1^2 \cdot 3^2}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11}$$

$$+ \&c. - t^1 + \frac{1^2 \cdot t^3}{2 \cdot 3} - \frac{1^2 \cdot 3^2 \cdot t^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot t^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot t^9}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}$$

$$+ \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot t^{11}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} - \&c. \text{ ob necessariam Seriei evanescentiam hypothefi fa-}$$

cta Cofinus Radio Aequalis, nempe Arcus x nullescentis. Quibus effectis sequentes nanciscimur Series convergentiae munere locupletatas, et mira necessitudine iunctas

$$\text{Cofin} = 1 - \frac{(\text{Arc } C)^2}{1.2} + \frac{(\text{Arc } C)^4}{1.2.3.4} - \frac{(\text{Arc } C)^6}{1.2.3.4.5.6} + \frac{(\text{Arc } C)^8}{1.2.3.4.5.6.7.8} - \frac{(\text{Arc } C)^{10}}{1.2.3.4.5.6.7.8.9.10} + \frac{(\text{Arc } C)^{12}}{1.2.3.4.5.6.7.8.9.10.11.12} \&c:$$

$$\text{Arc } C = (2 ($$

$$\text{Arc } C = 1 + \frac{1^2}{2.3} + \frac{1^2.3^2}{2.3.4.5} + \frac{1^2.3^2.5^2}{2.3.4.5.6.7} + \frac{1^2.3^2.5^2.7^2}{2.3.4.5.6.7.8.9} + \frac{1^2.3^2.5^2.7^2.9^2}{2.3.4.5.6.7.8.9.10.11} + \&c:$$

tanta equidem ubertate argumentum istud pinguescit.

168. Sed in re sublimiori Trigonometrica iam praecipuis potiti doctrinis paullisper immoremur Formulae $z^{\pm v \sqrt{-1}} = \text{Cofin. Arc } C. (Lz)^v \pm (\text{Sin. Arc } C. (Lz)^v) \sqrt{-1}$ rursus operam navantes. Dum Index v , seu potius Indicis Factor realis Unitatem peraequaverit abibit Aequatio in $z^{\pm \sqrt{-1}} = \text{Cofin. Arc } C. Lz \pm (\text{Sin. Arc } C. Lz) \sqrt{-1}$; quamobrem excitabitur $z =$

$$(\text{Cofin. Arc } C. Lz \pm (\text{Sin. Arc } C. Lz) \sqrt{-1})^{\pm \sqrt{-1}} = (\text{Cofin. Arc } C. Lz \pm (\text{Sin. Arc } C. Lz) \sqrt{-1})^{\mp \sqrt{-1}} (*)$$

Quod ne absurdum putetur adeundam cenfeo demonstrationem ex Capite III. erutam: namque ibidem Aequatio reapse identica prostat $z = 1 + \frac{(Lz)^1}{1} + \frac{(Lz)^2}{1.2} + \frac{(Lz)^3}{1.2.3} + \frac{(Lz)^4}{1.2.3.4} + \frac{(Lz)^5}{1.2.3.4.5} +$

$$\frac{(Lz)^6}{1.2.3.4.5.6} + \frac{(Lz)^7}{1.2.3.4.5.6.7} + \frac{(Lz)^8}{1.2.3.4.5.6.7.8} + \frac{(Lz)^9}{1.2.3.4.5.6.7.8.9} + \frac{(Lz)^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: , \text{ eaque numeri praecedentis 160. dat } z = (\text{Cofin. Arc } C. Lz \pm (\text{Sin. Arc } C. Lz) \sqrt{-1})^{\pm \sqrt{-1}} = \text{Cofin. Arc } C. ((Lz) \cdot \mp \sqrt{-1})$$

$$\pm (\text{Sin. Arc } C. (Lz) \cdot \mp \sqrt{-1}) \sqrt{-1} = 1 + \frac{(Lz)^2}{1.2} + \frac{(Lz)^4}{1.2.3.4} + \frac{(Lz)^6}{1.2.3.4.5.6} + \frac{(Lz)^8}{1.2.3.4.5.6.7.8} + \&c:$$

(*) Hoc ipso in Capite ad calcem, et identidem Capite VII. non pauca tractantur isti expressioni non dissimilia.

$$\text{Arc C.} = (2(1 - \text{Cofin}))^{\frac{1}{2}} \left(1 + \frac{1^2(1 - \text{Cofin})^1}{2^2 \cdot 3} + \frac{1^2 \cdot 3^2(1 - \text{Cofin})^2}{2^3 \cdot 3 \cdot 4 \cdot 5} + \frac{1^2 \cdot 3^2 \cdot 5^2(1 - \text{Cofin})^3}{2^4 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2(1 - \text{Cofin})^4}{2^5 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \&c. \right),$$

vel tandem, si mavis,

$$- (\text{Cofin})^1 - \frac{1^2(\text{Cofin})^3}{2 \cdot 3} - \frac{1^2 \cdot 3^2(\text{Cofin})^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{1^2 \cdot 3^2 \cdot 5^2(\text{Cofin})^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2(\text{Cofin})^9}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2(\text{Cofin})^{11}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \&c.$$

tanta

$$\frac{(\text{Lz})^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \frac{(\text{Lz})^{10}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} + \&c. + \frac{(\text{Lz})^1}{1} + \frac{(\text{Lz})^3}{1 \cdot 2 \cdot 3} + \frac{(\text{Lz})^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{(\text{Lz})^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{(\text{Lz})^9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \&c.,$$

evolutis Cofinus, Sinusque-recti ex-

pressionibus. Hanc iucundam aequipollentiam qui sedulo meditetur non negligat quaeso animadversionem universaliter Realis valoris Formulae quamvis Imaginariis illaqueatae, ac prima fronte impossibilis (Cofin. Arc C. Lz $\pm$ (Sin

Arc C. Lz) $\sqrt{-1}$) $\mp \sqrt{-1}$. Exempla dent Bases C atque B Systematum Loga-

rithmicorum Hyperbolici, et Tabularis, quibus positis loco z sequentes vale-

bunt Aequalitates, videlicet C = (Cofin. Arc C. 1 $\pm$ (Sin. Arc C. 1) $\sqrt{-1}$) $\mp \sqrt{-1}$,

sive 2,718281828459 &c. = (0,540302305843 &c. $\pm$ (0,841470984805 &c.)

$\sqrt{-1}$) $\mp \sqrt{-1}$, nec non B = 10 = (Cofin. Arc C. 2,302585092994 &c. $\pm$

(Sin. Arc C. 2,302585092994 &c.) $\sqrt{-1}$) $\mp \sqrt{-1}$, quas sine methodi supe-

rius traditae robore nequidem coniectando divinare fas esset.

169. Curiositatis plenissima Imaginariorum elisio alteram quoque prius

expositam afficit Aequationem generaliore $z^{\pm v} = \text{Cofin. Arc C.} ((\text{Lz})$

$(\mp v \sqrt{-1})) \pm (\text{Sin. Arc C.} (\text{Lz}) (\mp v \sqrt{-1})) \sqrt{-1}$; nec de eius omnimoda

veritate dubitari unquam poterit, cum sponte fluat ab exaltatione praeteri-

tae simplicioris ad Potestatem Ordinis $\pm v$. En Realium omnium Magnitu-

dinum possibilium positivarum, quas denotet $z^{\pm v}$ universis idonea Potestas

significandis Numeris positivis, symbolum habemus ope Sinuum-rectorum,

Cofinumque Imaginariii Circuli Arcuum, cuius Radius sit Unitas; en

totius Harmoniae Hyperbolen inter, atque Circulum iacta fundamenta, quam theoricen, non amplius reconditam, nec ab Infiniti sinu petitam, explicare per summa capita nobis est animus. Necesse autem, ut praemittam, opinor, quod in quaestionibus Logometricis, cum Magnitudines solummodo versentur numericae, Arcus Circulares consoni sint Sectoribus respondentibus a centro computatis; ideoque loco Sinuum, et Cofinuum Arcuum ipsorum eadem generatim affectiones, legesque depraedicari poterunt de Sinubus, Cofinibus, et aliis Lineis omnibus Trigonometricis perinde ac si eae non ad Arcus pertinerent, sed potius ad Circulares Sectores, quorum Areae per medietatem Unitatis, vel Radii divisae Circulares revera Arcus restituunt.

170. Ne autem facillimum Hyperbolae, Circulique commercium obscuritatem vel minimam patiaturs repetenda manet proprietas in calce Capitis II. fusius explicita, quae definit universalem Aequalitatem differentiae

Quadratorum $\left(\frac{C^x + C^{-x}}{2}\right)^2$, et $\left(\frac{C^x - C^{-x}}{2}\right)^2$ quo ad Unitatis Quadra-

tum. Dum igitur in Schemate XXII.^o delineetur Hyperbola Apollonia-na Aequilatera, cuius Semi-Transversum Latus sit Unitas AB, Areaeque Sectorum ABC, ABC' &c: positivorum, vel ex adverbo negativorum ABC'',

ABC''' &c: arithmetice progredientium divisae per $\frac{AB}{2}$, nempe earum Du-
pla, significantur Numeris Indicibus x, erit vi praenotatae affectionis

$\frac{C^x + C^{-x}}{2} = AD, AD' \&c: = \text{Cofin. Hyperb. } x$, nec non $\frac{C^x - C^{-x}}{2} = DC,$

DC', seu DC'', DC''' &c: = Sin. Hyperb. x, quae denominationes debitam conti-
nent analogiam rei Circulari illustrandae admodum opportunam. Quin etiam, quum

generalius habeatur $\frac{z^x + z^{-x}}{2} = \text{Cofin. Hyperb. } (Lz)x$, atque $\frac{z^x - z^{-x}}{2}$

= Sin. Hyperb. (Lz)x, hoc in casu Sectores Hyperbolae per $\frac{AB}{2}$ divisi, sci-

licet pari ritu duplicati, quos haud improprie nuncupares Pseudo-Arcus Hyperbolae, non Numeros x, verum xLz, seu L(z^x) incedentes Progref-

sione Arithmetica repraesentarent. Hinc z^x = Cofin. Hyperb. xLz + Sin. Hyperb. xLz, atque z^{-x} = Cofin. Hyperb. xLz - Sin. Hyperb. xLz, ac

simpliciore ratione tam C^x = Cofin. Hyperb. x + Sin. Hyperb. x, quam C^{-x}

$C^{-x} = \text{Cofin. Hyperb. } x - \text{Sin. Hyperb. } x$. Quisque nunc videt oculis prope admotum Circuli, et Aequilaterae Hyperbolae foedus; attamen foedus fictitium, foedus non in intima rei substantia, sed in forma Analyticae expressionis positum, uno verbo dicam foedus inter Reales, et Imaginarios Numeros Trigonometricos simillima Formula designatos.

171. Magnitudo itaque Exponentialis $C^{\pm x}$ Numeris Cofin. Hyperb. $x \pm \text{Sin. Hyperb. } x$, nimirum $CE, C'E' \&c: CF, C'F' \&c:$ ab Hyperbola Apolloniana derivatis exprimi poterit adeo, ut hae Summae Differentiaeve Cofinum, Sinuumque Hyperbolicorum idem Logarithmicum Systema una cum Quotis Sectorum Hyperbolae per $\frac{1}{2}$ divisus, aut cum Sectoribus duplicatis, componant ac illud superius nuncupatum Hyperbolicum, vel Naturale. Nec mirum, quandoquidem Numeri Unitate AB maiores $CE, C'E' \&c:$, et ii infra Unitatem positi $CF, C'F' \&c:$ vel $C''E, C'''E' \&c:$ servata semper eadem Ratione Geometrica aequae proportionaliter incedunt, eiusdemque valoris arithmetici sunt uti Numeri positivi $AR, AR' \&c:$ Unitatem propriam AP superantes, nec non Numeri quoque positivi AR'', AR''' ab Unitate ipsa AP infra deficientes, simul ac Hyperbolica Quadrilinea positiva $BPRC, BPR'C' \&c:$, sive Sectores $ABC, ABC' \&c:$, atque negativa Quadrilinea $BPR''C'', BPR'''C''' \&c:$, vel Sectores negativi $ABC'', ABC''' \&c:$ Areis suis Arithmetice progredientes eundem pariter habeant valorem numericum si ad Unitatem $APBS$ referantur, vel respectiva ipsorum Dupla duplae Unitati AB^2 comparentur. Profecto constat ex affectione similium Triangulorum, quae Schema latius demonstrat, nihil differre inter se Proportiones Numericas $AR''':AR'':AP:AR:AR' \dots \dots \dots E'C''':EC'':BT:EC:E'C' \dots \dots \dots$, ipsamet manente, uti supra, Quadrilineorum, aut Sectorum Proportione Arithmetica, praeterquam in assumptae Unitatis arbitraria Magnitudine: nam primum AP , aut $AS = BP$, deindeque $AB = BT = BM = AH$, cuius Ratio ad AP eadem est ac illa constans $\sqrt{2}:1$, Unitatem Linearem oculis subiiciunt (*).

172. In elementari igitur Conicorum passione totum latet haecenus nimis iactatum sublimioris communionis Circuli, atque Hyperbolae negotium, vel

(*) Consule etiam numerum 353.

vel nonnullorum stylo transitus ab Reali ad Imaginariam Logometriam, et vicissim. Attamen rite cavendum, ne ultra verborum significationem Imaginaria, ac Realia Systemata habeantur quasi duo revera diversa, quemadmodum forte Dalembertius adseruit, quae nihilominus tacitae Continuitatis Lege coniuncta Circulum reddant BNHK Aequilaterae Hyperbolae Apollonianae concentricae veluti partem accessionemve illi geometrica necessitate copulandam. Quo facto vacuum inter oppositos Hyperbolae Ramos adeo compleri supponeretur, ut Circulum atque Hyperbolam eadem comprehenderet continua Analytica Formula $\pm \Phi \sqrt{-1} = L(x \pm \sqrt{1-x^2} \cdot \sqrt{-1})$ designans variabili Φ , dum x manserit infra Unitatem, duplos Circulares Sectores, dum supra e-contra Hyperbolicos, quibus Cofinus x responderent, versa tunc illa expressione in $\pm \Phi = L(x \pm \sqrt{x^2-1})$, Circulo etiam atque Hyperbolae simul significandae idoneam; et haec sententia in priore *Miscellaneorum Taurinensium* Volumine praestantissimis viris Foncenexio et Lagrangio placuisse videtur. Quid autem sibi vult imaginaria haec Aequatio Circulares Sinus, Cofinus, Sectoresque complectens $\pm \Phi \sqrt{-1} = L(\text{Cofin. 2 Sect. Circ. } \Phi \pm (\text{Sin. 2 Sect. Circ. } \Phi) \sqrt{-1})$ nisi Realem Aequationem identicam $\pm \Phi = L(\text{Cofin. 2 Sect. Hyperb. } \Phi \pm \text{Sin. 2 Sect. Hyperb. } \Phi)$, cuius prima nuper exposita Symbolum sit, vel ad Calculi commodum, sive universalitatem procurandam obliqua, fictaque commentatio? Ita, qui scriberet Algebraicae Curvae Aequationem Localem $y = \pm (\sqrt{x^2-1}) \sqrt{-1}$, adparenter Hyperbolen, verumtamen reapse Circulum nunciaret; atque eodem pacto $y = \pm (\sqrt{1-x^2}) \sqrt{-1}$ non ad Circulum, sed ad Hyperbolen, tametsi intempestive contorta, Aequalitas pertineret. Ego vero non Circulum, et Hyperbolam, at postremam tantum lego videoque Curvam tam in Aequatione Realibus Numeris composita $\pm \Phi = L(\text{Cofin. Hyperb. 2 Sect. } \Phi \pm \text{Sin. Hyperb. 2 Sect. } \Phi)$, eamque proprio diceret vocabulo *Naturalem*, quam in altera *Artificiali* $\pm \Phi \sqrt{-1} = L(\text{Cofin. Circ. 2 Sect. } \Phi \pm (\text{Sin. Circ. 2 Sect. } \Phi) \sqrt{-1})$, quae Lineis insertis Trigonometricis Circulum mentitur, ac *Exponentialiter* ope elidendae Imaginarietae repraesentare studet formam fictitiam induens ad instar emblematis, nempe $C^{\pm \Phi \sqrt{-1}} = \text{Cofin. 2 Sect. Circ. } \Phi \pm (\text{Sin. 2 Sect. Circ. } \Phi) \sqrt{-1}$. Velo namque disrupto Aequatio $C^{\pm \Phi \sqrt{-1}} = \text{Cofin. 2 Sect. Circ. } \Phi \pm (\text{Sin. 2 Sect. Circ. } \Phi) \sqrt{-1}$ idem sonat eliminatis Imaginariis

$$\pm \Phi \sqrt{-1} \\ \text{ac } C \frac{\Phi}{\sqrt{-1}} = \text{Cofin. 2 Sect. Circ. } \frac{\Phi}{\sqrt{-1}} \pm \left(\text{Sin. 2 Sect. Circ. } \frac{\Phi}{\sqrt{-1}} \right) \sqrt{-1},$$

five $C^{\pm \Phi} = \text{Cofin. 2 Sect. Hyperb. } \Phi \pm \text{Sin. 2 Sect. Hyperb. } \Phi$; ita ut generatim eadem sint Magnitudines Aequationibus expressae Cofin. 2 Sect. Circ.

$$\frac{\Phi}{\sqrt{-1}} = \text{Cofin. 2 Sect. Hyperb. } \Phi = \text{Cofin. 2 Sect. Circ. } \frac{\Phi}{\sqrt{-1}}, \text{ et}$$

$$\left(\text{Sin. 2 Sect. Circ. } \frac{\Phi}{\sqrt{-1}} \right) \sqrt{-1} = \text{Sin. 2 Sect. Hyperb. } \Phi = \left(\text{Sin. 2 Sect.}$$

$$\text{Circ. } -\Phi \sqrt{-1} \right) \sqrt{-1}. \text{ Vicissim Cofin. 2 Sect. Circ. } \Phi = \text{Cofin. 2 Sect. Hy-}$$

$$\text{perb. } \Phi \sqrt{-1} = \text{Cofin. 2 Sect. Hyperb. } -\frac{\Phi}{\sqrt{-1}}, \text{ eodemque pacto Sin. 2 Sect.}$$

$$\text{Circ. } \Phi = \left(\text{Sin. 2 Sect. Hyperb. } \Phi \sqrt{-1} \right) \cdot -\sqrt{-1} =$$

$$\frac{\text{Sin. 2 Sect. Hyperb. } \Phi \sqrt{-1}}{\sqrt{-1}}, \text{ scilicet } \left(\text{Sin. 2 Sect. Circ. } \Phi \right) \sqrt{-1} = \text{Sin.}$$

2 Sect. Hyperb. $\Phi \sqrt{-1}$, et tanta equidem facilitate, ut non amplius egeat Lector exemplis hisce, aliisve similibus adducendis.

173. Univerſa ergo Imaginariae Logometriae machina, totiusque rei cardo ſuſtinetur mera Aequationum identitate, quae non verum Circulum, ſed Circulum fictitium, ideſt veram Hyperbolen Aequilateram reſpicit, nec geminas, ſed unicam Curvam, unicum Logarithmorum Systema Hyperbolicum vel Naturale, unicum Modulum, qui ſit Unitas, unica ſymptomata legesque decernit. Exinde profluit, quod analogia, commercium, harmonia Hyperbolen inter Parilateram Apollonianam, Ellipſinque Aequilateram, ſive Circulum, ſubſtantialiter evaneſcat, utpote quae ſoli imaginarietati, et Calculi mendo innitatur, atque elegantis compendii amore concinnata eo redeat, ut tantummodo ſit fictitia conſtans Proportio $\tau \theta 1 : \sqrt{-1}$ Ordinatum applicatis Hyperbolae, et Circuli, ac e-converſo, eidem Abſciſſae reſpondentibus quaſi competens, cuius Proportionis, quamvis facillimae, ac ſine Logarithmorum ambage ab Aequationibus $y = \pm \sqrt{1-x^2} = \sqrt{x^2-1} \cdot \sqrt{-1}$, $y = \pm \sqrt{x^2-1}$ derivandae, aliquem non ſpernendum uſum in VI. ſequente Capite contemplabimur. Interim ea, quae ab Infinitesimalis Algorithmi penu tam laborioſe excitarunt non obſcuri nominis Analyſtae Bougainvillius

in Cal-

in *Calculi Integralis Tractatu*, Valmesleyus in *Memorabilibus Berolinensis Academiae* ad annum MDCCLV. relatis, Philosophiae Neutonianae Interpretes in *Integralis Calculi Elementis*, Riccatus in *Analyticis Institutionibus*, et quae nuperrime quadriennio non adhuc elapso Pessutius, Calandrellius, Anonymusque in *Opellis* Romae, ac Liburni editis varia opinione disserta aut acriter disceptata dederunt post Ioannem Bernoullium, Eulerum, Foncenexium, Dalembertium, Lagrangium, nova non modo perfundere luce iuvabit, quin etiam insperata simplicitate, et relictis, fortasse fallacibus, Infiniti latebris obtinere, vel modesto studio emendare.

174. De aequipollentia expressionum Cofin. 2^o Sect. Circ. $\frac{\Phi}{\sqrt{-1}}$, et Cofin. 2^o Sect. Hyperb. Φ , similiterque Sin. 2^o Sect. Hyperb. Φ , ac $\left(\text{Sin. 2^o Sect. Circ. } \frac{\Phi}{\sqrt{-1}} \right) \sqrt{-1}$, et idcirco de intima Realitate adparenter impossibilium il-

lorum Sinuum, Cofinuumve Circularium, ob destructionem Imaginarietatis, qui vel minimum dubitaret, suspicioni remedium is afferet si meminerit esse

$$\text{Cofin. 2^o Sect. Circ. } \Phi = 1 - \frac{\Phi^2}{1.2} + \frac{\Phi^4}{1.2.3.4} - \frac{\Phi^6}{1.2.3.4.5.6} + \frac{\Phi^8}{1.2.3.4.5.6.7.8} - \frac{\Phi^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.; \text{ quamobrem Cofin. 2^o Sect. Circ. } \frac{\Phi}{\sqrt{-1}}, \text{ sive}$$

$$\text{Cofin. 2^o Sect. Circ. } -\Phi \sqrt{-1} = 1 - \frac{\Phi^2}{1.2} + \frac{\Phi^4}{1.2.3.4} - \frac{\Phi^6}{1.2.3.4.5.6} + \frac{\Phi^8}{1.2.3.4.5.6.7.8} - \frac{\Phi^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c. = \frac{C^{\Phi} + C^{-\Phi}}{2} \text{ per ea, quae num^o.}$$

78. demonstrantur, aut demum = Cofin. 2^o Sect. Hyperb. Φ . Eadem ratione,

quum fit $\left(\text{Sin. 2^o Sect. Circ. } \frac{\Phi}{\sqrt{-1}} \right) \sqrt{-1}$, aut aequipollenter $\left(\text{Sin. 2^o Sect.}$

$$\text{Circ. } -\Phi \sqrt{-1} \sqrt{-1} = \sqrt{-1} \left(\frac{\Phi^1}{\sqrt{-1}} + \frac{\Phi^3}{1.2.3. \sqrt{-1}} + \frac{\Phi^5}{1.2.3.4.5. \sqrt{-1}} + \right.$$

$$\left. \frac{\Phi^7}{1.2.3.4.5.6.7. \sqrt{-1}} + \frac{\Phi^9}{1.2.3.4.5.6.7.8.9. \sqrt{-1}} + \&c. \right), \text{ profluet Aequatio } \left(\text{Sin. 2^o Sect.}$$

Sect.

$$\text{Sect. Circ. } \frac{\Phi}{\sqrt{-1}} \sqrt{-1} = \Phi^1 + \frac{\Phi^3}{1.2.3} + \frac{\Phi^5}{1.2.3.4.5} + \frac{\Phi^7}{1.2.3.4.5.6.7} + \dots$$

$$\frac{\Phi^9}{1.2.3.4.5.6.7.8.9} + \&c: = \frac{C^\Phi - C^{-\Phi}}{2} = \text{Sin. 2 Sect. Hyperb. } \Phi, \text{ quam Imagi-}$$

nariorum elisionem renovatis experimentis in Formula conversa, seu Aequalitatis

Cosin. 2 Sect. Circ. $\Phi = \text{Cosin. 2 Sect. Hyperb. } \Phi \sqrt{-1}$, sive in altera Sin.

2 Sect. Circ. $\Phi = (\text{Sin. 2 Sect. Hyperb. } \Phi \sqrt{-1}) \cdot \sqrt{-1} =$

$$\frac{\text{Sin. 2 Sect. Hyperb. } \Phi \sqrt{-1}}{\sqrt{-1}} \text{ denuo Serierum praecognitarum adiumento obti-}$$

nebimus. Et re quidem vera omnes iam norunt Cosin. 2 Sect. Hyperb. $\Phi =$

$$1 + \frac{\Phi^2}{1.2} + \frac{\Phi^4}{1.2.3.4} + \frac{\Phi^6}{1.2.3.4.5.6} + \frac{\Phi^8}{1.2.3.4.5.6.7.8} + \frac{\Phi^{10}}{1.2.3.4.5.6.7.8.9.10} + \dots$$

$$+ \&c:, \text{ quocirca erit Cosin. 2. Sect. Hyperb. } \Phi \sqrt{-1} = 1 - \frac{\Phi^2}{1.2} +$$

$$\frac{\Phi^4}{1.2.3.4} - \frac{\Phi^6}{1.2.3.4.5.6} + \frac{\Phi^8}{1.2.3.4.5.6.7.8} - \frac{\Phi^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: = \text{Cosin.}$$

2 Sect. Circ. Φ ; praetereaque, cum habeatur Sin. 2 Sect. Hyperb. $\Phi =$

$$\frac{\Phi^1}{1} + \frac{\Phi^3}{1.2.3} + \frac{\Phi^5}{1.2.3.4.5} + \frac{\Phi^7}{1.2.3.4.5.6.7} + \frac{\Phi^9}{1.2.3.4.5.6.7.8.9} + \&c:, \text{ nul-}$$

la manebit difficultas etiam in Coronide percipienda definiente novam Ae-

$$\text{qualitatem } \tau\delta \text{ Sin. 2 Sect. Hyperb. } \Phi \sqrt{-1} = \frac{\Phi^1 \sqrt{-1}}{1} - \frac{\Phi^3 \sqrt{-1}}{1.2.3} + \frac{\Phi^5 \sqrt{-1}}{1.2.3.4.5} -$$

$$\frac{\Phi^7 \sqrt{-1}}{1.2.3.4.5.6.7} + \frac{\Phi^9 \sqrt{-1}}{1.2.3.4.5.6.7.8.9} + \&c:, \text{ ex qua constat altera}$$

$$\text{Sin. 2 Sect. Hyperb. } \Phi \sqrt{-1} = \frac{\Phi^1}{1} - \frac{\Phi^3}{1.2.3} + \frac{\Phi^5}{1.2.3.4.5} - \frac{\Phi^7}{1.2.3.4.5.6.7} +$$

$$\frac{\Phi^9}{1.2.3.4.5.6.7.8.9} - \&c: = \text{Sin. 2 Sect. Circ. } \Phi.$$

175. Summopere ad dicendorum illustrationem harumce Formularum identitas, aequipollentia, vel nomine magis idoneo univoca expressio usu-veniet, et perquam maximi momenti est illam meditari, ac omni facie in

censum subacta veluti contrectare; adeo ut ipsam quoque repetitionis vitium singulari hoc in casu tantae rei moli melius explicandae intelligendaeque con-
ducat. Sint itaque Lectorum animo altius, quam poterit, reposita Theore-
mata, quae Exponentiales Magnitudines Indicibus realibus, vel imaginariis
affectas communi foedere iungunt, geminaque exhibet Formula, scilicet

$$C^{\pm \phi \sqrt{-1}} = \text{Cofin. Circ. } \phi \pm (\text{Sin. Circ. } \phi) \sqrt{-1} = \text{Cofin. Hyperb. } \phi \sqrt{-1}$$

$$\pm \text{Sin. Hyperb. } \phi \sqrt{-1} = \text{Cofin. Hyperb. } \phi \sqrt{-1} \pm \left(\frac{\text{Sin. Hyperb. } \phi \sqrt{-1}}{\sqrt{-1}} \right) \sqrt{-1},$$

$$\text{et } C^{\pm \phi} = \text{Cofin. Hyperb. } \phi \pm \text{Sin. Hyperb. } \phi = \text{Cofin. Circ. } \frac{\phi}{\sqrt{-1}} \pm$$

$$\left(\text{Sin. Circ. } \frac{\phi}{\sqrt{-1}} \right) \sqrt{-1} = \text{Cofin. Circ. } \phi \sqrt{-1} \pm (\text{Sin. Circ. } \phi \sqrt{-1}) \sqrt{-1}.$$

Magnitudinem igitur quamlibet Indice praeditam Imaginario $C^{\pm x \sqrt{-1}}$, at-

que universalius $z^{\pm x \sqrt{-1}}$ Sinubus, ac Cofinubus Hyperbolicis, veluti sym-
bolo, exprimere quoque fas erit, nimirum ita, $\text{Cofin. Hyperb. } x \sqrt{-1} \pm \text{Sin.}$
 $\text{Hyperb. } x \sqrt{-1}$, vel $\text{Cofin. Hyperb. } (Lz) x \sqrt{-1} \pm \text{Sin. Hyperb. } (Lz) x \sqrt{-1}$,

et ex opposito $C^{\pm x}$, aut $z^{\pm x}$ in Sinubus, Cofinubusque expressa Circula-
ribus evadet $\text{Cofin. Circ. } \frac{x}{\sqrt{-1}} \pm \left(\text{Sin. Circ. } \frac{x}{\sqrt{-1}} \right) \sqrt{-1}$, five Cofin. Circ.

$\frac{(Lz)x}{\sqrt{-1}} \pm \left(\text{Sin. Circ. } \frac{(Lz)x}{\sqrt{-1}} \right) \sqrt{-1}$; adeo ut rebus sic stantibus facile evincatur

esse non tantum $\pm x$ Logarithmum Numeri $C^{\pm x}$, vel generatim Numeri

$B^{\pm x}$ in quacumque alia Basi Systematis Logometrici, verum etiam Loga-
rithmum Numeri ipsiusmet Realis, attamen diversimode concinnati, Cofin.

$\text{Circ. } \frac{x}{\sqrt{-1}} \pm \left(\text{Sin. Circ. } \frac{x}{\sqrt{-1}} \right) \sqrt{-1}$, aut $\text{Cofin. Circ. } \frac{(L_B)x}{\sqrt{-1}} \pm \left(\text{Sin. Circ.}$

$\frac{(L_B)x}{\sqrt{-1}} \right) \sqrt{-1}$, Imaginariis signis mutuo destructis, uti contingit Cardanicae,

ac Moi-

ac Moivraeanis universalioribus Formulis, quarum mentio in Capite VII. saepenumero occurret.

176. Theorice istam certatim alia confirmant argumenta non indirecta, quin imo fortasse nobilioris originis. Quum etenim ex num°. 154.

prorsus constet de veritate Aequationum $\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^\infty = C^{x\sqrt{-1}}$, at-

que $\left(1 - \frac{x\sqrt{-1}}{\infty}\right)^\infty = C^{-x\sqrt{-1}}$, nanciscemur Sin. Circ. $x =$

$$\frac{C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}}{2\sqrt{-1}} = \frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^\infty - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^\infty}{2\sqrt{-1}}, \text{ et Cofin. Circ. } x =$$

$$= \frac{C^{x\sqrt{-1}} + C^{-x\sqrt{-1}}}{2} = \frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^\infty + \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^\infty}{2}. \text{ Facta igitur}$$

tur variabeli $x = \frac{z}{\sqrt{-1}} = -z\sqrt{-1}$, evadent Binomia ad Potestatem

Infiniti Ordinis exaltanda tam $\left(1 + \frac{z}{\infty}\right)^\infty$, quam $\left(1 - \frac{z}{\infty}\right)^\infty$, et id-

$$\text{circo Sin. Circ. } \frac{z}{\sqrt{-1}} = \text{Sin. Circ. } -z\sqrt{-1} = \frac{\left(1 + \frac{z}{\infty}\right)^\infty - \left(1 - \frac{z}{\infty}\right)^\infty}{2\sqrt{-1}}$$

$$= \frac{C^z - C^{-z}}{2\sqrt{-1}} = \frac{\text{Sin. Hyperb. } z}{\sqrt{-1}}, \text{ seu } \left(\text{Sin. Circ. } \frac{z}{\sqrt{-1}}\right) \sqrt{-1} = (\text{Sin. Circ.}$$

$$-z\sqrt{-1}) \sqrt{-1} = \text{Sin. Hyperb. } z, \text{ nec non Cofin. Circ. } \frac{z}{\sqrt{-1}}, \text{ vel}$$

$$\text{Cofin. Circ. } -z\sqrt{-1} = \frac{\left(1 + \frac{z}{\infty}\right)^\infty + \left(1 - \frac{z}{\infty}\right)^\infty}{2} = \frac{C^z + C^{-z}}{2} =$$

Cofin. Hyperb. z ; quae omnia adamussim cum superioribus congruunt. En Curva Hyperbolica Apolloniana ope Circuli Imaginarii per Calculorum non

inutilem

inutilem iocum exposita, et versa vice Circulus in Imaginaria Hyperbola prope nativis depictus coloribus, atque adeo vividis, ut non fabulosas Algebraicas Metamorphoses, sed Veritatem ipsam non admodum cauti tractare videantur. Somniantibus autem non impares ii censerii merentur, qui effingant Areas Circularium Sectorum ad Areas Hyperbolicorum se habere in constanti Ratione $1 : \sqrt{-1}$; namque universae reales Magnitudines nusquam imaginaria necti possunt, sive impossibili Proportionem. Nec minus errarent illi, qui hanc imaginariam Curvarum transformationem unicis tantum convenire arbitrarentur Aequilaterae Hyperbolae, et Circulo, ac totum hoc imaginarium commercium

Aequatione haecenus memorata contineri $C^{\pm x} = \text{Cofin. 2 Sect. Hyperb. } x \pm \text{Sin. 2 Sect. Hyperb. } x$, et $C^{\pm x\sqrt{-1}} = \text{Cofin. 2 Sect. Circ. } x \pm (\text{Sin. 2 Sect. Circ. } x) \sqrt{-1}$ ipsius harmonica, vel potius aequipollente. (*)

177. Attentius etenim perscrutantes symbolicae illius Harmoniae indolem derivationemque sponte ac natura ipsa duce inveniemus suum etiam Imaginariis hisce Logarithmis contingere quasi-Modulum, et amice coniurare inter se, praeter Hyperbolicum aut Naturale utriusque Logometriae Systema, cetera quoque sine numero ab universis deducta Ellipsibus, Hyperbolicisque Scalenis. Sector namque Conicae Ellipseos ad illum Circuli circumscripti aut inscripti concentrici eodem Transverso Axe praediti, et eidem Abscissae respondentis constantem servat Rationem, eamque immutatis Ellipsibus exhibet infinities variam; quod aequae praedicari potest de Hyperbolae Apolloniana Scalenae Sectoribus respectu habito ad circumscriptam inscriptamve Aequilateram. Esto M generatim recensitae Rationis $M:1$ Exponens, quae ad unguem congruit Axiom Coniugatorum Rationi, nunc maioris, nunc utcumque minoris Inaequalitatis. Singularem in Schemate XXIII. Ellipseos, atque Hyperbolae eosdem Axes habentium delineatum Systematis casum pro universali sumamus. Ratio Axeos Coniugati BD ad Transversum EF sit peculiaris cuiusque Systematis Modulus M , qui Unitas fuit in primum comparatis Hyperbola, Ellipsique Aequilateris. Denotet insuper variabilis z duplas Areas Sectorum AFH , AFH' &c: in Scalena Hyperbola

(*) Confer cum hisce ea, quae addidimus in Num°. 300; sed nusquam obliviscere siglas Φ , x , z , aliasve Notas stare pro Duplis Sectorum, aut veris falsisque Arcibus Circuli, et Aequilaterae Hyperbolae.

Hyperbola GFLQEP, cuius Semi-Axis AF peraequet de more Unitatem, et erunt homologi Aequilaterae Conicae Curvae KFNSER Sectors pariter duplicati x , idest AFI, AFN, aequales $\frac{z}{M}$, quam Proportionem $M:1$ sequuntur

etiam universaliter Recti-Sinus Hyperbolarum $TH:TI, T'H':T'N$ &c: iisdem adscriptos Cofinibus AT, AT' &c:, Sectors Elliptici AFA relati ad respectivos Circulares AFC , nec non Recti-Sinus Elliptici OAD ad vere Trigonometricos OC , eodem Cofinu AO , vel FO Sinu-verso gaudentes. His in antecessum positis Aequationem notissimam repeto potius quam adstruam

$$\left(C^{\frac{1}{M}}\right)^{\pm z} = C^{\pm x} = \text{Cofin. 2 Sect. Hyperb. Aeq. } x \pm \text{Sin. 2 Sect. Hyperb. Aeq. } x = \text{Cofin. 2 Hyperb. Scal. Sect. } z \pm (\text{Sin. 2 Hyperb. Scal.}$$

$$\text{Sect. } z) \frac{1}{M}, \text{ ipsiusque imaginariam veluti tesseram } \left(C^{\frac{1}{M}}\right)^{\pm z\sqrt{-1}} =$$

$$C^{\pm x\sqrt{-1}} = \text{Cofin. 2 Sect. Circ. } x \pm (\text{Sin. 2 Sect. Circ. } x)\sqrt{-1} = \text{Cofin.}$$

$$\text{2 Ellip. Sect. } z \pm (\text{Sin. 2 Ellip. Sect. } z) \frac{\sqrt{-1}}{M}, \text{ a quibus universa profluunt,}$$

variaram combinationum ope, symptomata superius eruta, nunc innumeris aptanda Modulis, tyronumque sagacitati experiundae satis idonea.

178. Hyperboles autem Apolloniana Exponentiali Aequatione significan-

$$\text{da non tota continetur in priore reali } \left(C^{\frac{1}{M}}\right)^{\pm v} = \text{Cofin. 2 Sect. Hy-}$$

$$\text{perb. } v \pm (\text{Sin. 2 Sect. Hyperb. } v) \frac{1}{M}, \text{ vel aequipollente imaginaria}$$

$$\left(C^{\frac{1}{M}}\right)^{\pm v\sqrt{-1}} = \text{Cofin. 2 Ellip. Sect. } v \pm (\text{Sin. 2 Ellip. Sect. } v) \frac{\sqrt{-1}}{M}.$$

Ad complectendos omnes Hyperbolae Conicae Ramos, exhaustiendamque huiusce Loci Geometrici universalitatem manca ac mutila foret utraque Aequatio, quum duobus solummodo referatur Ramis; quod facillimum est intellectu. Non vero temere adscribendus illius Aequationis defectus impotentis Calculi vitio, proptereaquod Analystarum, non Analyseos, errori tribui e-con-

tra deberet in re tantae molis allucinatio. Unitate quippe loco Moduli re-
stituta, quum eadem redeant pro ceteris, memoremus oportet demonstratas
fuisse vi unius Cartesiani Algorithmi in praecedente num°. 80, et postmo-
dum quo ad Indices Imaginarios in num°. 156., veritati semper consonas

Aequationes $\left(\frac{C^x + C^{-x}}{2}\right)^2 - \left(\frac{C^x - C^{-x}}{2}\right)^2 = 1^2$, atque

$$\left(\frac{C^x \sqrt{-1} + C^{-x} \sqrt{-1}}{2}\right)^2 + \left(\frac{C^x \sqrt{-1} - C^{-x} \sqrt{-1}}{2\sqrt{-1}}\right)^2 = 1^2, \text{ quin etiam}$$

universaliores $\left(\frac{z^x + z^{-x}}{2}\right)^2 - \left(\frac{z^x - z^{-x}}{2}\right)^2 = 1^2$, nec non

$$\left(\frac{z^x \sqrt{-1} + z^{-x} \sqrt{-1}}{2}\right)^2 + \left(\frac{z^x \sqrt{-1} - z^{-x} \sqrt{-1}}{2\sqrt{-1}}\right)^2 = 1^2; \text{ quibus postre-}$$

mis sicuti diversa alicuius momenti inaedificari nequeunt Theoremata, id
unum ad maximam generalitatem tutandam praenotasse sufficiat. Radix autem
Quadratica geminum semper praebens valorem, positivum unum, alterum ne-
gativum, vel melius diceres geminam eiusdem Magnitudinis qualitatem,
rite recteque usurpata veras hoc pacto Aequationes progignet. Ac primum

$$\left(\frac{C^x + C^{-x}}{2}\right)^2 = (\text{Cofin. Hyperb. } x)^2 \text{ dat } \pm \left(\frac{C^x + C^{-x}}{2}\right) = \text{Cofin. Hy-}$$

$$\text{perb. } x, \text{ eodemque ritu } \left(\frac{C^x - C^{-x}}{2}\right)^2 = (\text{Sin. Hyperb. } x)^2 \text{ ducit ad al-}$$

$$\text{teram } \pm \left(\frac{C^x - C^{-x}}{2}\right) = \text{Sin. Hyperb. } x, \text{ ab quarum iam familiari combinatione}$$

$$\text{simpliciores } \pm C^x = \text{Cofin. Hyperb. } x \pm \text{Sin. Hyperb. } x, \text{ et } \pm C^{-x} = \text{Cofin.}$$

Hyperb. $x - \text{Sin. Hyperb. } x$, compendioque perquam maximo $\pm C^{\pm x} =$
Cofin. Hyperb. $x \pm \text{Sin. Hyperb. } x$, five aliarum Curvarum more $\pm C^x =$
Cofin. Hyperb. $x \pm \text{Sin. Hyperb. } x$ oriuntur. Haec ultima Aequatio du-
plici Signo affecta quatuor Hyperbolae Ramos, nempe totam simul Curvam,
designat, estque ideo inter Exponentiales Aequalitates vera, ut aiunt, Lo-
calis, cum regulis Analyseos ac Geometriae consentanea. Dum Signum eli-
gatur

gatur superius, duo revera habebuntur Curvae Puncta in una oppositarum
 descriptae Hyperbolae partium, scilicet respondentia $+C^x$, et $+C^{-x}$,
 vel $+Cofin. Hyperb. x + Sin. Hyperb. x$, ac $+Cofin. Hyperb. x -$
 $Sin. Hyperb. x$, aut in XXII. Schemate EC, EC'', residuaque in adversa
 parte Hyperbolae Puncta praebebit Signum inferius ope valorum $-C^x$,
 et $-C^{-x}$, seu $-Cofin. Hyperb. x - Sin. Hyperb. x = GL$, nec non
 $-Cofin. Hyperb. x + Sin. Hyperb. x = GL'$, quemadmodum negativos,
 et positivos Sinus ac Cofinus persequentibus summa facilitate constabit. Iis-
 dem omnino principiis renovatis aequae concluditur bisgeminus valor Radi-
 cum Quadraticarum extractione acquisitus $\pm C^x \sqrt{-1} = Cofin. Circ. x +$
 $(Sin. Circ. x) \sqrt{-1}$ ex eo, quod $+C^x \sqrt{-1} = +Cofin. Circ. x + (Sin.$
 $Circ. x) \sqrt{-1}$, $+C^{-x} \sqrt{-1} = Cofin. Circ. x - (Sin. Circ. x) \sqrt{-1}$, dein-
 deque $-C^x \sqrt{-1} = -Cofin. Circ. x - (Sin. Circ. x) \sqrt{-1}$, et $-$
 $C^{-x} \sqrt{-1} = -Cofin. Circ. x + (Sin. Circ. x) \sqrt{-1}$. Quadruplex Expo-
 nentialis Aequationis veluti Radix denuo comprobatur, quod, dum de Cir-
 culo agatur Imaginario, ipsiusque affectionibus, reapse Hyperbola Realis
 Apolloniana sit ibidem abscondita, et Imaginariarum Magnitudinum Notis
 non parum monstrose involuta. Aequatio igitur $\pm C^x = y$, quomodocum-
 que prima fronte videatur non unicam Curvam, sed duarum Curvarum,
 gaudentium Aequationibus respectivis $C^x = y$, et $-C^x = y$ Systema com-
 ponere, uni tamen Hyperbolae Continuae est addicta, quae ope Quadratura-
 rum repraesentabitur ab Aequatione plenissima $C^{2x} = y^2$, construere fa-
 cillime poterit. Criterium exinde illud in resolutione consistens Aequatio-
 nis Localis, quae Factores Rationales admittat, ad distinguendos ab unicis
 Curvis Systemata Curvarum invicem cumularum, exceptionem patitur,
 saltem in casu (*), quo Curvae non amplius sint Algebraicae, sed Transcen-
 dentes, et a Quadraturis Arearum Curvilinearum non quadrabilium origi-
 nem ducant. Nec mirum, quum hae Quadraturae nusquam eliminatam si-
 nant

(*) Perlege numerum 384.

nant ab Aequationibus illis Localibus *Transcendentalitatem*, quae multo magis, quam *Irrationalitas*, obest applicationi Legum, Canonumve ab Analyftis praefcriptorum in diiudicandam Curvarum Algebraicarum formam, Crurumque Continuitatem, mutuumve nexum adeo, ut Quadraturae intelligi debeant veluti modus *fui generis* pro Curvis gignendis, in quo rationem Continuitatis Ramorum ab Arearum genitricium Continuitate deducere tantummodo liceat.

179. Insigne tamen gravissimae aequipollentiae, de qua toties sermo fuit, argumentum digesta per Infinitas Series Exponentialia confirmant. Est

$$\begin{aligned} \text{enim } \pm C^x = y = \pm & \left(1 + \frac{x^1}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} \right. \\ & + \frac{x^6}{1.2.3.4.5.6} + \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^9}{1.2.3.4.5.6.7.8.9} \\ & \left. + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c. \right) \text{ Aequatio Transcendens Aequilaterae Apollonii} \end{aligned}$$

Hyperbolae, cuius Abscissae x minus usitato more sint Sectorum Areae Centralium per $\frac{1}{2}$, Semissem medietatis Lateris Transversi, divisae, et Ordinatum applicatae y , Sinuum, Cosinumque Hyperbolicorum Summis, vel Differentiis coalescant. Duplicitate Signi interim neglecta, propterea quod non valorem, sed qualitatem aut existendi modum Radicum afficiat, resolvenda proponatur

$$\begin{aligned} \text{tur Aequatio alias assumpta, nimirum } 0 = y - 1 = C^x - 1 = \\ x \left(1 + \frac{x^1}{1.2} + \frac{x^2}{1.2.3} + \frac{x^3}{1.2.3.4} + \frac{x^4}{1.2.3.4.5} + \frac{x^5}{1.2.3.4.5.6} + \frac{x^6}{1.2.3.4.5.6.7} \right. \\ \left. + \frac{x^7}{1.2.3.4.5.6.7.8} + \frac{x^8}{1.2.3.4.5.6.7.8.9} + \frac{x^9}{1.2.3.4.5.6.7.8.9.10} + \&c. \right); \text{ eruntque va-} \end{aligned}$$

lores x adeo comparati, ut reddant $C^x - 1 = 0$, five $C^x = 1$, cui conditioni primum satisfacit $x = 0$, id confirmante alterius membri Factore, deindeque $x = \pm 2n P \sqrt{-1}$, nuncupatis P semisse Circularis Peripheriae Radio 1 descriptae, et n quolibet Numero positivo, quia vi toties exposito-

$$\begin{aligned} \text{rum } C^{\pm 2n P \sqrt{-1}} &= \text{Cofin. Circ } (\pm 2n P) \pm (\text{Sin. Circ } (\pm 2n P)) \sqrt{-1} \\ &= 1 \end{aligned}$$

== 1 ex Elementis Trigonometricis. Universae igitur Radices Infinito-

mialis elegantissimae Aequalitatis $0 = 1 - \frac{x^1}{1.2} + \frac{x^2}{1.2.3} - \frac{x^3}{1.2.3.4} +$

$\frac{x^4}{1.2.3.4.5} - \frac{x^5}{1.2.3.4.5.6} + \frac{x^6}{1.2.3.4.5.6.7} - \frac{x^7}{1.2.3.4.5.6.7.8} + \frac{x^8}{1.2.3.4.5.6.7.8.9} -$

$\frac{x^9}{1.2.3.4.5.6.7.8.9.10} + \&c:$ facillima methodo inventae habebuntur Imagina-

riae - simplices; quamobrem Infinitinomialium ipsum in Reales Factores statim

resolvi posse quisquis prima fronte arbitraretur $\left(1 - \frac{x^2}{4P^2}\right) \left(1 - \frac{x^2}{16P^2}\right)$

$\left(1 - \frac{x^2}{36P^2}\right) \left(1 - \frac{x^2}{64P^2}\right) \left(1 - \frac{x^2}{100P^2}\right) \left(1 - \frac{x^2}{144P^2}\right) \left(1 - \frac{x^2}{196P^2}\right)$

&c: sine Limite, vel $\left(1 - \frac{x^2}{2^2P^2}\right) \left(1 - \frac{x^2}{4^2P^2}\right) \left(1 - \frac{x^2}{6^2P^2}\right)$

$\left(1 - \frac{x^2}{8^2P^2}\right) \left(1 - \frac{x^2}{10^2P^2}\right) \left(1 - \frac{x^2}{12^2P^2}\right) \left(1 - \frac{x^2}{14^2P^2}\right) \&c:$, aut, desi-

gnante Π integram Circumferentiam, $\left(1 - \frac{x^2}{\Pi^2}\right) \left(1 - \frac{x^2}{2^2\Pi^2}\right)$

$\left(1 - \frac{x^2}{3^2\Pi^2}\right) \left(1 - \frac{x^2}{4^2\Pi^2}\right) \left(1 - \frac{x^2}{5^2\Pi^2}\right) \left(1 - \frac{x^2}{6^2\Pi^2}\right) \left(1 - \frac{x^2}{7^2\Pi^2}\right) \&c:$

quorum progressiva Lex, siue Numeri in Denominatoribus impares, siue pa-

res considerentur, satis evidenter elucet. Tanta equidem simplicitas ad hosce

Factores investigandos, qui a repertis Radicibus ducunt originem, $2P\sqrt{-1} - x = 0$, $2P\sqrt{-1} + x = 0$, $4P\sqrt{-1} - x = 0$, $4P\sqrt{-1} + x = 0$,

$6P\sqrt{-1} - x = 0$, $6P\sqrt{-1} + x = 0$ &c: vel $1 - \frac{x}{2P\sqrt{-1}} = 0$, $1 + \frac{x}{2P\sqrt{-1}} = 0$,

$1 - \frac{x}{4P\sqrt{-1}} = 0$, $1 + \frac{x}{4P\sqrt{-1}} = 0$, $1 - \frac{x}{6P\sqrt{-1}} = 0$, $1 + \frac{x}{6P\sqrt{-1}} = 0$ &c: ultra spem foret, quum methodum Cotesia-

nam, herculeasque ab Eulero vires in Capite IX^o. prioris de *Introductione in*

Analysin Infinitorum Voluminis adhibitas ferme eluserit.

180. Verum, tametsi Factores superius recensiti in falsum abire videantur ex eo, quod Infinitinomium decomponendum, praeter Potestates τx pari Indice praeditas, illas etiam imparis Exponentis complectatur, differre ii nequeunt a veris nisi discrimine Infinitesimo: nam, si distarent finita, et utcumque parvula magnitudine Radices Aequationis ex veris Factoribus depromptae, non amplius iam inventis congruerent, earumque veritas supra tam firmiter aedificata labefactaretur. Porro, sicuti Formula universalis praeteritos

exhibens Binomiales Factores erat $1 + \frac{x^2}{(2nP)^2}$, altera habebit necessariam,

et infinite-parum discrepantem a prima formam Trinomiale $1 + \frac{rx}{\infty} +$

$\frac{x^2}{(2nP)^2}$, in qua statim detegitur esse indeterminatum Coefficientem $r = 1$, quia

$\frac{rx}{\infty}$ per $\frac{\infty}{2}$, vel per Factorum Trinomialium numerum multiplicatus restituere

debebit secundum Infinitinonii propositi terminum $\frac{x}{2}$, cui signanter $\tau \delta \frac{x}{\infty} \cdot \frac{\infty}{2}$,

nullumque aliud, aequale fiet. Ne vero dubium excitetur circa Radicum aequipollentiam Arithmetica, quae a posterioribus profluunt Trinomiis, periculum libenter facio istorum resolutionis. Praebet notissima haec resolu-

lutio $1 + \frac{x}{\frac{(2nP)^2 + 2nP\sqrt{-1}}{2 \cdot \infty}} = 0$, in cuius expressionis generali Denomina-

core, dum valor Radicum quaeratur Arithmeticus, pars Infinitesima $\frac{(2nP)^2}{2 \cdot \infty}$

tuto omitti poterit, quo facto novae Radices cum antea actis consensu miro coincidunt. Quin etiam denuo proderit comprobare Summam ex Infinitis nume-

ro Coefficientibus $\frac{1}{\frac{(2nP)^2 + 2nP\sqrt{-1}}{2 \cdot \infty}}$ conflata efficere $\frac{1}{2}$, scilicet iuxta Alge-

brae regulas Coefficientem secundi termini ad toties citatum Infinitinomium spectantis. Quilibet etenim praefatorum Coefficientium Binarius

$$\frac{1}{\frac{(2P)^2 - 2P\sqrt{-1}}{2 \cdot \infty}} \text{ et } \frac{1}{\frac{(2P)^2 + 2P\sqrt{-1}}{2 \cdot \infty}}, \frac{1}{\frac{(4P)^2 - 4P\sqrt{-1}}{2 \cdot \infty}} \text{ et } \frac{1}{\frac{(4P)^2 + 4P\sqrt{-1}}{2 \cdot \infty}},$$

$$\frac{1}{(6P)^2 - 6P\sqrt{-1}} \text{ et } \frac{1}{(6P)^2 + 6P\sqrt{-1}}, \frac{1}{(8P)^2 - 8P\sqrt{-1}} \text{ atque } \frac{1}{(8P)^2 + 8P\sqrt{-1}} \&c;$$

rite ordinatus reddit una cum aliis Summam

$$\frac{(2P)^2}{2^2 \cdot \infty^2} + \frac{(4P)^2}{2^2 \cdot \infty^2} +$$

$$\frac{(6P)^2}{2^2 \cdot \infty^2} + \frac{(8P)^2}{2^2 \cdot \infty^2} \&c.; \text{ nimirum } \left(\frac{1}{\infty} + \frac{1}{\infty} + \frac{1}{\infty} + \frac{1}{\infty} \right.$$

$$\left. + \frac{1}{\infty} + \&c. \right), \text{ quae Fractio } \frac{1}{\infty}, \text{ cum repeti debeat per vices } \frac{\infty}{2}, \text{ nempe}$$

per Numerum Binariorum, progignit tandem $\frac{1}{2}$, quemadmodum fui-

mus superius polliciti. Pari iucunditate, ita ratiocinando, detegeremus Radicum ipsarum, quae partim infinitesimis, partim finitis Numeris componuntur, Binarios, Ternarios, Quadernarios, Quinarios, ceterasque affines Combinationes, singulas simul sumptas, conficere, uti Algebra docet,

$$\frac{1}{6}, \frac{1}{24}, \frac{1}{120} \&c.; \text{ nihil obstante incomparabilium aut Infinitesimarum Ma-}$$

gnitudinum commixtione.

181. Duo non admodum levia hic supersunt adnotanda. Vellem non fu-
geret Lectorum ingenium eximia certe Aequationum Infinitis numero ter-
minis coalescentium proprietas, quod non semper, nec Analytica necessitate,
ut Polynomiis contingit ex finito terminorum numero existentibus, inventae
Radices Finitae praebeant Infinitinomii, seu primi Membri Aequationis, Fa-
ctores, quia salvo Radicum valore Arithmetico, at non rigide Analytico,
misceri tamen cum Finitis Magnitudinibus inassignabiles, vel Infinitesimae
possunt. Hae in multiplicatione Factorum numero Infinitorum infinities re-
petitae finitum constituere Numerum valent, et idcirco negligi nequeunt,
utcumque in singulis Radicibus nullius sint valoris. Isti non ingratae con-
templationi aliam subiungo summi, ac fortassis inopinati, discriminis inter

Aequa-

$$\text{Aequationem } 0 = 1 + \frac{x^1}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^7}{1.2.3.4.5.6.7} + \&c. = \left(1 + \frac{x}{\infty}\right)^{\infty}, \text{ alteramque vix im-}$$

$$\text{mutatam, et sola Unitate minutam, } \frac{x^1}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^7}{1.2.3.4.5.6.7} + \&c. = \left(1 + \frac{x}{\infty}\right)^{\infty} - 1 = 0,$$

cuius plenissimam nuper dedimus enodationem. Prior enim Aequatio, veluti alias inuimus, omnes habet Radices aequales $x = -\infty$, Reales, negativas, et numero ac magnitudine Infinitas, habetque primum eius Membrum numero Infinitos Factores Binomiales aequales ex Finita, et Infinitesima quan-

titate conflatos $1 + \frac{x}{\infty}$: dum e-contra Aequatio secunda, praeter unicam Radicem x nullescentem, ceteras numero Infinitas tribuit universim Imaginarias, et dimidiatim positivas, ac negativas, ipsiusque primum Membrum, eliminato Factore simplici x , componitur Factoribus Binomialibus numero Infinitis, at inaequalibus, et Imaginariis, atque adeo existentibus, ut dimidio numero Factores Trinomiales diversos efforment Reales Finitis, et In-

$$\text{finitesimis coalescentes } 1 + \frac{x}{\infty} + \frac{x^2}{1^2 \Pi^2}, 1 + \frac{x}{\infty} + \frac{x^2}{2^2 \Pi^2}, 1 + \frac{x}{\infty} + \frac{x^2}{3^2 \Pi^2}, 1 + \frac{x}{\infty} + \frac{x^2}{4^2 \Pi^2}, 1 + \frac{x}{\infty} + \frac{x^2}{5^2 \Pi^2}, \&c.: \text{ Erit igitur } \frac{x^1}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^7}{1.2.3.4.5.6.7} + \&c. = x \left(1 + \frac{x}{\infty} + \frac{x^2}{1^2 \Pi^2}\right) \left(1 + \frac{x}{\infty} + \frac{x^2}{2^2 \Pi^2}\right) \left(1 + \frac{x}{\infty} + \frac{x^2}{3^2 \Pi^2}\right) \left(1 + \frac{x}{\infty} + \frac{x^2}{4^2 \Pi^2}\right) \left(1 + \frac{x}{\infty} + \frac{x^2}{5^2 \Pi^2}\right) \left(1 + \frac{x}{\infty} + \frac{x^2}{6^2 \Pi^2}\right) \left(1 + \frac{x}{\infty} + \frac{x^2}{7^2 \Pi^2}\right) \&c. = \left(1 + \frac{x}{\infty}\right)^{\infty} - 1 = C^x - 1; \text{ nec non}$$

$$C^{\pm 2nPV\sqrt{-1}} = C^{\pm 2nPV\sqrt{-1}} - \frac{(2nP)^2}{2 \cdot \infty}.$$

182. Quae haecenus exposuimus, etsi minus communia videri possint, ad unam tantummodo referuntur Aequationem angustis nimium limitibus circumscriptam $C^x - 1 = 0$. Universaliora quaerentes Aequationem $C^x - m = 0$, five

$$(1 - m) + \frac{x^1}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^9}{1.2.3.4.5.6.7.8.9} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c. = 0, \text{ vel tandem maiori compendio } \left(1 + \frac{x}{\infty}\right)^{\infty}$$

$- m = 0$, quod elementum m Numeros omnes positivos significet, resolvendam proponamus. Cum in III. Capite, ac signanter num°. 88., ostensum

fit esse $C^{Lm} = m$, ideoque $C^{Lm} - m = 0$, ambigi non poterit unam Infinitinomiae Aequationis propositae Radicem positivam ac Realem fore Lm , unumque primi Membri Factorem ab hoc reperto valore pendere. Ceteris inveniendis Radicibus idoneam viam stravit simplicioris praeteritae Aequationis iam obtenta resolutio, utpote quae semper nos doceat Aequationem

Exponentialem $C^{\pm(2nP)\sqrt{-1}} = 1$, et idcirco $C^{Lm} \times C^{\pm(2nP)\sqrt{-1}} = m.1$

$= m$, nimirum $C^{Lm \pm (2nP)\sqrt{-1}} - m = 0$, cum qua dum primum assignata comparetur $C^x - m = 0$, profluent ipsius universae Radices compendiosa satis Formula exhibitae $x = Lm \pm (2nP)\sqrt{-1}$. Investigatae igitur Radices singillatim evadunt Lm , unica Realis et positiva, $Lm + 2P\sqrt{-1}$, $Lm - 2P\sqrt{-1}$, $Lm + 4P\sqrt{-1}$, $Lm - 4P\sqrt{-1}$, $Lm + 6P\sqrt{-1}$, $Lm - 6P\sqrt{-1}$, $Lm + 8P\sqrt{-1}$, $Lm - 8P\sqrt{-1}$, $Lm + 10P\sqrt{-1}$, $Lm - 10P\sqrt{-1}$ &c.; omnes Imaginariae-compositae, et quoad partes tantum Imaginarias alternatim positivae ac negativae, vel, si ita placeat, Lm , $Lm + \Pi\sqrt{-1}$, $Lm - \Pi\sqrt{-1}$, $Lm + 2\Pi\sqrt{-1}$, $Lm - 2\Pi\sqrt{-1}$, $Lm + 3\Pi\sqrt{-1}$, $Lm - 3\Pi\sqrt{-1}$, $Lm + 4\Pi\sqrt{-1}$, $Lm - 4\Pi\sqrt{-1}$, $Lm + 5\Pi\sqrt{-1}$, $Lm - 5\Pi\sqrt{-1}$ &c.; quae vertuntur in 0, $\pm 2P\sqrt{-1}$, $\pm 4P\sqrt{-1}$, $\pm 6P\sqrt{-1}$, $\pm 8P\sqrt{-1}$, $\pm 10P\sqrt{-1}$ &c.; aut 0, $\pm \Pi\sqrt{-1}$, $\pm 2\Pi\sqrt{-1}$, $\pm 3\Pi\sqrt{-1}$, $\pm 4\Pi\sqrt{-1}$, $\pm 5\Pi\sqrt{-1}$ &c. dum $m = 1$, et anteriora confirmant.

183. Binomium ergo unicum, atque numero Infinita Trinomia completeness repertas Radices ex praecognitis manabunt $Lm - x = 0$, $(Lm)^2$

Nn

$+ (2P)^2$

$+ (2P)^2 - 2(Lm)x + x^2 = 0$, $(Lm)^2 + (4P)^2 - 2(Lm)x + x^2 = 0$,
 $(Lm)^2 + (6P)^2 - 2(Lm)x + x^2 = 0$, $(Lm)^2 + (8P)^2 - 2(Lm)x + x^2 = 0$,
 $(Lm)^2 + (10P)^2 - 2(Lm)x + x^2 = 0$ &c.; eaque summam
 repraesentantur ope solius expressionis generalissimae $(Lm)^2 + (2nP)^2 - 2(Lm)x + x^2 = 0$.
 Haec, dum $n = 0$, tribuit Quadratum $(Lm)^2 - 2(Lm)x + x^2$, cuius Radix, ut
 Analystae omnes norunt, Binomium $Lm - x$, vel $x - Lm$ initio assignatum constituit.
 Videant vero, quos aequus amavit Iupiter, ocio maximo fruente Mathematicum
 sublimiorum cultores, an Infinitinomii

$$(1 - m) + \frac{x^1}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^9}{1.2.3.4.5.6.7.8.9} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \text{\&c.}$$

Factoribus satisfiat consequente modo eos disponendo

$$\left(1 - m + \frac{1 - m \cdot x}{-Lm}\right) \left(1 + \left(\frac{1}{\infty} - \frac{2Lm}{(Lm)^2 + (2P)^2}\right)x + \frac{x^2}{(Lm)^2 + (2P)^2}\right)$$

$$\left(1 + \left(\frac{1}{\infty} - \frac{2Lm}{(Lm)^2 + (4P)^2}\right)x + \frac{x^2}{(Lm)^2 + (4P)^2}\right)$$

$$\left(1 + \left(\frac{1}{\infty} - \frac{2Lm}{(Lm)^2 + (6P)^2}\right)x + \frac{x^2}{(Lm)^2 + (6P)^2}\right)$$

$$\left(1 + \left(\frac{1}{\infty} - \frac{2Lm}{(Lm)^2 + (8P)^2}\right)x + \frac{x^2}{(Lm)^2 + (8P)^2}\right)$$

$$\left(1 + \left(\frac{1}{\infty} - \frac{2Lm}{(Lm)^2 + (10P)^2}\right)x + \frac{x^2}{(Lm)^2 + (10P)^2}\right) \text{\&c.}, \text{ quorum universi,}$$

praeter primum, sub hac generali Formula $1 + \left(\frac{1}{\infty} - \frac{2Lm}{(Lm)^2 + (2nP)^2}\right)x +$

$\frac{x^2}{(Lm)^2 + (2nP)^2}$ continentur. Argumentum, quod mihi geminum obversatur,

Factorum veritati admodum favens, innixum est experimento Producti omnium
 terminorum, quibus non inest variabilis x , priori Infinitinomii termino
 $1 - m$ admissim aequalis, ac transformationis cuiusque ex Factoribus, ca-
 su $m = 1$, five $Lm = 0$, in illos simpliciores iam superius determinatos.
 Et re quidem vera Factor Trinomialis universaliter expressus $1 +$

$$\left(\frac{1}{\infty} - \frac{2Lm}{(Lm)^2 + (2nP)^2}\right)x + \frac{x^2}{(Lm)^2 + (2nP)^2}$$

$\left(\frac{1}{\infty} - \frac{2Lm}{(Lm)^2 + (2nP)^2}\right)x + \frac{x^2}{(Lm)^2 + (2nP)^2}$ tunc vertitur in $1 + \frac{x}{\infty} +$

$\frac{x^2}{(2nP)^2}$, quemadmodum supra in num°. 180., ipseque Factor simplex $1 - m$

$+ \left(\frac{1-m}{-Lm}\right)x$, vel potius, evoluta Logarithmo ad indeterminationem Fra-

ctionis $\frac{0}{0}$ vincendam, $1 - m +$

$\frac{1-m \cdot x}{(1-m) \left(1 - \frac{m-1}{2} + \frac{m-1^2}{3} - \frac{m-1^3}{4} + \frac{m-1^4}{5} \&c:\right)}$, five $1 - m$

$+ \frac{x}{1 - \frac{m-1}{2} + \frac{m-1^2}{3} - \frac{m-1^3}{4} + \frac{m-1^4}{5} \&c:}$ abit in $1 - 1 + \frac{x}{1}$

$= x$, congruitque retroactis observationibus. Sed nihilominus timeo, vel prudentia vel pusillanimitas incusetur, quiddam forte peregrinum in Infinitinomio latitare perturbans confectam analyticam evolutionem; ideoque rem difficillimam aliorum examini cedo libenter, immo in eam perpoliendam stimulos addo quammaximos.

184. Aequationem Binomiam Exponentialem $C^x - m$, aut $C^x - y = 0$ nemo ignorat Logarithmos x , et Numeros y invicemnectere. Ista exemplum praestat Aequationis Infinitinomialis $1 - y + \frac{x^1}{1} - \frac{x^2}{1.2} + \frac{x^3}{1.2.3}$

$+ \frac{x^4}{1.2.3.4} - \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} - \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^8}{1.2.3.4.5.6.7.8} -$

$\frac{x^9}{1.2.3.4.5.6.7.8.9} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: = 0$, et huius conversae, pari-

ter Infinitinomialae, $0 = -x + \frac{y-1^1}{1} - \frac{y-1^2}{2} + \frac{y-1^3}{3} - \frac{y-1^4}{4}$

$+ \frac{y-1^5}{5} - \frac{y-1^6}{6} + \frac{y-1^7}{7} - \frac{y-1^8}{8} + \frac{y-1^9}{9} - \frac{y-1^{10}}{10} + \&c:$

$\pm (2nP)\sqrt{-1}$, quarum prima unicam habet y , et altera x numero Infinitas Radices. Novas acquireret vires haec periucunda ratiocinatio statim ac in Capite

in Capite VII°. datum erit ostendere eandem Radicum seu valorum Logarithmi cuiuslibet Numeri infinitam multiplicitem ab Aequatione etiam

notissima $\infty \left(1^{\frac{1}{\infty}} \cdot \frac{1}{1+z} - 1 \right) - v = 0$ derivari, quae dato Numero $1 \rightarrow z$ perducit ad v eius Logarithmum noscendum.

185. In pretio erit nonnihil elegans Confectarium, quod ex nostra methodo sponte profuit, definitque non modo $0, \pm 2P\sqrt{-1}, \pm 4P\sqrt{-1}, \pm 6P\sqrt{-1}, \pm 8P\sqrt{-1}, \pm 10P\sqrt{-1} \&c.$, seu generatim $\pm 2nP\sqrt{-1}$, verum etiam $Lm, Lm \pm 2P\sqrt{-1}, Lm \pm 4P\sqrt{-1}, Lm \pm 6P\sqrt{-1}, Lm \pm 8P\sqrt{-1}, Lm \pm 10P\sqrt{-1} \&c.$, ac compendiosius $Lm \pm 2nP\sqrt{-1}$, priori casu pro Logarithmo Hyperbolico ≈ 1 , ac posteriori Numeri m cuiuscumque obtineri. Tanto etenim labore, ac tanta solertiae copia, veluti fertur etiam Capite IX°. hos multiplices Logarithmos Eulero contigit extricare, ut singularem penes Geometras acquisiverint celebritatem. At, dum ab Hyperbolicorum Systemate ad alios Systematis cuiuscumque Logarithmos transitum facere libeat, omni carebit difficultate, quod, posito M Modulo, infinities multiplices Logarithmi Numeri m evadant $M (Lm \pm 2nP\sqrt{-1})$, ideoque $M (\pm 2nP\sqrt{-1})$ respectu habito ad Unitatem. Constat hoc ex toties notatis Aequalitatibus $M = \frac{1}{L_B}$ in Systemate Logarithmico Basi B

praedito, et $B^x = C^{xL_B} = C^{xL_B} \times C^{\pm 2nP\sqrt{-1}} = C^{xL_B \pm 2nP\sqrt{-1}}$; quam-

obrem fit $C^{xL_B \pm 2nP\sqrt{-1}} - m = 0$, atque $x = lm = \frac{Lm \pm 2nP\sqrt{-1}}{L_B}$

$= \frac{Lm \pm 2nP\sqrt{-1}}{L_B}$. Reiectis itaque primis Logarithmis non-Naturalibus

nanciscimur $\frac{L_1}{\sqrt{-1}} = \pm 2nP$, nimirum $= 0 = \Pi = 2\Pi = 3\Pi =$

$4\Pi = 5\Pi = 6\Pi = 7\Pi \&c.$ in Infinitum, atque etiam $= -\Pi = -2\Pi = -3\Pi = -4\Pi = -5\Pi = -6\Pi = -7\Pi \&c.$ sine Limite. Ecquid? Fortassis hasce Aequationes, quae Theorema celeberrimum aemulantur ab Ioanne Bernoullio proditum, et in Capite hoc ipso fusius infra perpensum, veluti quoddam incomprehensibile Analyseos Oraculum interpretabimur, mysteriique nomine in re Mathematica abusi, perinde ac si reapse forent

rent invicem aequales $0, \pm \Pi, \pm 2\Pi, \pm 3\Pi \&c.$, adparenti contradictioni me-

delam ferre censebimus? Nil aliud vere sibi vult haec expressio $\frac{L_1}{\sqrt{-1}}$ prae-

terquam $\tau^d Ly$ omni casu Functionem esse Multiformem; adeo ut $\frac{Ly}{\sqrt{-1}} =$

$\frac{x}{\sqrt{-1}}$, et simul $= \frac{x \pm 2n P \sqrt{-1}}{\sqrt{-1}}$, nimirum $= \frac{x}{\sqrt{-1}} = \frac{x}{\sqrt{-1}} + \Pi = \frac{x}{\sqrt{-1}} +$

$2\Pi = \frac{x}{\sqrt{-1}} + 3\Pi = \frac{x}{\sqrt{-1}} + 4\Pi = \frac{x}{\sqrt{-1}} + 5\Pi \&c. = \frac{x}{\sqrt{-1}} - \Pi =$

$\frac{x}{\sqrt{-1}} - 2\Pi = \frac{x}{\sqrt{-1}} - 3\Pi = \frac{x}{\sqrt{-1}} - 4\Pi = \frac{x}{\sqrt{-1}} - 5\Pi \&c.$, ubi elidi

nequit Imaginarietas, excepto unico casu $y = 1$, ideoque Logarithmi eius x nullescentis. Ista vero $\tau^d Ly$ Multiformitas tantummodo in medio ponit Multiplicitatem Radicum illius Aequationis, cuius ope ex Numero cognito Logarithmus eruitur: qua de re Ly , vel L_1 non unum Realem, sed innumeros designat valores, praeter ipsum, Imaginarios, nec est Numerus determinatus, verumtamen Algebraico universali sensu indeterminatus, utpote qui sit Radix infinities diversa Infinitinomiae Aequationis. Imaginarii autem $\tau^d L_1$, et universalissime Ly valores, sive potius Infinitinomii, a cuius resolutione pendent, innumerae Radices, ideo Circulares complectantur necesse est Peripherias, vel earum Multipla, quia dupla Sectorum Circuli Imaginarii positiva, et negativa, sive Sectorum Realis Aequilaterae Hyperbolae Apolloniana, sunt Logarithmi Hyperbolici summae differentiaeve Cofinuum Circularium, Sinuumque Imaginariorum, scilicet, Imaginarietate destructa, veri Numeri y . Inde evenit, quod, uti supra monuimus, ac praesertim in num.^{is} 174. 175., Aequatio $C^x = y$, aut $x = Ly$, pertinens ad Hyperbolam Realem transferri, ludente quasi Geometra, possit ad fictitium Cir-

culum, eam sic concinnando, $x = L \left(\text{Cofin. Circ. } \frac{x}{\sqrt{-1}} + \left(\text{Sin. Circ. } \frac{x}{\sqrt{-1}} \right) \sqrt{-1} \right)$, quae Imaginariis eliminatis veram Hyperbolen exprimit, ipsamque exhibere pergit (etiamsi loco x subrogetur $x \pm (n\Pi \sqrt{-1})$), quum evidenter pateat elementaris Trigonometriae non ignaris esse $x \pm n\Pi \sqrt{-1}$

$$= L \left(\left(\text{Cofin. Circ. } \frac{x}{\sqrt{-1}} \rightarrow (\text{Sin. Circ. } \frac{x}{\sqrt{-1}}) \sqrt{-1} \right) (\text{Cofin. Circ. } \pm n\pi \rightarrow (\text{Sin. Circ. } \pm n\pi) \sqrt{-1}) \right) = L \left(\left(\text{Cofin. Circ. } \frac{x}{\sqrt{-1}} \rightarrow (\text{Sin. Circ. } \frac{x}{\sqrt{-1}}) \sqrt{-1} \right) \times 1 \right).$$

186. Si Logarithmi optarentur Imaginarii Hyperbolen pariter indicantes Apollonianam in quolibet selecto Systemate, memoria repetamus oportet Aequationem reductionis $y = B^x = C^{xL_B} = C^{L(B^x)}$, de qua toties actum in superioribus. Oritur enim spontanea quasi deductio alterius

$$\text{Aequalitatis } xL_B = L \left(\text{Cofin. Circ. } \frac{xL_B}{\sqrt{-1}} \rightarrow (\text{Sin. Circ. } \frac{xL_B}{\sqrt{-1}}) \sqrt{-1} \right), \text{ nimirum realiter } xL_B = L \left(\text{Cofin. Hyperb. } xL_B \rightarrow \text{Sin. Hyperb. } xL_B \right), \text{ vel } x = M.L \left(\text{Cofin. Hyperb. } \frac{x}{M} \rightarrow \text{Sin. Hyperb. } \frac{x}{M} \right) \text{ adeo, ut Logarithmo in Systemate non-Hyperbolico, sed Moduli } M, \text{ aut Baseos } B, \text{ nempe } ly, \text{ conveniat Numerus aequae Realis } \text{Cofin. Circ. } \frac{xL_B}{\sqrt{-1}} \rightarrow (\text{Sin. Circ. } \frac{xL_B}{\sqrt{-1}}) \sqrt{-1}, \text{ atque } x \pm \frac{n\pi\sqrt{-1}}{L_B} \text{ Logarithmo universaliori respondeat Numerus ipse } (\text{Cofin. Circ. } \frac{xL_B}{\sqrt{-1}} \rightarrow (\text{Sin. Circ. } \frac{xL_B}{\sqrt{-1}}) \sqrt{-1}). (\text{Cofin. Circ. } \pm n\pi \rightarrow (\text{Sin. Circ. } \pm n\pi) \sqrt{-1}), \text{ scilicet idem } (\text{Cofin. Circ. } \frac{xL_B}{\sqrt{-1}} \rightarrow (\text{Sin. Circ. } \frac{xL_B}{\sqrt{-1}}) \sqrt{-1}), \text{ aut } \frac{\text{Cofin. Hyperb. } xL_B \rightarrow \text{Sin. Hyperb. } xL_B}{r}, \text{ ob alte-}$$

rum Factorem Unitati aequalem. Quomocunque igitur permutetur Logometricum Systema, erunt innumeri Logarithmi $x \pm \frac{n\pi\sqrt{-1}}{L_B}$, seu $x \pm M(n\pi\sqrt{-1}) = ly$ in Aequatione comprehensi $B^x = y$ ad Hyperbolam relata $\pm x = l(\text{Cofin. Hyperb. } xL_B \pm \text{Sin. Hyperb. } xL_B) = mL(\text{Cofin. Hyp. } xL_B \pm \text{Sin. Hyp. } xL_B)$. Nec putes hoc adversari iis, quae supra

recen-

recensuimus; propterea quod $x = ly$, vel $x \pm \frac{n\pi\sqrt{-1}}{LB} = ly = \frac{Ly}{LB} =$

$$\frac{Ly \pm n\pi\sqrt{-1}}{LB} \text{ praeberet } x = \frac{L(C^{xLB})}{LB} =$$

$$\frac{L(\text{Cofin. Hyperb. } xLB + \text{Sin. Hyperb. } xLB)}{LB} = L\left(\left(\text{Cofin. Hyperb. } xLB + \right.\right.$$

$$\left.\text{Sin. Hyperb. } xLB\right)^{\frac{1}{LB}}) = L\left(\text{Cofin. Hyperb. } \frac{xLB}{LB} + \text{Sin. Hyperb. } \frac{xLB}{LB}\right)$$

$= L(\text{Cofin. Hyperb. } x + \text{Sin. Hyperb. } x) = L(C^x)$, ratiocinationis periodo, ut par est, ad veritatem quaquaversus confirmandam in se redeunte. Postrema conclusio, quae novitatem aliquam praefert, derivatur a

generaliori Theoremate $(\text{Cofin. Hyperb. } x \pm \text{Sin. Hyperb. } x)^v = (C^{\pm x})^v = C^{\pm vx} = \text{Cofin. Hyperb. } vx \pm \text{Sin. Hyperb. } vx$, ad instar expressionum analogis Lineis Trigonometricis praeditarum.

$$187. \text{ Factores etiam Infinitinomialis Aequationis } 0 = \frac{(LB)^1 x^1}{1} + \frac{(LB)^2 x^2}{1.2} + \frac{(LB)^3 x^3}{1.2.3} + \frac{(LB)^4 x^4}{1.2.3.4} + \frac{(LB)^5 x^5}{1.2.3.4.5} + \frac{(LB)^6 x^6}{1.2.3.4.5.6} + \frac{(LB)^7 x^7}{1.2.3.4.5.6.7} + \&c., \text{ five}$$

Radices Aequationis $B^x - 1 = 0$, facillima ratione investigabuntur quum sit

$$B^x - 1 = C^{xLB} - 1 = xLB \left(1 - \frac{xLB}{\pi\sqrt{-1}}\right) \left(1 + \frac{xLB}{\pi\sqrt{-1}}\right) \left(1 - \frac{xLB}{2\pi\sqrt{-1}}\right)$$

$$\left(1 + \frac{xLB}{2\pi\sqrt{-1}}\right) \left(1 - \frac{xLB}{3\pi\sqrt{-1}}\right) \left(1 + \frac{xLB}{3\pi\sqrt{-1}}\right) \&c. = xLB \left(1 + \frac{(LB)^2 x^2}{\pi^2}\right)$$

$$\left(1 + \frac{(LB)^2 x^2}{2^2 \pi^2}\right) \left(1 + \frac{(LB)^2 x^2}{3^2 \pi^2}\right) \&c. \text{ fallaciter, si Radicum valori fidendum foret, attamen vere } = xLB \left(1 + \frac{(LB)x}{\infty} + \frac{(LB)^2 x^2}{\pi^2}\right) \left(1 + \frac{(LB)x}{\infty} + \frac{(LB)^2 x^2}{2^2 \pi^2}\right)$$

$$\left(1 + \frac{(LB)x}{\infty} + \frac{(LB)^2 x^2}{3^2 \pi^2}\right) \left(1 + \frac{(LB)x}{\infty} + \frac{(LB)^2 x^2}{4^2 \pi^2}\right) \left(1 + \frac{(LB)x}{\infty} + \frac{(LB)^2 x^2}{5^2 \pi^2}\right) \&c.$$

$$\text{in Infinitum. Consimili pacto Formulae fluere Factorum Trinomialium hypo-}$$

potheticorum

potheticorum Infinitinomii $1 - m + \frac{(L_B)^1 x^1}{1} + \frac{(L_B)^2 x^2}{1.2} + \frac{(L_B)^3 x^3}{1.2.3} +$

$\frac{(L_B)^4 x^4}{1.2.3.4} + \frac{(L_B)^5 x^5}{1.2.3.4.5} + \frac{(L_B)^6 x^6}{1.2.3.4.5.6} + \&c: = B^x - m = C^{(L_B)x} - m$ sub-

stituendo in iis numeri 183. loco x Indicem alterum xL_B . Hinc resolutio

derivaretur Aequationis oecumenicae $\left(1 + \frac{(L_B)x}{\infty}\right)^\infty - m = 0$ in respecti-

vos sine numero Factores, vel $\left(1 + \frac{(Lz)x}{\infty}\right)^\infty - y = 0$, cuius singularif-

fima tantum species est $\left(1 + \frac{(L_B)x}{\infty}\right)^\infty - 1 = 0$; recurritque observa-

tio

&c: Cofin.Circ.(z-2y) + (Sin.Circ.(z-2y)) $\sqrt{-1}$, ..., Cofin.Circ.(z-y) + (Sin.Circ.(z-y)) $\sqrt{-1}$, ..., Cofin.Circ.z +

&c: $(x-2y)\sqrt{-1}$ $(x-y)\sqrt{-1}$ x.

Istae Series, tametsi Logarithmos habeant omnimode Imaginarios, non uni-
versos aequae Numeros Imaginarios habent; sed, quoties z , vel $z \pm ry =$

$(L_B)(x \pm ry) = \pm nP = \pm \frac{n\Pi}{2}$, five $x \pm ry = \frac{\pm nP}{L_B}$, Numeri Imagina-
riis interpositi evadent Reales, nimirum ± 1 , dum $\pm n$ evanescat, aut sit
par

&c: ± 1 , -1 , ± 1 , -1 , ± 1 , -1
&c: $\frac{3\Pi\sqrt{-1}}{L_B}$, $\frac{5P\sqrt{-1}}{L_B}$, $\frac{2\Pi\sqrt{-1}}{L_B}$, $\frac{3P\sqrt{-1}}{L_B}$, $\frac{\Pi\sqrt{-1}}{L_B}$, $\frac{P\sqrt{-1}}{L_B}$

quod in Hyperbolicae Basis seu Antonomastici-Numeri C hypothesi simpli-
cius

&c: ± 1 , -1 , ± 1 , -1 , ± 1 , -1
&c: $-\frac{3\Pi\sqrt{-1}}{L_B}$, $-\frac{5P\sqrt{-1}}{L_B}$, $-\frac{2\Pi\sqrt{-1}}{L_B}$, $-\frac{3P\sqrt{-1}}{L_B}$, $-\frac{\Pi\sqrt{-1}}{L_B}$, $-\frac{P\sqrt{-1}}{L_B}$

absque demonstrationis necessitate.

tio insperatae propemodum diversitatis Aequationem inter $\left(1 + \frac{(LB)x}{\infty}\right)^{\infty}$

$= 0$, alteramque vix discrepantem $\left(1 + \frac{(LB)x}{\infty}\right)^{\infty} - 1 = 0$, prouti

de homologo argumento alias monuimus.

188. Logarithmi Imaginarii ab Exponentiali generalissima Aequatione

deducti $B^{x\sqrt{-1}} = C^{(LB)x\sqrt{-1}} = \text{Cofin. Circ. } (LB)x + (\text{Sin. Circ. } (LB)x)\sqrt{-1}$,

vel posito $(LB)x = z$, maioris causa simplicitatis, nec non $(LB)v = y$, tribuunt Serierum infra-scriptarum Logometricam relationem, Trigonometriae subsidio determinandam,

&c: Cofin.

$\text{Sin. (Circ. } z)\sqrt{-1} \dots \text{Cofin. Circ. } (z+y) + (\text{Sin. Circ. } (z+y))\sqrt{-1} \dots \text{Cofin. Circ. } (z+2y) + (\text{Sin. Circ. } (z+2y))\sqrt{-1} \dots$ &c:

$\sqrt{-1} \dots (x+v)\sqrt{-1} \dots (x+2v)\sqrt{-1} \dots$ &c:.

Istae

par Numerus integer, ac ex aduerso -1 , dum $\pm n$ imparibus adscribatur Numeris integris. Sic, innumeris intermediis in priore Numerorum Serie commixtis Imaginariis, emergit Systema hoc Logometricum indefinita praeditum Base B

&c: -1

$\dots, -1, -1, -1, -1, -1, -1, -1 \dots$ &c:
 $\dots, 0, \frac{P\sqrt{-1}}{LB}, \frac{\Pi\sqrt{-1}}{LB}, \frac{3P\sqrt{-1}}{LB}, \frac{2\Pi\sqrt{-1}}{LB}, \frac{5P\sqrt{-1}}{LB}, \frac{3\Pi\sqrt{-1}}{LB} \dots$ &c: ,

quod in

cuius redditur

$\dots, -1, -1, -1, -1, -1, -1, -1 \dots$ &c:
 $\dots, 0, P\sqrt{-1}, \Pi\sqrt{-1}, 3P\sqrt{-1}, 2\Pi\sqrt{-1}, 5P\sqrt{-1}, 3\Pi\sqrt{-1} \dots$ &c: ,

189. Non itaque solummodo $\frac{L(-1)}{\sqrt{-1}} = P$, uti est Ioannis Bernoullii

peringeniosum Theorema, veruntamen universalius $\frac{L(-1)}{\sqrt{-1}} = \pm (2n+1)P$,

ubi n Nullitatem, aut Numerum quemlibet denotat positivum, quin etiam in Systemate per Basin B designato, vel potius per Modulum $\frac{1}{L_B}$, Aequationes

$$\frac{1(-1)}{\sqrt{-1}} = \frac{P}{L_B}, \text{ et amplissime } \frac{1(-1)}{\sqrt{-1}} = \frac{\pm (2n+1)P}{L_B} \text{ obtinentur. Re-}$$

cipit quoque novum robur Aequalitas in praecedente num°. 185. iam tra-

dita $\frac{L_1}{\sqrt{-1}} = \pm n\Pi$, et generatim aptatur cuicunque Systemati Logarithmi-

co hac methodo, $\frac{1_1}{\sqrt{-1}} = \pm \frac{n\Pi}{L_B}$. Ipsa praeunte Serie concipere ulte-

rius in promptu est Logarithmos Numerorum, qui sint Medietates Geometri-
cae inter alternos $+1$ et -1 sine Limite successivos, scilicet Num-
erorum impossibilium $\pm \sqrt{-1}$. Si de Systemate imprimis loquamur Hyperbo-
lico

$$\&c: +1, +\sqrt{-1}, -1, -\sqrt{-1}, +1, +\sqrt{-1}, -1, -\sqrt{-1}, +1,$$

$$\&c: -4P\sqrt{-1}, -\frac{7P}{2}\sqrt{-1}, -3P\sqrt{-1}, -\frac{5P}{2}\sqrt{-1}, -2P\sqrt{-1}, -\frac{3P}{2}\sqrt{-1}, -P\sqrt{-1}, -\frac{P}{2}\sqrt{-1}, 0,$$

$$\&c: -\frac{4P}{L_B}\sqrt{-1}, -\frac{7P}{2L_B}\sqrt{-1}, -\frac{3P}{L_B}\sqrt{-1}, -\frac{5P}{2L_B}\sqrt{-1}, -\frac{2P}{L_B}\sqrt{-1}, -\frac{3P}{2L_B}\sqrt{-1}, -\frac{P}{L_B}\sqrt{-1}, -\frac{P}{2L_B}\sqrt{-1}, 0,$$

Huiusce distributionis origo quaerenda est non in Progressionum Geometri-
carum rigidis Legibus, nam ordini ipsi *Progressionali* Signorum dispositio,
et commixtae Realibus Imaginariae Magnitudines evidenter repugnant, sed
in Formula Numerorum oecumenica Cofin. Circ. $\Phi + (\text{Sin. Circ. } \Phi) \sqrt{-1}$,

quippe quae, dum $\Phi = +\frac{P}{2}, +\frac{5P}{2}, +\frac{9P}{2}, +\frac{13P}{2} \&c;$, vel $-$

$$\frac{3P}{2}$$

lico habebitur $L(\pm\sqrt{-1}) = \pm \frac{P}{2}\sqrt{-1}, \pm \frac{3P}{2}\sqrt{-1}, \pm \frac{5P}{2}\sqrt{-1}, \pm$

$\frac{7P}{2}\sqrt{-1}, \pm \frac{9P}{2}\sqrt{-1}, \&c., \pm \frac{(2n+1)P}{2}\sqrt{-1}$ ita, ut enascatur

$$\frac{L(\pm\sqrt{-1})}{\sqrt{-1}} = \pm \frac{(2n+1)P}{2}, \text{ nec non aliis in Systematibus } \frac{L(\pm\sqrt{-1})}{\sqrt{-1}} = \pm$$

$$\frac{(2n+1)P}{2LB} = \pm M(2n+1) \frac{P}{2}. \text{ Fallax autem fuisset deduxisse, eadem For-}$$

mula duce, atque ex Legibus Geometrico - Arithmeticae Progressionis,

$$\frac{L(\sqrt{-1})}{\sqrt{-1}}, \text{ vel potius } \frac{L(\sqrt{\sqrt{-1}})}{\sqrt{-1}} = \pm \frac{(2n+1)P}{2}, \text{ five maiore universali-}$$

$$\text{tate } \frac{L(\sqrt{\sqrt{-1}})}{\sqrt{-1}} = \pm \frac{(2n+1)P}{4LB}. \text{ Verum, ne Signorum peculiaris distinctio}$$

obscuritatem progignat, atque ut Geometricarum sine fine Medietatum experi-
mentum Lectoribus relictum facilius reddatur, geminas Logarithmicas Series
ordine necessario disponam.

&c: — +

$$+ \sqrt{-1}, -1, -\sqrt{-1}, +1, +\sqrt{-1}, -1, -\sqrt{-1}, +1 \&c:$$

$$+ \frac{P}{2}\sqrt{-1}, + P\sqrt{-1}, + \frac{3P}{2}\sqrt{-1}, + 2P\sqrt{-1}, + \frac{5P}{2}\sqrt{-1}, + 3P\sqrt{-1}, + \frac{7P}{2}\sqrt{-1}, + 4P\sqrt{-1} \&c:$$

$$+ \frac{P}{2LB}\sqrt{-1}, + \frac{P}{LB}\sqrt{-1}, + \frac{3P}{2LB}\sqrt{-1}, + \frac{2P}{LB}\sqrt{-1}, + \frac{5P}{2LB}\sqrt{-1}, + \frac{3P}{LB}\sqrt{-1}, + \frac{7P}{2LB}\sqrt{-1}, + \frac{4P}{LB}\sqrt{-1} \&c:.$$

Huiusce

$$\frac{3P}{2}, \frac{7P}{2}, \frac{11P}{2}, \frac{15P}{2} \&c:, \text{ abit in } +\sqrt{-1}, \text{ et vicissim verti-}$$

$$\text{tur in } -\sqrt{-1}, \text{ quoties } \Phi = +\frac{3P}{2}, +\frac{7P}{2}, +\frac{11P}{2}, +\frac{15P}{2} \&c:, \text{ five}$$

$$-\frac{P}{2}, -\frac{5P}{2}, -\frac{9P}{2}, -\frac{13P}{2} \&c:.$$

Eadem praedicanda de universis
Logarithmorum Systematibus; quandoquidem facile dictu est intuituque dif-
ferentiam

ferentiam omnem in consideratione π & Lb , vel $\frac{1}{M}$ tantummodo versari.

190. Speculatio non parum iucunda longius protrahi, aliisque Numeris Imaginariis extendi posset si Finitorum Calculi praesidio ea, quae Val-

mesleyus Infinitesima tractans de Logarithmis gemini fictitii Numeri $\frac{\pm 1 + \sqrt{-1}}{\sqrt{2}}$

protulit, repetamus (*). Ad hoc etenim, ut illos significet Numeros familiaris expressio canonica Cosin. Circ. $\Phi \rightarrow (\text{Sin. Circ. } \Phi) \sqrt{-1}$, debet Arcus, Sectorve duplus, Φ medietatem Arcus Quadrantalıs, vel Aream Quadrantis, nimirum $\frac{P}{4}$ aut $2\left(\frac{P}{4} \cdot \frac{1}{2}\right)$ peraequare; quo posito erit Cosin.

Circ. $\Phi = \text{Sin. Circ. } \Phi = \frac{1}{\sqrt{2}}$. Similiter, facto Arcu $\Phi = \frac{3P}{4}$, profluet

Sin.

$$\&c: \frac{-1 - \sqrt{-1}}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{+1 - \sqrt{-1}}{\sqrt{2}}, +1, \frac{+1 + \sqrt{-1}}{\sqrt{2}}, +$$

$$\&c: -\frac{11P\sqrt{-1}}{4}, -\frac{5P\sqrt{-1}}{2}, -\frac{9P\sqrt{-1}}{4}, -2P\sqrt{-1}, -\frac{7P\sqrt{-1}}{4}, -$$

$$\&c: -\frac{11P\sqrt{-1}}{4Lb}, -\frac{5P\sqrt{-1}}{2Lb}, -\frac{9P\sqrt{-1}}{4Lb}, -\frac{2P\sqrt{-1}}{Lb}, -\frac{7P\sqrt{-1}}{4Lb}, -$$

$$+1, \frac{+1 + \sqrt{-1}}{\sqrt{2}}, +\frac{1}{\sqrt{2}}, \frac{-1 + \sqrt{-1}}{\sqrt{2}}, -1, \frac{-1 - \sqrt{-1}}{\sqrt{2}}, -$$

$$, 0, +\frac{P\sqrt{-1}}{4}, +\frac{P\sqrt{-1}}{2}, +\frac{3P\sqrt{-1}}{4}, +P\sqrt{-1}, +\frac{5P\sqrt{-1}}{4}, +$$

$$, 0, +\frac{P\sqrt{-1}}{4Lb}, +\frac{P\sqrt{-1}}{2Lb}, +\frac{3P\sqrt{-1}}{4Lb}, +\frac{P\sqrt{-1}}{Lb}, +\frac{5P\sqrt{-1}}{4Lb}, +$$

(*) Consulantur *Acta Berolinensis Scientiarum Academiae* in Volumine anni MDCCLV., ubi occurrit Dissertatio sub titulo *Methodi ad inveniendos cuiuslibet Numeri positivi, negativi, quin etiam impossibilis, Logarithmos.*

Sin. Circ. $\Phi = \frac{1}{\sqrt{2}}$, veluti supra, at Cofin. Circ. $\Phi = \frac{-1}{\sqrt{2}}$. Prima ergo

in hypothese erit Cofin. Circ. $\Phi + (\text{Sin. Circ. } \Phi) \sqrt{-1} = \frac{1 + \sqrt{-1}}{\sqrt{2}}$, atque

$\frac{-1 + \sqrt{-1}}{\sqrt{2}}$ in altera. Nec solummodo $\frac{P}{4}$, sive $\frac{3P}{4}$ antea actis Aequalitati-

bus satisfaciunt, quandoquidem manent iidem, eodemque Signo notati Co-

sinus, restique Sinus $\pi \pm n\pi \pm \frac{P}{4}$, vel in casu secundo $\pm n\pi \pm \frac{3P}{4}$;

quamobrem, lacunis completis, pinguius evadet superiorum Serierum Logometricarum syntagma.

$$\begin{aligned} & \sqrt{-1}, \frac{-1 + \sqrt{-1}}{\sqrt{2}}, \frac{-1 - \sqrt{-1}}{\sqrt{2}}, \sqrt{-1}, \frac{+1 - \sqrt{-1}}{\sqrt{2}}, +1, \\ & \frac{3P\sqrt{-1}}{2}, \frac{5P\sqrt{-1}}{4}, -P\sqrt{-1}, \frac{3P\sqrt{-1}}{4}, -\frac{P\sqrt{-1}}{2}, -\frac{P\sqrt{-1}}{4}, 0, \\ & \frac{3P\sqrt{-1}}{2LB}, \frac{5P\sqrt{-1}}{4LB}, -\frac{P\sqrt{-1}}{LB}, -\frac{3P\sqrt{-1}}{4LB}, -\frac{P\sqrt{-1}}{2LB}, -\frac{P\sqrt{-1}}{4LB}, 0, \\ & \sqrt{-1}, \frac{+1 - \sqrt{-1}}{\sqrt{2}}, +1, \frac{+1 + \sqrt{-1}}{\sqrt{2}}, +\sqrt{-1}, \frac{-1 + \sqrt{-1}}{\sqrt{2}} \&c: \\ & \frac{3P\sqrt{-1}}{2}, +\frac{7P\sqrt{-1}}{4}, +2P\sqrt{-1}, +\frac{9P\sqrt{-1}}{4}, +\frac{5P\sqrt{-1}}{2}, +\frac{11P\sqrt{-1}}{4} \&c: \\ & \frac{3P\sqrt{-1}}{2LB}, +\frac{7P\sqrt{-1}}{4LB}, +\frac{2P\sqrt{-1}}{LB}, +\frac{9P\sqrt{-1}}{4LB}, +\frac{5P\sqrt{-1}}{2LB}, +\frac{11P\sqrt{-1}}{4LB} \&c: . \end{aligned}$$

Hic

Hic denuo patet Logarithmos Imaginarios legitimam sequi Arithmeticam, sive Aequidifferentium Magnitudinum Progressionem, non autem verè Geometricam Numeros partim Reales, partimque Imaginarios. Mentiuntur ta-

Q q

men

men ii Numeri Geometricam Progressionem, quia ex Formula manant Cofin. Circ.

$\Phi \rightarrow (\text{Sin. Circ. } \Phi) \sqrt{-1}$, in qua, si loco Φ pars aliquota $\frac{P}{n}$ Semiperiphæriæ sub-

rogetur, sunt, analytice loquendo, in Progressione Geometrica Numeri

Cofin. Circ. $0 \rightarrow (\text{Sin. Circ. } 0) \sqrt{-1}$, vel 1; Cofin. Circ. $\frac{P}{n} \rightarrow (\text{Sin.}$

Circ. $\frac{P}{n}) \sqrt{-1}$; Cofin. Circ. $\frac{2P}{n} \rightarrow (\text{Sin. Circ. } \frac{2P}{n}) \sqrt{-1}$; Cofin. Circ.

$\frac{3P}{n} \rightarrow (\text{Sin. Circ. } \frac{3P}{n}) \sqrt{-1}$; Cofin. Circ. $\frac{4P}{n} \rightarrow (\text{Sin. Circ. } \frac{4P}{n}) \sqrt{-1}$;

Cofin. Circ. $\frac{5P}{n} \rightarrow (\text{Sin. Circ. } \frac{5P}{n}) \sqrt{-1}$ &c.; scilicet ex praeostensis

num°. 160. huius Capitis, in quo sumus, 1; Cofin. Circ. $\frac{P}{n} \rightarrow (\text{Sin. Circ.}$

$\frac{P}{n}) \sqrt{-1}$; $(\text{Cofin. Circ. } \frac{P}{n} \rightarrow (\text{Sin. Circ. } \frac{P}{n}) \sqrt{-1})^2$; $(\text{Cofin. Circ. } \frac{P}{n} \rightarrow (\text{Sin.}$

Circ. $\frac{P}{n}) \sqrt{-1})^3$; $(\text{Cofin. Circ. } \frac{P}{n} \rightarrow (\text{Sin. Circ. } \frac{P}{n}) \sqrt{-1})^4$; $(\text{Cofin. Circ.}$

$\frac{P}{n} \rightarrow (\text{Sin. Circ. } \frac{P}{n}) \sqrt{-1})^5$ &c. adeo, ut illius prope Monstri Geometrica-

rum

$$\&c: 1, -\sqrt{-1}, -\sqrt{-1}, \frac{1}{1+\sqrt{-1}}, +1, +\sqrt{-1},$$

$$\&c: \Delta \mp 12\theta, \Delta \mp 11\theta, \Delta \mp 10\theta, \Delta \mp 9\theta, \Delta \mp 8\theta, \Delta \mp 7\theta,$$

$$+1, +\sqrt{-1}, +\sqrt{-1}, \frac{-1}{1+\sqrt{-1}}, -1, -\sqrt{-1}, -\sqrt{-1},$$

$$x \rightarrow z\sqrt{-1}$$

scu

$$\Delta, \Delta \pm \theta, \Delta \pm 2\theta, \Delta \pm 3\theta, \Delta \pm 4\theta, \Delta \pm 5\theta, \Delta \pm 6\theta,$$

rum Progressionum quaerenda sit ratio non in Numerorum valoribus, qui cum mendosi Calculi abortu in Imaginariis quasi ludentis comparari debent, veruntamen in Analyticae tantum Formulae singulari caractere, et natura.

191. Ex hac Formulae indole fit, ut generatim Cofin. Circ. $v \pm (\text{Sin. Circ. } v) \sqrt{-1}$, Cofin. Circ. $v+z \pm (\text{Sin. Circ. } v+z) \sqrt{-1}$, Cofin. Circ. $v+2z \pm (\text{Sin. Circ. } v+2z) \sqrt{-1}$, nimirum $C^{\pm v \sqrt{-1}}$, $C^{\pm v+2z \sqrt{-1}}$,

$C^{\pm v+2z \sqrt{-1}}$, praebeant extremorum Productum $(\text{Cofin. Circ. } v \pm (\text{Sin. Circ. } v) \sqrt{-1}) (\text{Cofin. Circ. } v+2z \pm (\text{Sin. Circ. } v+2z) \sqrt{-1})$, videlicet $C^{\pm v \sqrt{-1}} \times C^{\pm (v+2z) \sqrt{-1}}$, seu $C^{\pm (2v+2z) \sqrt{-1}}$ = Quadrato intermedii

termini $(\text{Cofin. Circ. } v+z \pm (\text{Sin. Circ. } v+z) \sqrt{-1})^2$, quippe quum τd $(C^{\pm (v+z) \sqrt{-1}})^2$ aut idem $C^{\pm (2v+2z) \sqrt{-1}}$, veluti antea, peraequet.

Sin autem Progressiones Geometricae a Formula illa speciali Analytica, uti fas est, abludentes effingerentur, quas inter locum obtinet Diagramma sub-

nexum &c: —

$$+\sqrt{-1}, \frac{-1}{+\sqrt{+\sqrt{-1}}}, -1, -\sqrt{+\sqrt{-1}}, -\sqrt{-1}, \frac{+1}{-\sqrt{+\sqrt{-1}}}, +1,$$

$x+z\sqrt{-1}$

seu

$$\Delta \mp 6\Theta, \Delta \mp 5\Theta, \Delta \mp 4\Theta, \Delta \mp 3\Theta, \Delta \mp 2\Theta, \Delta \mp \Theta, \Delta,$$

$$\frac{+1}{-\sqrt{+\sqrt{-1}}}, +1, +\sqrt{+\sqrt{-1}}, +\sqrt{-1}, \frac{-1}{+\sqrt{+\sqrt{-1}}}, -1 \text{ \&c:}$$

$$\Delta \pm 7\Theta, \Delta \pm 8\Theta, \Delta \pm 9\Theta, \Delta \pm 10\Theta, \Delta \pm 11\Theta, \Delta \pm 12\Theta \text{ \&c:},$$

Unitates

Unitates negativae, ac positivae, quae alterna periodo sine fine recurrunt, nec non alternatim quoque periodicae Magnitudines Imaginariae $\pm \sqrt{-1}$, aliaeve interspersae, quoslibet Logarithmos Reales Imaginarios simplices, aut compositos recipere possent absque Circularis Areae, ipsiusve Functionum consideratione. Quibus praemissis concluditur $\text{Log.}(\pm 1)$, atque, si velis, $\text{Log.}(\pm z)$ latissimo sensu esse indeterminatum, utpote qui arbitrio Geometrae innumerorum valorum sit capax ita, ut recte Dalembertius in primo *Opusculorum* Volumine adseruerit, Seriem Nullitatum Arithmeticam Logarithmos, quamvis evanescentes, Numerorum Geometrice progredientium posse exhibere (*). Vitium itaque ratiocinationis Analyticae censendum foret pro aris, et focus pugnare de praecisa π $\text{Log.}(\pm 1)$ in universis Logometricis Systematibus, quae excudi possint plenissime, expressione, quum unicus solummodo, specialisque casus Problematis huiusce Indeterminati, solutionemque indefinitam habentis ad instar $\frac{0}{0}$, vel $\infty - \infty$, sive $0 \cdot \infty$ &c., sint superius exposi-

tae Magnitudines ab Circulo pendentes $\pm n\pi \sqrt{-1}$, vel $\pm \left(\frac{2n+1}{2}\right) \pi \sqrt{-1}$ sensusque expers existimari debeat definiti illius Logarithmi absolutissima omnibus numeris investigatio

192. Etsi geminis Progressionibus, quas memorat numerus praecedens 190., tam Geometricae, quam Arithmeticae, condiciones adsint Systema Logometricum efficientes, ac in Geometrica commisceantur innumeris Quantitatibus Imaginariis Reales quoque ± 1 sine Limite recurrentes, non ideo Systematis Curva Logistica delineari poterit, nec eam quidem prouti Punctorum Coniugatorum absque numero congeriem adfirmare fas erit. Quum etenim universae Abscissae, seu Logarithmi comitem habeant Imaginarietatem, deficit Abscissarum Axis, nempe deficit Curva, deficiuntque non solum Ordinatum-applicatae Imaginariae, verum etiam, deficientibus Abscissis, Reales. Locus igitur illius Systematis Logarithmici Imaginarius est, haud secus ac Systema, de quo toties loquenti. Ridicula est ergo quaestio de Subtangente Systematis, interea dum nulla sit Curva illud exhibens. Aliqui eam Subtangenter nihilominus volunt Realem, et signanter Unitatem, uti in
vero

(*) Meminisse proderit Numeri 147.

vero Systemate Hyperbolico, pro Subtangente determinant; alii Imaginariam Subtangente et $= \sqrt{-1}$ deprædicant. Postremis argumenta omnia defunctur inter infinite multiplices Numeros Imaginarios feligant potius Numerum impossibilem $\sqrt{-1}$, quam alterum. Quin immo duceret eorum ratiocinatio ad definiendam, contra ipsorum votum, Unitatem positivam pro Subtangente illius admodum curiosi Systematis. Liceat aliquantisper præoccupare in Capite VIII, plenius enucleata. Systematis Numeros, vel Ordina-

tim-applicatas Logisticae Imaginariae offert Formula Exponentialis $C^{\Phi\sqrt{-1}}$

$$= 1 + \frac{\Phi\sqrt{-1}}{1} - \frac{\Phi^2}{1.2} - \frac{\Phi^3\sqrt{-1}}{1.2.3} + \frac{\Phi^4}{1.2.3.4} - \frac{\Phi^5\sqrt{-1}}{1.2.3.4.5} - \frac{\Phi^6}{1.2.3.4.5.6} \\ - \&c.; \text{ five universalior } B^{\Phi\sqrt{-1}} = 1 + \frac{(LB)\Phi\sqrt{-1}}{1} - \frac{(LB)^2\Phi^2}{1.2} -$$

$$\frac{(LB)^3\Phi^3\sqrt{-1}}{1.2.3} + \frac{(LB)^4\Phi^4}{1.2.3.4} - \&c.; \text{ simulque Imaginarii Arcus } \Phi\sqrt{-1} \text{ conve-}$$

nientes indicant Logarithmos. In oecumenica nunc Subtangente expressio-

ne $\frac{ydx}{dy}$ loco y subrogentur $C^{\Phi\sqrt{-1}}$ vel $B^{\Phi\sqrt{-1}}$; loco dx $\tau\delta (d\Phi)\sqrt{-1}$;

et demum vice dy , seu $d(C^{\Phi\sqrt{-1}})$ ac $d(B^{\Phi\sqrt{-1}})$, ponantur $\frac{d\Phi\sqrt{-1}}{1} -$

$$\frac{\Phi^1 d\Phi}{1} - \frac{\Phi^2 d\Phi\sqrt{-1}}{1.2} + \frac{\Phi^3 d\Phi}{1.2.3} - \frac{\Phi^4 d\Phi\sqrt{-1}}{1.2.3.4} - \frac{\Phi^5 d\Phi}{1.2.3.4.5} - \&c.; \text{ aut}$$

$$\frac{(LB)(d\Phi)\sqrt{-1}}{1} - \frac{(LB)^2\Phi^1 d\Phi}{1} - \frac{(LB)^3\Phi^2 d\Phi\sqrt{-1}}{1.2} + \frac{(LB)^4\Phi^3 d\Phi}{1.2.3} - \&c.; \text{ scilicet}$$

$$d\Phi\sqrt{-1} \left(1 + \frac{\Phi\sqrt{-1}}{1} - \frac{\Phi^2}{1.2} - \frac{\Phi^3\sqrt{-1}}{1.2.3} + \frac{\Phi^4}{1.2.3.4} - \frac{\Phi^5\sqrt{-1}}{1.2.3.4.5} - \&c. \right),$$

$$\text{five } C^{\Phi\sqrt{-1}} (d\Phi)\sqrt{-1}, \text{ et } (LB)(d\Phi)\sqrt{-1} \left(1 + \frac{(LB)\Phi^1\sqrt{-1}}{1} - \frac{(LB)^2\Phi^2}{1.2} \right.$$

$$\left. - \frac{(LB)^3\Phi^3\sqrt{-1}}{1.2.3} + \&c. \right), \text{ videlicet } B^{\Phi\sqrt{-1}} (LB)(d\Phi)\sqrt{-1}. \text{ Illa tum For-}$$

$$\text{mula evadet } \frac{C^{\Phi\sqrt{-1}} (d\Phi)\sqrt{-1}}{C^{\Phi\sqrt{-1}} (d\Phi)\sqrt{-1}} = +1, \text{ atque } \frac{B^{\Phi\sqrt{-1}} (d\Phi)\sqrt{-1}}{B^{\Phi\sqrt{-1}} (LB)(d\Phi)\sqrt{-1}} =$$

$\frac{+1}{L_B} = M$, quae Subtangentes, Modulive Realibus Logarithmicis Curvis

quantitate, ac qualitate respondent. Quousque autem Geometrarum patientia tot paradoxa abutentur? Mea sententia favente tota corrui contradictionis praefatae suspicio de Subtangente aut Moduli communione in Imaginariis, atque Realibus Logarithmicis Curvis recens praesumpta. Curva namque, cuius Aequatio prima fronte videtur Imaginaria, $C^{\Phi\sqrt{-1}} = \text{Cofin. Circ. } \Phi + (\text{Sin. Circ. } \Phi)\sqrt{-1}$, propter Impossibilia Magnitudinum elisionem nihil est aliud nisi Aequatio $C^{\Phi} = \text{Cofin. Hyperb. } \Phi + \text{Sin. Hyperb. } \Phi$, scilicet Aequatio ad Realem Curvam Logisticam Subtangente $+1$ praeditam.

Transformatio Aequationis $\pm C^{\pm\Phi} = \text{Cofin. Hyperb. } \Phi \pm \text{Sin. Hyperb. } \Phi$ in alteram aequipollentem, prouti toties superius, ac praecipue in num^o.

172. monuimus, $\pm C^{\pm\Phi\sqrt{-1}} = \text{Cofin. Circ. } \Phi \pm (\text{Sin. Circ. } \Phi)\sqrt{-1}$, nec prioris significationem, nec denotatae Curvae naturam permutare valet. Sic, qui Parabolam Apollonii Aequatione $ax\sqrt{-1} = y^2\sqrt{-1}$ repraesentare mallet, non ideo Imaginariam signaret Curvam, et eius Subtangente aequae Impossibilem adserere posset, veruntamen Subtangente eam quaerens inveniret Realem $2x$, propterea quod data Aequatio reapse congruit alteri $ax = y^2$. Destructio Imaginariorum, quae partitionis subsidio hic facillime contingit, longiore certe opus habet ratiocinationis circuitu ad Aequationum

$\pm C^{\pm\Phi\sqrt{-1}} = \text{Cofin. Circ. } \Phi \pm (\text{Sin. Circ. } \Phi)\sqrt{-1}$, et $\pm$

$C^{\pm\Phi} = \text{Cofin. Hyperb. } \Phi \pm \text{Sin. Hyperb. } \Phi$, vel potius $\pm B^{\pm\Phi\sqrt{-1}} = \pm$

$C^{\pm(L_B)\Phi\sqrt{-1}} = \text{Cofin. Circ. } (L_B)\Phi \pm (\text{Sin. Circ. } (L_B)\Phi)\sqrt{-1}$, atque

$\pm B^{\pm\Phi} = \pm C^{\pm(L_B)\Phi} = \text{Cofin. Hyperb. } (L_B)\Phi \pm \text{Sin. Hyperb. } (L_B)\Phi$

perfectam identitatem respective demonstrandam; quo vix facto non mirum, si in postremo quoque casu universa Logometrica Systemata comprehendente enascatur Subtangens

$\frac{+1}{(L_B)}$. Curva enim Aequatione non Imaginaria, sed

intimius

intimius Reali $\pm B$ $\pm \Phi \sqrt{-1}$ $= \text{Cofin. Circ. (LB)} \Phi \pm (\text{Sin. Circ. (LB)} \Phi) \sqrt{-1}$ insignitur.

193. Iis omnibus, de quibus usque a numo. 179. dissertum est, in unum coniunctis, obtinetur variis argumentis locupletata proprietas, in Aequatione scilicet ipsa Hyperbolen Apollonianam Aequilateram, sive Logisticam ab eadem genitam Systematis Hyperbolici exprimente $\pm C^x = y =$

$$\pm \left(1 + \frac{x^1}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^8}{1.2.3.4.5.6.7.8} + \&c. \right) = \text{Cofin. Hyperb. } x + \text{Sin. Hyperb. } x,$$

$$\text{quae resoluta praebet } x = \pm \frac{\sqrt{y-1}^1}{1} \mp \frac{\sqrt{y-1}^2}{2} \pm \frac{\sqrt{y-1}^3}{3} \mp \frac{\sqrt{y-1}^4}{4} \pm \frac{\sqrt{y-1}^5}{5} \mp \frac{\sqrt{y-1}^6}{6} \pm \frac{\sqrt{y-1}^7}{7} \mp \frac{\sqrt{y-1}^8}{8} \pm \frac{\sqrt{y-1}^9}{9} \mp \frac{\sqrt{y-1}^{10}}{10} \pm \&c., \text{ vel}$$

Ly , contineri etiam $x = Ly \pm n\pi \sqrt{-1}$. Aequatio igitur $\pm C^x = y$,

sive $\pm C^{Ly} = y = \text{Cofin. Hyperb. (Ly)} + \text{Sin. Hyperb. (Ly)}$ ad amuf-

sim congruit cum altera $\pm C^{Ly \pm n\pi \sqrt{-1}} = y = \text{Cofin. Hyperb. ((Ly) \pm n\pi \sqrt{-1})} + \text{Sin. Hyperb. ((Ly) \pm n\pi \sqrt{-1})}$. Atqui Formula nos docet Exponentialis esse Cofin. Hyperb. ((Ly) \pm n\pi \sqrt{-1}) + Sin. Hyperb. ((Ly) \pm n\pi \sqrt{-1}) = (Cofin. Hyperb. (Ly) + Sin. Hyperb. (Ly)) (Cofin. Hyperb. (\pm n\pi \sqrt{-1}) + Sin. Hyperb. (\pm n\pi \sqrt{-1})). Igitur erit Cofin. Hyperb. (\pm n\pi \sqrt{-1}) + Sin. Hyperb. (\pm n\pi \sqrt{-1}) = Cofin. Circ. (\pm n\pi) + (Sin. Circ. (\pm n\pi)) \sqrt{-1}: nimirum hae duae Analyticae expressiones aequipollentia donantur, idque comprobat superius exposita, et ad unguem in numo.

172. exemplis plurimis communita. Interim turpe foret negligere elegantissimam Circularis Circumferentiae affectionem, qua fit, ut quodlibet eius Multiplicum Imaginarium, positivum seu negativum, $\pm n\pi \sqrt{-1}$ pro x in

$$\text{Infinitinomio } x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} + \frac{x^6}{2.3.4.5.6} + \frac{x^7}{2.3.4.5.6.7} + \frac{x^8}{2.3.4.5.6.7.8} + \&c. \text{ subrogatum, veluti scribitur in } \pm n\pi \sqrt{-1}$$

$$\frac{n^2 \Pi^2}{2} \mp \frac{n^3 \Pi^3 \sqrt{-1}}{2.3} + \frac{n^4 \Pi^4}{2.3.4} \pm \frac{n^5 \Pi^5 \sqrt{-1}}{2.3.4.5} - \frac{n^6 \Pi^6}{2.3.4.5.6} \mp \frac{n^7 \Pi^7 \sqrt{-1}}{2.3.4.5.6.7}$$

$$+ \frac{n^8 \Pi^8}{2.3.4.5.6.7.8} \pm \&c: , \text{ reddat Infinitinomialium istud nullefcens, quod acci-}$$

dere nequit nifi univerfaliter locum habeant confequentes geminae Aequali-

$$\text{tates } n\Pi - \frac{n^3 \Pi^3}{6} + \frac{n^5 \Pi^5}{120} - \frac{n^7 \Pi^7}{5040} + \&c: = 0, \text{ atque } \frac{n^2 \Pi^2}{2} - \frac{n^4 \Pi^4}{24}$$

$$+ \frac{n^6 \Pi^6}{720} - \frac{n^8 \Pi^8}{40320} + \&c: = 0: \text{ fiquidem id exigit neceffaria Imagina-}$$

riorum ab Realibus feparatio. Confimili iure evanefcet Infinitinomialium 2 +

$$\frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^7}{1.2.3.4.5.6.7} +$$

$$\frac{x^8}{1.2.3.4.5.6.7.8} + \&c: \text{ dum vice } x \text{ fubftituatur } \pm \lambda P \sqrt{-1}, \text{ pofito } \lambda \text{ quo-}$$

$$\text{cumque Numero impare ita, ut fit perpetuo } 2 \pm \lambda P \sqrt{-1} - \frac{\lambda^2 P^2}{2} \mp$$

$$\frac{\lambda^3 P^3 \sqrt{-1}}{2.3} + \frac{\lambda^4 P^4}{2.3.4} \pm \frac{\lambda^5 P^5 \sqrt{-1}}{2.3.4.5} - \frac{\lambda^6 P^6}{2.3.4.5.6} \mp \frac{\lambda^7 P^7 \sqrt{-1}}{2.3.4.5.6.7} +$$

$$\frac{\lambda^8 P^8}{2.3.4.5.6.7.8} \pm \&c: = 0, \text{ ideoque } 2 - \frac{\lambda^2 P^2}{2} + \frac{\lambda^4 P^4}{24} - \frac{\lambda^6 P^6}{720} + \frac{\lambda^8 P^8}{40320}$$

$$- \&c: = 0, \text{ nec non } \lambda P - \frac{\lambda^3 P^3}{6} + \frac{\lambda^5 P^5}{120} - \frac{\lambda^7 P^7}{5040} + \&c: = 0: \text{ nun-}$$

quam fatis contemplanda proprietas, quae Transcendentem Numerum 6,283185307179586 &c:, ipfiusve medietatem 3,141592653589793 &c: diftinguit.

194. Sicuti uberrima rationum penu ditare nobis contigit argumentum Harmoniae Circuli, atque Hyperbolae, vel minus recte intellectum, vel non fuis coloribus depictum, vel, quod maxime dedecet graviffimam Analyfin, fabulis commixtum, advertiffe proderit valores Logarithmorum univerfos ope Circuli nuper expreffos Imaginarij ad Aequationem Hyperbolae $\pm$

$$C^x = y = \pm \left(1 + \frac{x^1}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^8}{1.2.3.4.5.6.7.8} + \&c: \right)$$

$$\frac{x^6}{1.2.3.4.5.6} + \frac{x^7}{1.2.3.4.5.6.7} + \&c:) = \text{Cofin. Hyperb. } x + \text{Sin. Hyperb. } x \text{ perti-}$$

nere, non secus ac ad Aequationem Curvae Parabolaeidos $y = a + bx' + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + ix^8 + \&c:$ spectarent, praeter Reales Radices R, etiam Imaginariae, si quae forent, $A \pm B\sqrt{-1}$, utcunque per Cofinus, Sinusque rectos Circulares expositae. Neminem etenim latet, Dalemberthio, Eulero, Foncenexio &c: simul demonstrantibus, Radices cuiuslibet Aequationis, siue Imaginarias, siue Reales, complecti Formulam oecumenicam $A \pm B\sqrt{-1}$; nec difficile conceptu est ipsam Formulam $A \pm B\sqrt{-1}$ verti semper posse in aequipollentem $\sqrt{A^2+B^2}$ (Cofin. Circ. $v \pm (\text{Sin. Circ. } v) \sqrt{-1}$), dummodo Arcus, duplusve Sector, v gaudeat Cofinu $\frac{A}{\sqrt{A^2+B^2}}$,

Fractione simplicissimae originis, quin etiam, si placeat, abire in $\sqrt{A^2+B^2}$ (Cofin. Hyperb. $v \sqrt{-1} \pm \text{Sin. Hyperb. } v \sqrt{-1}$), ubi v Sector sit duplus a Centro supputatus Aequilaterae Hyperbolae Semi-Latere Transverso i gaudentis, cui Cofinus impossibilis $\frac{A}{\sqrt{A^2+B^2}}$ responderet. Rebus sic stanti-

bus emerget conclusio, summi equidem momenti in Logometrica Facultate, quod Logarithmi omnes Imaginarii Unitati, vel cuilibet Numero positivo post unicum Logarithmum Realem in praecedentibus adscripti non diversum constituent, sed idem Systema Hyperbolicum vel Naturale, quod praeterea Hyperbolen sapiant Theoremata Bernoullii, atque ipsis analogae; et quod demum Logarithmi Imaginarii praefati simul cum Reali unam, eandemque Curvam Hyperbolen Conicam, unicam, eandemque Aequationem indigent, quae, cum Infiniti sit Gradus, necessario Radices, seu Logarithmos innumeros tribuere debeat iuxta vulgares Algebrae regulas. Veluti igitur aberraret a veritate quisquis male ratiocinando in Problemate duarum Medietatum geometricae Proportionalium, scilicet in Aequatione resolvenda $x^3 - a^2b = 0$ lippis, et tonforibus nota, opinaretur geminos Imaginarios valores $\left(\frac{-1 \pm \sqrt{-3}}{2}\right) \sqrt[3]{a^2b}$, vel $\sqrt[3]{a^2b} \left(\text{Cofin. Circ. } \left(\pm \frac{n\pi}{3}\right) \pm \left(\text{Sin. Circ. } \left(\pm \frac{n\pi}{3}\right)\right) \sqrt{-1}\right)$, praeter Realem Radicem $\sqrt[3]{a^2b}$, diverso Problema-

ti, diversae Aequationi, Curvae aut Loco Mesolabico a Parabola Cubica diverso respondere, haud aliter abluderent ii, quibus superius explicita displicuerint. Nec ad Aequationem solummodo $\pm C^x = y$, veruntamen ad eo magis canonicam $\pm B^x = y$, quippe quae vertatur in aliam priori

confinilem, et Indice tantum discrepantem $\pm C^{(Lb)x} = y$, sermo omnis in antecessum confectus sese porrigit. Et haec facillima unius Formulae in alteram conversio nobis est causa sufficiens, cur deinceps unicam tantum

expressionem $\pm C^{\pm x}$ rerum dicendarum numero undique excrescente coacti tractabimus, Lectori aliquando cedentes substitutionem $\tau \tilde{x} \pm (Lb)x$, vel $\pm$

$L(B^x)$ vice Indicis x , atque idcirco Exponentialium formae $\pm B^{\pm x}$, seu

$\pm (C^{Lb})^x$, vel $\pm (C^{\frac{1}{M}})^x$ aequae facilem exetasin.

195. Ordo lucubrationis incoeptae nos vocat ad inspiciendam resolutionem canonicae Formulae Cofinum, Sinumque rectum, tam Circularem, quam Hyperbolicum, exhibentis. Esto itaque primum Cofin. Hyperb. $x =$

$$1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} +$$

$$\&c: = \frac{C^x + C^{-x}}{2} = 0, \text{ quaeranturque Infinitinomiali Radices, seu, quod}$$

idem est, casus omnes evanescitiae Cofin. Hyperb. x , scilicet $\tau \tilde{x} C^x + C^{-x}$.

Facile inventu protinus obversatur nullum dari Realem Indicem x , quo posito nullefcet expressio $C^x + \frac{1}{C^x}$; idque etiam evincit Conicae Hyperbolae

indoles, quae Cofinus certe suos evanescere nusquam finit. Universas ergo Infinitinomialium habebit Radices Imaginarias, easque ita comparatas, ut $C^x = \frac{1}{C^x}$, ac vicissim, vel $C^{2x} = -1$. Superius autem observavimus

huic Aequationi Exponentiali omnimode satisfacere $C^{\pm (2n+1)P\sqrt{-1}} =$

Cofin. Circ. $(\pm 2n+1)P + (\text{Sin. Circ. } (\pm 2n+1)P) \sqrt{-1} = -1$. Ra-

dices ideo, hisce praemissis, Imaginariae repertae nunc manent; quin immo et Binomiales secundi Ordinis Reales Infinitinomiali Factores ab illarum no-

tione

tione deducti. Primae erunt $x = + \frac{P\sqrt{-1}}{2}$, $x = - \frac{P\sqrt{-1}}{2}$, $x = +$

$\frac{3P\sqrt{-1}}{2}$, $x = - \frac{3P\sqrt{-1}}{2}$, $x = + \frac{5P\sqrt{-1}}{2}$, $x = - \frac{5P\sqrt{-1}}{2}$, $x = +$

$\frac{7P\sqrt{-1}}{2}$, $x = - \frac{7P\sqrt{-1}}{2}$, $x = + \frac{9P\sqrt{-1}}{2}$, $x = - \frac{9P\sqrt{-1}}{2}$ &c.; et exin-

de memorati Factores invenientur $(1 + \frac{4x^2}{P^2})(1 + \frac{4x^2}{9P^2})(1 + \frac{4x^2}{25P^2})$

$(1 + \frac{4x^2}{49P^2})(1 + \frac{4x^2}{81P^2})(1 + \frac{4x^2}{121P^2})$ &c: sine Limite, (vel potius

$(1 + \frac{x^2}{1^2.Q^2})(1 + \frac{x^2}{3^2.Q^2})(1 + \frac{x^2}{5^2.Q^2})(1 + \frac{x^2}{7^2.Q^2})(1 + \frac{x^2}{9^2.Q^2})$

$(1 + \frac{x^2}{11^2.Q^2})(1 + \frac{x^2}{13^2.Q^2})$ &c.; in Infinitum terminorum Lege pro-

tracta, specieque Q Circularis Circumferentiae Quadrantalem Arcum indi-

cante. Eodem ritu, quum Formula Hyperbolici Sinus-recti tribuat Aequa-

litem $0 = \frac{C^x - C^{-x}}{2} = C^x - \frac{1}{C^x} = \frac{x^1}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} +$

$\frac{x^7}{1.2.3.4.5.6.7} + \frac{x^9}{1.2.3.4.5.6.7.8.9} + \frac{x^{11}}{1.2.3.4.5.6.7.8.9.10.11} + \text{&c:} = x(1 + \frac{x^2}{1.2.3}$

$+ \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6.7} + \frac{x^8}{1.2.3.4.5.6.7.8.9} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10.11} + \text{&c:})$,

nulloque modo $C^x = \frac{1}{C^x}$, five $C^{2x} = 1$, scilicet Cosin. Circ. $\frac{2x}{\sqrt{-1}}$

(Sin. Circ. $\frac{2x}{\sqrt{-1}})$ $\sqrt{-1} = 1$ evanescere possit praeter suppositionem $\frac{2x}{\sqrt{-1}}$

$= \pm nI = \pm 2nP$, vel $x = \pm nP\sqrt{-1}$, quaesitae Infinitinonii propo-

siti Radices erunt $x = 0$, $x = + P\sqrt{-1}$, $x = - P\sqrt{-1}$, $x = +$

$2P\sqrt{-1}$, $x = - 2P\sqrt{-1}$, $x = + 3P\sqrt{-1}$, $x = - 3P\sqrt{-1}$, $x = +$

$4P\sqrt{-1}$, $x = - 4P\sqrt{-1}$, $x = + 5P\sqrt{-1}$, $x = - 5P\sqrt{-1}$ &c.;

quarum Radicum progressionem inutile foret longius producere. Infiniti-

nomium itaque elegantissimum $x \left(1 + \frac{x^2}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} + \right.$

$\frac{x^8}{1.2.3.4.5.6.7.8.9} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10.11} + \&c. \Big) \text{ Hyperbolicos Sinus, de qui-}$

bus loquimur, unico in casu Sectoris x nulescentis annihilari demon-
strat, id comprobante Conicae ipsius Curvae natura, et innumeros in Fa-

ctores Reales apte dissolvitur $x \left(1 + \frac{x^2}{P^2} \right) \left(1 + \frac{x^2}{4P^2} \right) \left(1 + \frac{x^2}{9P^2} \right)$

$\left(1 + \frac{x^2}{16P^2} \right) \left(1 + \frac{x^2}{25P^2} \right) \left(1 + \frac{x^2}{36P^2} \right) \left(1 + \frac{x^2}{49P^2} \right) \&c.$, seu $x \left(1 + \right.$

$\frac{x^2}{1^2.P^2} \Big) \left(1 + \frac{x^2}{2^2.P^2} \right) \left(1 + \frac{x^2}{3^2.P^2} \right) \left(1 + \frac{x^2}{4^2.P^2} \right) \left(1 + \frac{x^2}{5^2.P^2} \right) \left(1 + \right.$

$\frac{x^2}{6^2.P^2} \Big) \left(1 + \frac{x^2}{7^2.P^2} \right) \&c.$, ex quorum inexhaustibili multiplicatione constat,

uti Geometris omnibus innotescit.

196. Ponatur $x = zLB$, eritque $C^x = C^{zLB} = B^z$: quamobrem

$$\frac{B^z + B^{-z}}{2} = 1 + \frac{(LB)^2 z^2}{1.2} + \frac{(LB)^4 z^4}{1.2.3.4} + \frac{(LB)^6 z^6}{1.2.3.4.5.6} + \frac{(LB)^8 z^8}{1.2.3.4.5.6.7.8} +$$

$$\frac{(LB)^{10} z^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c. = \left(1 + \frac{(LB)^2 z^2}{1^2.Q^2} \right) \left(1 + \frac{(LB)^2 z^2}{3^2.Q^2} \right) \left(1 + \frac{(LB)^2 z^2}{5^2.Q^2} \right)$$

$$\left(1 + \frac{(LB)^2 z^2}{7^2.Q^2} \right) \left(1 + \frac{(LB)^2 z^2}{9^2.Q^2} \right) \left(1 + \frac{(LB)^2 z^2}{11^2.Q^2} \right) \left(1 + \frac{(LB)^2 z^2}{13^2.Q^2} \right) \&c. = \text{Cosin. Hy-}$$

$$\text{perb. } (LB)z; \text{ ac praeterea } \frac{B^z - B^{-z}}{2} = \text{Sin. Hyperb. } D(B^z) = (LB)z$$

$$\left(1 + \frac{(LB)^2 z^2}{1.2.3} + \frac{(LB)^4 z^4}{1.2.3.4.5} + \frac{(LB)^6 z^6}{1.2.3.4.5.6.7} + \frac{(LB)^8 z^8}{1.2.3.4.5.6.7.8.9} + \right.$$

$$\frac{(LB)^{10} z^{10}}{1.2.3.4.5.6.7.8.9.10.11} + \&c. \Big) = (LB)z \left(1 + \frac{(LB)^2 z^2}{1^2.P^2} \right) \left(1 + \frac{(LB)^2 z^2}{2^2.P^2} \right)$$

$$\left(1 + \frac{(LB)^2 z^2}{3^2.P^2} \right) \left(1 + \frac{(LB)^2 z^2}{4^2.P^2} \right) \left(1 + \frac{(LB)^2 z^2}{5^2.P^2} \right) \left(1 + \frac{(LB)^2 z^2}{6^2.P^2} \right) \left(1 + \frac{(LB)^2 z^2}{7^2.P^2} \right)$$

$\&c.$; nimirum utraque expressio in Factores resoluta se offeret. Radices,

qui

qui petat, inveniet facillime primo Infinitinomio $= 0$ satisfacere $\frac{x}{L_B}$ feu

$$z = \pm \frac{\lambda Q \sqrt{-1}}{L_B}, \text{ alterique } z = \pm \frac{nP \sqrt{-1}}{L_B} \text{ pro optata Formula universali.}$$

197. Nec nostram methodum renuent Eulerianae expressiones praecedentibus magis compositae, inter quas statim occurrit

$$\frac{C^x - 2 \text{Cofin. Circ. } A + C^{-x}}{2} = 1 - \text{Cofin. Circ. } A + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: . \text{Primum etenim,}$$

ideoque etiam alterum Aequationis Membrum, in Nihilum abit tum, quum

$$\frac{C^x + C^{-x}}{2} = + \text{Cofin. Circ. } A, \text{ videlicet dum } x = (\pm 2nP \pm A) \sqrt{-1}$$

$$\text{ex eo, quod } \frac{C^{(\pm 2nP \pm A) \sqrt{-1}} + C^{-(\pm 2nP \pm A) \sqrt{-1}}}{2} = \text{Cofin. Circ. } (\pm 2nP \pm A)$$

$= \text{Cofin. Circ. } A$, uti poscit exposita Aequalitas. Repertis sine Limite universis Radicibus, quae prodeunt subrogando vice n omnes Numeros integros, emergent una cum ipsis dati Infinitinomii Factores; scilicet

$$\frac{C^x - 2 \text{Cofin. Circ. } A + C^{-x}}{2(1 - \text{Cofin. Circ. } A)} = \frac{\text{Cofin. Hyperb. } x - \text{Cofin. Circ. } A}{1 - \text{Cofin. Circ. } A} = 1 +$$

$$\left(\frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: \right) / (1 - \text{Cofin. Circ. } A)$$

$$= \left(1 + \frac{x^2}{A^2} \right) \left(1 + \frac{x^2}{(\Pi + A)^2} \right) \left(1 + \frac{x^2}{(\Pi - A)^2} \right) \left(1 + \frac{x^2}{(2\Pi + A)^2} \right)$$

$$\left(1 + \frac{x^2}{(2\Pi - A)^2} \right) \left(1 + \frac{x^2}{(3\Pi + A)^2} \right) \left(1 + \frac{x^2}{(3\Pi - A)^2} \right) \left(1 + \frac{x^2}{(4\Pi + A)^2} \right)$$

$$\left(1 + \frac{x^2}{(4\Pi - A)^2} \right) \&c: . \text{Eadem maxime simplici ratione Aequatio ab Eule-}$$

$$\text{ro neglecta resolvetur } \frac{C^x - 2(\text{Sin. Circ. } A) \sqrt{-1} - C^{-x}}{2} = \text{Sin. Hyperb. } x$$

T t

— (Sin.

$$-(\text{Sin. Circ. } A)\sqrt{-1} = -\text{Sin. Circ. } A\sqrt{-1} + \frac{x^1}{1} + \frac{x^3}{1.2.3} - \frac{x^5}{1.2.3.4.5} + \frac{x^7}{1.2.3.4.5.6.7} - \frac{x^9}{1.2.3.4.5.6.7.8.9} + \frac{x^{11}}{1.2.3.4.5.6.7.8.9.10.11} + \&c: \text{nam-}$$

$$\text{que illa cum altera congruit } \frac{C^{(\pm 2nP + A)\sqrt{-1}} - C^{-(\pm 2nP + A)\sqrt{-1}}}{2\sqrt{-1}} = \text{Sin.}$$

Circ. A, quae est identica Aequalitas, nec non ipsamet est cum subsequente

$$\frac{C^{(\pm 2n + 1 \cdot P - A)\sqrt{-1}} - C^{-(\pm 2n + 1 \cdot P - A)\sqrt{-1}}}{2\sqrt{-1}} = \text{Sin. Circ. } A,$$

quae pariter ex principiis antea positis perfectam porrigit aequipollentiam. Nanciscemur igitur Radices numero Infinitas $x = (\pm 2nP + A)\sqrt{-1}$, atque $x = (\pm 2n + 1 \cdot P - A)\sqrt{-1}$, quae omnes Aequationi satisfaciunt, sed Reales Factores nequaquam suppeditare potis sunt, prouti Infinitinomialium ipsum ob terminum Imaginarium nullo modo eliminandum $-(\text{Sin. Circ. } A)\sqrt{-1}$ evidenter confirmat. Ceteras autem Sinuum, Cosinuumve Hyperbolicorum, et Circularium combinationes, quae novas Aequationum formas praestarent ad instar superioris methodi resolvendas, Lectorum cedo sagacitati, quum omnia longius persequi non sit animus, Transcendensque Functio Exponen-

tialis $C^{B+x} \pm C^{A-x}$ nobis potissimum contemplanda obviam veniat. Por-

ro sit universalis Aequatio $C^{B+x} + C^{A-x} = 0$, quin etiam, quod in idem recidit, $\frac{C^{B+x} + C^{A-x}}{m} = 0$. Nulla indiget peculiari demonstratione

Aequalitatem priorem, facto $B+x = A-x = z$, verti in aliam $C^z =$

$-C^z$, five $C^z = -C^z \cdot C^{\pm \lambda P \sqrt{-1}} = -C^{z \pm \lambda P \sqrt{-1}}$, quo fit

$C^{2z} = C^{2z \pm 2\lambda P \sqrt{-1}}$, et eapropter $2z = 2z \pm 2\lambda P \sqrt{-1}$, vel $B+x = A-x \pm \lambda P \sqrt{-1}$, istaque Aequatio postrema tribuit innumeras Radi-

ces qualesitas praesidio Formulae oecumenicae $x = \frac{A-B}{2} \pm \lambda Q \sqrt{-1}$. Ra-

lices

dices igitur investigandae a solutione universalissimi Trinomii Quadratici x^2

$$+ (B - A)x + \frac{(B - A)^2}{4P^2} = 0, \text{ originem ducunt, praebentque}$$

$$+ \lambda^2 \cdot Q^2$$

$$\frac{C^{B+x} + C^{A-x}}{C^B + C^A} = \text{Producto Factorum numero Infinitorum}$$

$$\left(1 + \frac{4(B-A)x + 4x^2}{(B-A)^2 + 1^2 \cdot P^2}\right) \left(1 + \frac{4(B-A)x + 4x^2}{(B-A)^2 + 3^2 \cdot P^2}\right) \left(1 + \frac{4(B-A)x + 4x^2}{(B-A)^2 + 5^2 \cdot P^2}\right)$$

$$\left(1 + \frac{4(B-A)x + 4x^2}{(B-A)^2 + 7^2 \cdot P^2}\right) \left(1 + \frac{4(B-A)x + 4x^2}{(B-A)^2 + 9^2 \cdot P^2}\right) \left(1 + \frac{4(B-A)x + 4x^2}{(B-A)^2 + 11^2 \cdot P^2}\right) \&c.$$

Mutatis mutandis reperientur innumerae quoque Radices alterius Aequa-

$$\text{tionis } \frac{C^{B+x} - C^{A-x}}{m} = 0, \text{ aut simplicius } C^{B+x} - C^{A-x} = 0, \text{ quum}$$

ambigi nequeat esse $C^z = C^z \cdot 1 = C^z \cdot C^{\pm 2nP\sqrt{-1}} = C^{z \pm 2nP\sqrt{-1}}$,
 quae conclusio reddit $z = z \pm 2nP\sqrt{-1}$, seu exhibet $B + x = A - x \pm$
 $2nP\sqrt{-1}$, idest $x = \frac{A-B}{2} \pm nP\sqrt{-1}$. Hisce positis habebitur

$$\frac{C^{B+x} - C^{A-x}}{C^B - C^A} = \left(1 + \frac{2x}{B-A}\right) \left(1 + \frac{4(B-A)x + 4x^2}{(B-A)^2 + 2^2 \cdot P^2}\right)$$

$$\left(1 + \frac{4(B-A)x + 4x^2}{(B-A)^2 + 4^2 \cdot P^2}\right) \left(1 + \frac{4(B-A)x + 4x^2}{(B-A)^2 + 6^2 \cdot P^2}\right) \left(1 + \frac{4(B-A)x + 4x^2}{(B-A)^2 + 8^2 \cdot P^2}\right)$$

$$\left(1 + \frac{4(B-A)x + 4x^2}{(B-A)^2 + 10^2 \cdot P^2}\right) \&c.; \text{ unde erit primum Membrum in numero Infinitos}$$

Factores artificio huiusmodi dispersitum (*).

198. Quamplurima profluunt a Theorematis postremo loco enucleatis utillima Confectaria, quae in Analyseos sublimioris Operibus, ac praesertim in Eulerianae *Introductionis* Volumine I. ad calcem Capitis IX. passim consulendi occasio non deerit. Sed ea, quum novarum non egeant methodorum

(*) Vide quam difficiliter Infiniti ope haec omnia solvat omni laude cumulandus Eulerus in num. 159., ac sequentibus.

rum, recensere singillatim inutile foret, et ab suscepto munere alienum. In-
 xerim, quam praesetulit numerus 79., pleno Marte confecimus resolutionem
 universalissimam Aequationum $\left(1 + \frac{x}{\infty}\right)^{\infty} \pm \left(1 - \frac{x}{\infty}\right)^{\infty} = 0$, nec non,

loco x subrogata quavis Functione $\Delta(z)$, Aequationum $\left(1 + \frac{\Delta(z)}{\infty}\right)^{\infty} \pm$
 $\left(1 - \frac{\Delta(z)}{\infty}\right)^{\infty} = 0$, atque insuper $\left(1 + \frac{(B+x)}{\infty}\right)^{\infty} \pm$
 $\left(1 + \frac{(A-x)}{\infty}\right)^{\infty} = 0$, $\left(1 + \frac{x}{\infty}\right)^{\infty} + \left(1 - \frac{x}{\infty}\right)^{\infty} = 2 \text{ Cofin. Circ.}$
 $A=0$, $\left(1 + \frac{x}{\infty}\right)^{\infty} - \left(1 - \frac{x}{\infty}\right)^{\infty} = 2 (\text{Sin. Circ. } A) \sqrt{-1} = 0$. Etiam

aliae consimili modo efformandae Aequalitates iisdem principiis antea pro-
 latis penitus subiicientur, dum vice x quamlibet variabilium Functionem

placeat illic substituere. Exinde fit, ut $\left(1 + \frac{z\sqrt{-1}}{\infty}\right)^{\infty} \pm$
 $\left(1 - \frac{z\sqrt{-1}}{\infty}\right)^{\infty} = 0$, aut $\left(1 + \frac{\Delta(z)\sqrt{-1}}{\infty}\right)^{\infty} \pm \left(1 - \frac{\Delta(z)\sqrt{-1}}{\infty}\right)^{\infty} = 0$,

ab iisque similiter derivandae, quasi superiorum Coronides nulla ineunte dif-
 ficultate resolutionem, ac in Factores evolutionem recipiant.

199. Et re quidem vera Aequationis Radices $\left(1 + \frac{z\sqrt{-1}}{\infty}\right)^{\infty} +$
 $\left(1 - \frac{z\sqrt{-1}}{\infty}\right)^{\infty} = 0$, vel ad maiorem universalitatem $\frac{C^{z\sqrt{-1}} + C^{-z\sqrt{-1}}}{m}$
 $= 0$, aut $\frac{2 \text{ Cofin. Circ. } z}{m} = 0$ facillime deducuntur ex repertis in num°. 195.,

eruntque $z = +Q$, $z = -Q$, $z = +3Q$, $z = -3Q$, $z = +5Q$,
 $z = -5Q$, $z = +7Q$, $z = -7Q$, $z = +9Q$, $z = -9Q$ &c.;
 Imaginariis omnibus deletis, nempe $z = \pm \lambda Q = \pm \frac{\lambda P}{2}$ adeo, ut, ficu-

ti hae Radices innumerae Aequationi conveniunt Infinitinomiali $1 - \frac{z^2}{1.2}$

$$+ \frac{z^4}{1.2.3.4} - \frac{z^6}{1.2.3.4.5.6} + \frac{z^8}{1.2.3.4.5.6.7.8} - \frac{z^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: = 0,$$

Infinitinomium ipsum propositum in $\left(1 + \frac{z}{1.Q}\right) \left(1 - \frac{z}{1.Q}\right) \left(1 + \frac{z}{3.Q}\right) \left(1 - \frac{z}{3.Q}\right) \left(1 + \frac{z}{5.Q}\right) \left(1 - \frac{z}{5.Q}\right) \left(1 + \frac{z}{7.Q}\right) \left(1 - \frac{z}{7.Q}\right) \left(1 + \frac{z}{9.Q}\right) \left(1 - \frac{z}{9.Q}\right) \left(1 + \frac{z}{11.Q}\right) \left(1 - \frac{z}{11.Q}\right) \left(1 + \frac{z}{13.Q}\right) \left(1 - \frac{z}{13.Q}\right) \&c:$
 five $\left(1 - \frac{z^2}{1^2.Q^2}\right) \left(1 - \frac{z^2}{3^2.Q^2}\right) \left(1 - \frac{z^2}{5^2.Q^2}\right) \left(1 - \frac{z^2}{7^2.Q^2}\right) \left(1 - \frac{z^2}{9^2.Q^2}\right) \left(1 - \frac{z^2}{11^2.Q^2}\right) \left(1 - \frac{z^2}{13^2.Q^2}\right) \&c:$ dissolvatur. Evanescit itaque

Cosinus Circularis, veluti memoratae demonstrant Radices, tum cum Quadranti positivo, vel negativo per Numerum quemlibet imparem multiplicato Arcus respondeat; idque corroborant universa Trigonometricae Facul-

tatis fundamenta. Agedum loco $\frac{C^{z\sqrt{-1}} + C^{-z\sqrt{-1}}}{m} = 0$ scribatur universa-

$$\text{lior } \frac{B^{z\sqrt{-1}} + B^{-z\sqrt{-1}}}{m} = 0, \text{ aut } B^{z\sqrt{-1}} + B^{-z\sqrt{-1}} = C^{(LB)z\sqrt{-1}} +$$

$- (LB)z\sqrt{-1} = 0$. Substitutio iuxta morem simplicissima extemplo decernit

$$\text{fore } \left(1 + \frac{(LB)z}{1.Q}\right) \left(1 - \frac{(LB)z}{1.Q}\right) \left(1 + \frac{(LB)z}{3.Q}\right) \left(1 - \frac{(LB)z}{3.Q}\right) \left(1 + \frac{(LB)z}{5.Q}\right) \left(1 - \frac{(LB)z}{5.Q}\right) \left(1 + \frac{(LB)z}{7.Q}\right) \left(1 - \frac{(LB)z}{7.Q}\right) \&c:, \text{ five } \left(1 - \frac{(LB)^2 z^2}{1^2.Q^2}\right) \left(1 - \frac{(LB)^2 z^2}{3^2.Q^2}\right) \left(1 - \frac{(LB)^2 z^2}{5^2.Q^2}\right) \left(1 - \frac{(LB)^2 z^2}{7^2.Q^2}\right) \left(1 - \frac{(LB)^2 z^2}{9^2.Q^2}\right) \&c: = 1 - \frac{(LB)^2 z^2}{1.2} + \frac{(LB)^4 z^4}{1.2.3.4} - \frac{(LB)^6 z^6}{1.2.3.4.5.6} + \frac{(LB)^8 z^8}{1.2.3.4.5.6.7.8} - \&c:, \text{ cuius}$$

Infinitinomii suppositi = 0 Radices Formula generalis $z = \frac{\pm \lambda Q}{LB} = \frac{\pm \lambda P}{L(B^2)}$

$$= \frac{\pm \lambda \Pi}{L(B^4)} \text{ determinabit.}$$

200. Praeterea ex num.^o eodem 195. Aequatio altera Infinitinomialis

$$\frac{z^1}{1} - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \frac{z^7}{1.2.3.4.5.6.7} + \frac{z^9}{1.2.3.4.5.6.7.8.9} - \&c: = 0,$$

$$\text{vel, separando Factorem, } z \left(1 - \frac{z^2}{1.2.3} + \frac{z^4}{1.2.3.4.5} - \frac{z^6}{1.2.3.4.5.6.7} + \right.$$

$$\left. \frac{z^8}{1.2.3.4.5.6.7.8.9} - \frac{z^{10}}{1.2.3.4.5.6.7.8.9.10.11} + \&c: \right) = \text{Sin. Circ. } z =$$

$$\frac{\left(1 + \frac{z\sqrt{-1}}{\infty} \right)^\infty - \left(1 - \frac{z\sqrt{-1}}{\infty} \right)^\infty}{2\sqrt{-1}} = 0 \text{ Radices absque ullo Limite habe-}$$

bit $z = \pm nP$, nimirum $0, +P, -P, +2P, -2P, +3P, -3P, +4P, -4P, +5P, -5P \&c:$, quas indirecto etiam tramite nobis suadet Trigonometria, dum in Circulo monstrat Arcuum sine numero hic recensitorum Sinus-rectos nulescere. Memoremus autem oportet resolutam Aequationem forma tantum, non substantia, discrepare ab ea,

$$\text{quam subnectere iuvat } C^{+z\sqrt{-1}} - C^{-z\sqrt{-1}}, \text{ aut } \frac{C^{+z\sqrt{-1}} - C^{-z\sqrt{-1}}}{2\sqrt{-1}}, \text{ vel}$$

$$\frac{C^{+z\sqrt{-1}} - C^{-z\sqrt{-1}}}{m} = 0; \text{ ita, ut aequae de hisce pronuntianda omnia iam}$$

superius exposita censeretur debeant. Factores qui optet praefati Infinitinomiali, is facile desiderium adimplebit Radices ipsas sub Binomiali specie repetens,

$$\text{cuius fit Unitas primus terminus. Idcirco obtinebitur } z \left(1 - \frac{z^2}{1.2.3} + \right.$$

$$\left. \frac{z^4}{1.2.3.4.5} - \frac{z^6}{1.2.3.4.5.6.7} + \frac{z^8}{1.2.3.4.5.6.7.8.9} - \frac{z^{10}}{1.2.3.4.5.6.7.8.9.10.11} + \&c: \right) =$$

$$z \left(1 + \frac{z}{1.P} \right) \left(1 - \frac{z}{1.P} \right) \left(1 + \frac{z}{2.P} \right) \left(1 - \frac{z}{2.P} \right) \left(1 + \frac{z}{3.P} \right) \left(1 - \frac{z}{3.P} \right)$$

$$\left(1 + \frac{z}{4.P} \right) \left(1 - \frac{z}{4.P} \right) \left(1 + \frac{z}{5.P} \right) \left(1 - \frac{z}{5.P} \right) \left(1 + \frac{z}{6.P} \right) \left(1 - \frac{z}{6.P} \right)$$

$$\left(1 - \frac{z}{6.P} \right) \left(1 + \frac{z}{7.P} \right) \left(1 - \frac{z}{7.P} \right) \&c: = z \left(1 - \frac{z^2}{1^2.P^2} \right) \left(1 - \frac{z^2}{2^2.P^2} \right)$$

$$\left(1 - \right.$$

$$\left(1 - \frac{z^2}{3^2.P^2}\right) \left(1 - \frac{z^2}{4^2.P^2}\right) \left(1 - \frac{z^2}{5^2.P^2}\right) \left(1 - \frac{z^2}{6^2.P^2}\right) \left(1 - \frac{z^2}{7^2.P^2}\right) \&c;$$

pronaque est deductio fore $\frac{B}{m} \frac{z\sqrt{-1} - B}{-z\sqrt{-1} - B}$, sive $\left(1 + \frac{(LB)z\sqrt{-1}}{\infty}\right)^\infty$

$$\left(1 - \frac{(LB)z\sqrt{-1}}{\infty}\right)^\infty, \text{ aut Infinitomii } (LB)^1 z^1 - \frac{(LB)^3 z^3}{1.2.3} + \frac{(LB)^5 z^5}{1.2.3.4.5} -$$

$$\frac{(LB)^7 z^7}{1.2.3.4.5.6.7} + \frac{(LB)^9 z^9}{1.2.3.4.5.6.7.8.9} - \frac{(LB)^{11} z^{11}}{1.2.3.4.5.6.7.8.9.10.11} + \&c. = 0 \text{ Radices ab}$$

universali Formula significatas $z = \frac{\pm nP}{LB}$, evolutosque Factores $(LB)z \times$

$$\left(1 + \frac{(LB)z}{1.P}\right) \left(1 - \frac{(LB)z}{1.P}\right) \left(1 + \frac{(LB)z}{2.P}\right) \left(1 - \frac{(LB)z}{2.P}\right) \left(1 + \frac{(LB)z}{3.P}\right) \left(1 - \frac{(LB)z}{3.P}\right)$$

$$\left(1 + \frac{(LB)z}{4.P}\right) \left(1 - \frac{(LB)z}{4.P}\right) \&c., \text{ scilicet a Producto } (LB)z \left(1 - \frac{(LB)^2 z^2}{1^2.P^2}\right)$$

$$\left(1 - \frac{(LB)^2 z^2}{2^2.P^2}\right) \left(1 - \frac{(LB)^2 z^2}{3^2.P^2}\right) \left(1 - \frac{(LB)^2 z^2}{4^2.P^2}\right) \left(1 - \frac{(LB)^2 z^2}{5^2.P^2}\right) \&c. \text{ in Infinitum}$$

continuando repraesentari.

201. Urgente reliquorum Theorematum copia oblivisci attamen nefas esset mirabilis cuiusdam in Aequationibus ab Imaginarietate turbatis affectionis, cui nisi ea, quae antecedente num.^o 180. protulimus, obviam irent, Algebra ipsa in fallaciae notam impingeret. Aequatio, de qua agere in praesentiarum suscipio, est $C \frac{z\sqrt{-1}}{1} - 1 = 0$, videlicet Cofin. Circ. $z + (\text{Sin. Circ. } z) \sqrt{-1} - 1 = 0$

$$= 1 + \frac{z^1 \sqrt{-1}}{1} - \frac{z^2}{1.2} - \frac{z^3 \sqrt{-1}}{1.2.3} + \frac{z^4}{1.2.3.4} - \frac{z^5 \sqrt{-1}}{1.2.3.4.5} + \frac{z^6}{1.2.3.4.5.6} - \frac{z^7 \sqrt{-1}}{1.2.3.4.5.6.7} + \frac{z^8}{1.2.3.4.5.6.7.8} - \frac{z^9 \sqrt{-1}}{1.2.3.4.5.6.7.8.9} + \frac{z^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c. = 0$$

eamque Imaginariam esse, sive impossibilem, adfirmantes omnes in unum convenient. Absque eo, quod specialis occurrat mentio, patebit idem quoque valere de ceteris Aequationibus $C \frac{z\sqrt{-1}}{B} - m = 0$, atque

$$C \frac{z\sqrt{-1}}{B} - m = 0, \text{ atque}$$

$B \frac{z\sqrt{-1}}{1} - 1 = 0$, $B \frac{z\sqrt{-1}}{m} - m = 0$, utpote quae vertantur in $C \frac{(LB)z\sqrt{-1}}{1}$

-1 aut $-m = 0$, analogumque idcirco resolutionis canonem admittant ab iis, quae in num°. 182. observari poterunt, derivandum. Mihi equidem non est paradoxon, quomodocumque prima fronte videatur, Infinitomii partim Realibus, partim Imaginariis terminis coalescentis, seu formae $\Delta + \Phi\sqrt{-1} - 1 = 0$, Radices universas, ac numero Infinitas haberi Reales, quum eae profecto satisfacere debeant duabus Aequationibus terminorum omnimoda Realitate praeditis $\Delta - 1 = 0$, et $\Phi\sqrt{-1} = 0$, scilicet $\Phi = 0$, aut potius nostra in hypothese Cofin. Circ. $z = 1$, ac Sin. Circ. $z = 0$, quas Aequationes disciplinae Trigonometricae summa simul solvunt facilitate, nempe ope unius Formulae generalissimae $z = \pm n\Pi$ nullam

Imaginarietatem involventis. Dum itaque detur Aequalitas $z \left(\sqrt{-1} - \frac{z}{1.2} \right.$

$$\left. - \frac{z^2\sqrt{-1}}{1.2.3} + \frac{z^3}{1.2.3.4} - \frac{z^4\sqrt{-1}}{1.2.3.4.5} + \frac{z^5}{1.2.3.4.5.6} - \frac{z^6\sqrt{-1}}{1.2.3.4.5.6.7} + \frac{z^7}{1.2.3.4.5.6.7.8} - \frac{z^8\sqrt{-1}}{1.2.3.4.5.6.7.8.9} + \frac{z^9}{1.2.3.4.5.6.7.8.9.10} - \&c. \right) = 0$$
 haec ha-

bebit Radices omnes Reales $z = 0$, $z = \pm\Pi$, $z = \pm 2\Pi$, $z = \pm 3\Pi$, $z = \pm 4\Pi$, $z = \pm 5\Pi$, $z = \pm 6\Pi$ &c.: En exemplum Imaginariae Aequationis solutionem admittentis infinitis modis realem, veluti ex adverso non rarae occurrunt Aequationes Reales, quae possibilem quamque respuant resolutionem. Paradoxon equidem maius consistit in Pseudo-Factorum a Ra-

dicibus illis genitorum Producto $z \left(1 + \frac{z}{\Pi} \right) \left(1 - \frac{z}{\Pi} \right) \left(1 + \frac{z}{2\Pi} \right)$

$$\left(1 - \frac{z}{2\Pi} \right) \left(1 + \frac{z}{3\Pi} \right) \left(1 - \frac{z}{3\Pi} \right) \left(1 + \frac{z}{4\Pi} \right) \left(1 - \frac{z}{4\Pi} \right) \left(1 + \frac{z}{5\Pi} \right)$$

$$\left(1 - \frac{z}{5\Pi} \right) \left(1 + \frac{z}{6\Pi} \right) \left(1 - \frac{z}{6\Pi} \right) \left(1 + \frac{z}{7\Pi} \right) \left(1 - \frac{z}{7\Pi} \right) \&c.; \text{ aut, confecta}$$

$$\text{binorum coacervatione, } z \left(1 - \frac{z^2}{1^2.\Pi^2} \right) \left(1 - \frac{z^2}{2^2.\Pi^2} \right) \left(1 - \frac{z^2}{3^2.\Pi^2} \right)$$

$$\left(1 - \frac{z^2}{4^2.\Pi^2} \right) \left(1 - \frac{z^2}{5^2.\Pi^2} \right) \left(1 - \frac{z^2}{6^2.\Pi^2} \right) \left(1 - \frac{z^2}{7^2.\Pi^2} \right) \&c. \text{ nequaquam}$$

$$= z$$

$$\begin{aligned}
 &= z \left(1 - \frac{z^1}{1.2\sqrt{-1}} - \frac{z^2}{1.2.3} + \frac{z^3}{1.2.3.4\sqrt{-1}} + \frac{z^4}{1.2.3.4.5} - \frac{z^5}{1.2.3.4.5.6\sqrt{-1}} \right. \\
 &\quad \left. - \frac{z^6}{1.2.3.4.5.6.7} + \frac{z^7}{1.2.3.4.5.6.7.8\sqrt{-1}} + \frac{z^8}{1.2.3.4.5.6.7.8.9} - \&c: \right) = z \left(1 + \frac{z^1\sqrt{-1}}{1.2} \right. \\
 &\quad \left. - \frac{z^2}{1.2.3} - \frac{z^2\sqrt{-1}}{1.2.3.4} + \frac{z^4}{1.2.3.4.5} + \frac{z^5\sqrt{-1}}{1.2.3.4.5.6} - \frac{z^6}{1.2.3.4.5.6.7} - \frac{z^7\sqrt{-1}}{1.2.3.4.5.6.7.8} \right. \\
 &\quad \left. + \frac{z^8}{1.2.3.4.5.6.7.8.9} + \frac{z^9\sqrt{-1}}{1.2.3.4.5.6.7.8.9.10} - \&c: \right) \text{ propterea, quod fieri ne-}
 \end{aligned}$$

queat Reales Factores universos invicem multiplicatos Imaginarium, ut nostro casu eveniret, Polynomium componere. Veruntamen iam eruditi fuimus ab num^o. 180. Radices Infinitinomialis illius Aequationis summo rigore

Analytico expositas tribui a Formula $1 - \frac{z}{\frac{(n\Pi)^2\sqrt{-1}}{2.\infty} \mp \sqrt{(n\Pi)^2 - \frac{(n\Pi)^4}{4.\infty^2}}}$

$= 0$, videlicet $1 - \frac{z}{\frac{(n\Pi)^2\sqrt{-1}}{2.\infty} \mp n\Pi} = 0$, quae praebet unamquamque

Radicem $z = \pm n\Pi + \frac{(n\Pi)^2\sqrt{-1}}{2.\infty}$, revera Realem, ob terminum Imagi-

narium $+ \frac{(n\Pi)^2\sqrt{-1}}{2.\infty} = 0$. $(n\Pi)^2\sqrt{-1}$, qui nempe singulus nullefcit, sed,

propter infinities repetitum in Factoribus terminum ipsam Infinitesimi valoris, nihilominus efficit Productum Imaginarium. Erit igitur ob argumenta eadem, quae prostant apud praecitatum numerum 180. fusius ibidem tractata,

$$\begin{aligned}
 &z \left(1 + \frac{z^1\sqrt{-1}}{1.2} - \frac{z^2}{1.2.3} - \frac{z^3\sqrt{-1}}{1.2.3.4} + \frac{z^4}{1.2.3.4.5} + \frac{z^5\sqrt{-1}}{1.2.3.4.5.6} - \frac{z^6}{1.2.3.4.5.6.7} \right. \\
 &\quad \left. - \frac{z^7\sqrt{-1}}{1.2.3.4.5.6.7.8} + \frac{z^8}{1.2.3.4.5.6.7.8.9} + \frac{z^9\sqrt{-1}}{1.2.3.4.5.6.7.8.9.10} - \&c: \right) =
 \end{aligned}$$

$$z \left(1 - \frac{z}{\frac{1^2.\Pi^2\sqrt{-1}}{2.\infty} \mp \Pi} \right) \left(1 - \frac{z}{\frac{1^2.\Pi^2\sqrt{-1}}{2.\infty} \mp \Pi} \right) \times$$

X X

(1 -

$$\left(1 - \frac{z}{2^2 \cdot \Pi^2 \sqrt{-1}} - 2\Pi\right) \left(1 - \frac{z}{2^2 \cdot \Pi^2 \sqrt{-1}} - 2\Pi\right) \times$$

$$\left(1 - \frac{z}{3^2 \cdot \Pi^2 \sqrt{-1}} - 3\Pi\right) \left(1 - \frac{z}{3^2 \cdot \Pi^2 \sqrt{-1}} - 3\Pi\right) \&c.; \text{ vel, Factores adhi-}$$

$$\text{bendo Trinomiales,} = z \left(1 + \frac{z\sqrt{-1}}{\infty} - \frac{z^2}{1^2 \cdot \Pi^2}\right) \left(1 + \frac{z\sqrt{-1}}{\infty} - \frac{z^2}{2^2 \cdot \Pi^2}\right)$$

$$\left(1 + \frac{z\sqrt{-1}}{\infty} - \frac{z^2}{3^2 \cdot \Pi^2}\right) \left(1 + \frac{z\sqrt{-1}}{\infty} - \frac{z^2}{4^2 \cdot \Pi^2}\right) \left(1 + \frac{z\sqrt{-1}}{\infty} - \frac{z^2}{5^2 \cdot \Pi^2}\right)$$

$$\left(1 + \frac{z\sqrt{-1}}{\infty} - \frac{z^2}{6^2 \cdot \Pi^2}\right) \left(1 + \frac{z\sqrt{-1}}{\infty} - \frac{z^2}{7^2 \cdot \Pi^2}\right) \&c.: \text{ Isti Factores, subti-}$$

li equidem, ac iam superius parata theoricè inventi, Radices easdem, quas supra adduximus, decernunt, et simul alteram conditionem integram tuentur progeniti Infinitinomii alternis terminis Realibus, atque Imaginariis gaudentis, quae postrema conditio priorum Factorum obstabat Analyticae veritati. Videant obiter quanto a veris Algebrae regulis distabunt in-

tervallo ii, qui auferint excogitare $0\sqrt{-1}$, vel $\frac{\sqrt{-1}}{\infty}$ singulariter consideratum non Nullitatem, sed Imaginarium quiddam in se continere, ridiculam fabulantes Proportionem $0:0\sqrt{-1}::1:\sqrt{-1}$, perinde ac si significationis capax, et rei idonea subsisteret.

202. Adiumento Imaginariorum Exponentialium Algorithmus nuperrimus cum universae Trigonometriae commoditate, et nitore perquammaximo mirum in modum contrahitur. Praebeam in Cofinibus, Sinubusque rectis tam Circularibus, quam Hyperbolicis sequens parvulum specimen, et signanter exemplorum inversum, quae in notissimo adparent Bougainvillii *Tractus* Prooemio. Imprimis Sin. Circ. $(y+z) =$

$$\frac{C^{(y+z)\sqrt{-1}} - C^{-(y+z)\sqrt{-1}}}{2\sqrt{-1}} = \left(\frac{C^{y\sqrt{-1}} - C^{-y\sqrt{-1}}}{2\sqrt{-1}} \right) \times$$

$$\left(\frac{C^{z\sqrt{-1}} + C^{-z\sqrt{-1}}}{2} \right) + \left(\frac{C^{y\sqrt{-1}} + C^{-y\sqrt{-1}}}{2} \right) \left(\frac{C^{z\sqrt{-1}} - C^{-z\sqrt{-1}}}{2\sqrt{-1}} \right),$$

id, quod

id, quod multiplicatio patefacit, nimirum $\text{Sin. Circ. } y \times \text{Cofin. Circ. } z \rightarrow \text{Cofin. Circ. } y \times \text{Sin. Circ. } z$. Ratione consimili in Factores dispescitur

$$\begin{aligned} \text{Cofin. Circ. } (y \rightarrow z) &= \frac{C^{(y+z)\sqrt{-1}} + C^{-(y+z)\sqrt{-1}}}{2} = \\ &= \left(\frac{C^{y\sqrt{-1}} + C^{-y\sqrt{-1}}}{2} \right) \left(\frac{C^{z\sqrt{-1}} + C^{-z\sqrt{-1}}}{2} \right) - \left(\frac{C^{y\sqrt{-1}} - C^{-y\sqrt{-1}}}{2\sqrt{-1}} \right) \times \\ &\quad \left(\frac{C^{z\sqrt{-1}} - C^{-z\sqrt{-1}}}{2\sqrt{-1}} \right) = \text{Cofin. Circ. } y \times \text{Cofin. Circ. } z - \text{Sin. Circ. } y \times \end{aligned}$$

$\text{Sin. Circ. } z$. Hisce positis absque ulteriore investigatione profluunt aliae Aequationes $\text{Cofin. Circ. } (y - z) = \text{Cofin. Circ. } y \times \text{Cofin. Circ. } z + \text{Sin. Circ. } y \times \text{Sin. Circ. } z$, nec non $\text{Sin. Circ. } (y - z) = \text{Sin. Circ. } y \times \text{Cofin. Circ. } z - \text{Cofin. Circ. } y \times \text{Sin. Circ. } z$: namque statim haec ex antedictis obtinentur simplicissima observatione praeunte, quae Cofinum τz Arcus $-z$ positivum, at ipsius Arcus Rectum-Sinum negativum determinat. Non absimili methodo, dum auferantur Indices Imaginari, proveniet Cofin. Hyperb.

$$\begin{aligned} (v \rightarrow x) &= \frac{C^{(v+x)} + C^{-(v+x)}}{2} = \left(\frac{C^v + C^{-v}}{2} \right) \left(\frac{C^x + C^{-x}}{2} \right) + \\ &\quad \left(\frac{C^v - C^{-v}}{2} \right) \left(\frac{C^x - C^{-x}}{2} \right) = \text{Cofin. Hyperb. } v \times \text{Cofin. Hyperb. } x + \text{Sin.} \end{aligned}$$

$\text{Hyperb. } v \times \text{Sin. Hyperb. } x$, et idcirco $\text{Cofin. Hyperb. } (v - x) = \text{Cofin. Hyperb. } v \times \text{Cofin. Hyperb. } x - \text{Sin. Hyperb. } v \times \text{Sin. Hyperb. } x$ ob Cofinum positivum, Rectumque Sinum negativum negativo Pseudo-Arcui Hyperbolico respondentes. Demum emerget $\text{Sin. Hyperb. } (v \rightarrow x) =$

$$\begin{aligned} \frac{C^{(v+x)} - C^{-(v+x)}}{2} &= \left(\frac{C^v - C^{-v}}{2} \right) \left(\frac{C^x + C^{-x}}{2} \right) + \left(\frac{C^v + C^{-v}}{2} \right) \\ &\quad \left(\frac{C^x - C^{-x}}{2} \right) = \text{Sin. Hyperb. } v \times \text{Cofin. Hyperb. } x + \text{Cofin. Hyperb. } v \times \end{aligned}$$

$\text{Sin. Hyperb. } x$, ac $\text{Sin. Hyperb. } (v - x) = \text{Sin. Hyperb. } v \times \text{Cofin. Hyperb. } x - \text{Cofin.}$

— Cofin. Hyperb. $v \times \text{Sin. Hyperb. } x$, a quorum complexu canonum gemina pendet Hyperbolae, et Circuli canonica Trigonometria.

203. Ne vero mutila sit utriusque, de qua agimus, harmonicae Trigonometriae disciplina eo redeamus, unde post numerum 167. digressi sumus; et interim hic loci Sinus - Versos, quemadmodum aiunt, Hyperbolicos nonnihil considerare iuvabit. Est etenim adeo elegans analogia Sinus inter Versos Circulares, atque Hyperbolicos, ut, dum Sin. V. Circ. $x =$

$$1 - \text{Cofin. Circ. } x = \frac{2 - C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}}{2} = \frac{x^2}{1.2} - \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} - \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c: =$$

$$2 - \left(1 + \frac{x\sqrt{-1}}{\infty} \right)^{\infty} - \left(1 - \frac{x\sqrt{-1}}{\infty} \right)^{\infty}, \text{ sit vicissim Sin. V. Hyperb. } x$$

$$= \text{Cofin. Hyperb. } x - 1 = \frac{C^x + C^{-x} - 2}{2} = \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: \text{ in Infinitum, } =$$

$$-2 + \left(1 + \frac{x}{\infty} \right)^{\infty} + \left(1 - \frac{x}{\infty} \right)^{\infty}, \text{ veluti Conicae Curvae natura ipsa}$$

demonstrat. Quin immo praedictorum ope Theorematum regredi quoque fas erit in Hyperbolica Trigonometria, prouti nobis feliciter in Circulo contigit, ab Sinibus Rectis, Versisve, et Cofinibus datis ad respectivos Sectores absque Calculi Infinitesimalis subsidio. Ac primum in Sinu - Verso periculum facientibus constabit in postremam Seriem numero terminorum

$$\text{Infinitam negative sumptam, nempe } - \left(\frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c: \right), \text{ aliam verti Circulo adscriptam}$$

$$\frac{x^2}{1.2} - \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} - \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c: \text{ tum}$$

quum

quum loco x Realis Imaginarium $x\sqrt{-1}$ subrogetur, quam metamorphosin substitutio non tantum memorata, verum etiam Formulae paullo superius adlatae consensu sane maximo firmant. Quibus positis, cum num^{us}. 165. nos

$$\text{doceat ab Serie } \frac{x^2}{1.2} - \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} - \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10}$$

$$- \&c: = v \text{ oriri conversam } x = (2v)^{\frac{1}{2}} \left(1 - \frac{1^2.v^1}{2^2.3} + \frac{1^2.3^2.v^2}{2^3.3.4.5} - \right.$$

$$\left. \frac{1^2.3^2.5^2.v^3}{2^4.3.4.5.6.7} + \frac{1^2.3^2.5^2.7^2.v^4}{2^5.3.4.5.6.7.8.9} - \&c: \right), \text{ regredi aequae poterimus ab Serie}$$

$$\frac{(x\sqrt{-1})^2}{1.2} - \frac{(x\sqrt{-1})^4}{1.2.3.4} + \frac{(x\sqrt{-1})^6}{1.2.3.4.5.6} - \frac{(x\sqrt{-1})^8}{1.2.3.4.5.6.7.8} + \frac{(x\sqrt{-1})^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c:$$

$$= -t \text{ ad alteram } x\sqrt{-1} = (-2t)^{\frac{1}{2}} \left(1 - \frac{1^2.t^1}{2^2.3} + \frac{1^2.3^2.t^2}{2^3.3.4.5} - \right.$$

$$\left. \frac{1^2.3^2.5^2.t^3}{2^4.3.4.5.6.7} + \frac{1^2.3^2.5^2.7^2.t^4}{2^5.3.4.5.6.7.8.9} - \&c: \right), \text{ scilicet, Imaginariis destructis, } x =$$

$$(2t)^{\frac{1}{2}} \left(1 - \frac{1^2.t^1}{2^2.3} + \frac{1^2.3^2.t^2}{2^3.3.4.5} - \frac{1^2.3^2.5^2.t^3}{2^4.3.4.5.6.7} + \frac{1^2.3^2.5^2.7^2.t^4}{2^5.3.4.5.6.7.8.9} - \&c: \right). \text{ Ista}$$

expressio haecenus, uti opinor, intentato molimine reperta convenit ad amussim Areae Sectoris a Centro supputati Hyperbolae Apolloniana Aequilaterae ac per medietatem Transversi Semi-Lateris, seu per $\frac{1}{2}$ divisi, et idcirco analogiam cum Circuli Arcubus instituit statim atque t eius Conicae Curvae Sinum-Versum, sive Arcus Sagittam significet. Habebitur ergo

$$\text{Sin. V.Hyp.} = \frac{(2 \text{ Sect. Hype.})^2}{1.2} + \frac{(2 \text{ Sect. Hype.})^4}{1.2.3.4} + \frac{(2 \text{ Sect. Hype.})^6}{1.2.3.4.5.6} + \frac{(2 \text{ Sect. Hype.})^8}{1.2.3.4.5.6.7.8} + \frac{(2 \text{ Sect. Hype.})^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c:$$

atque vicissim

$$2 \text{ Sect. Hyp.} = (2 \text{ Sin. V.Hyp.})^{\frac{1}{2}} \left(1 - \frac{1^2(\text{Sin. V.H.})^1}{2^2.3} + \frac{1^2.3^2(\text{Sin. V.H.})^2}{2^3.3.4.5} - \frac{1^2.3^2.5^2(\text{Sin. V.H.})^3}{2^4.3.4.5.6.7} + \frac{1^2.3^2.5^2.7^2(\text{Sin. V.H.})^4}{2^5.3.4.5.6.7.8.9} - \&c: \right)$$

ubi pro 2 Sect. Hyp. nequaquam veros Arcus Hyperbolicos, sed veluti Circularium symbola animo subintelligere licet.

204. Sinus deinceps Hyperbolicos Rectos pro viribus contemplaturū

mente volvamus oportet esse vi praecedentium $\frac{C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}}{2\sqrt{-1}} = \text{Sin.}$

$$\text{Cir. } x = \frac{x^1}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^9}{1.2.3.4.5.6.7.8.9} -$$

$$\frac{x^{11}}{1.2.3.4.5.6.7.8.9.10.11} + \&c.; \text{ quae Formula abit in Sin. Hyp. } x = \frac{C^x - C^{-x}}{2}$$

$$= \frac{x^1}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^9}{1.2.3.4.5.6.7.8.9} -$$

$$\frac{x^{11}}{1.2.3.4.5.6.7.8.9.10.11} + \&c.; \text{ dummodo vice } x \text{ ponatur } x\sqrt{-1}, \text{ postque fa-}$$

ctam substitutionem Series prior per $\sqrt{-1}$ dividatur. Quemadmodum igitur

$$\text{Aequatio directa } v = x^1 - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} +$$

$$\frac{x^9}{1.2.3.4.5.6.7.8.9} - \frac{x^{11}}{1.2.3.4.5.6.7.8.9.10.11} + \&c.; \text{ genuit in num}^\circ. 161. \text{ con-}$$

versam

$$\text{Sin. H.} = \frac{(2 \text{ Sect. H.})^1}{1} - \frac{(2 \text{ Sect. H.})^3}{1.2.3} + \frac{(2 \text{ Sect. H.})^5}{1.2.3.4.5} - \frac{(2 \text{ Sect. H.})^7}{1.2.3.4.5.6.7} +$$

$$2 \text{ Sect. H.} = (\text{Sin. H.})^1 - \frac{1^2 (\text{Sin. H.})^3}{1.2.3} + \frac{1^2.3^2 (\text{Sin. H.})^5}{1.2.3.4.5} - \frac{1^2.3^2.5^2 (\text{Sin. H.})^7}{1.2.3.4.5.6.7} +$$

et inibi nec Potestatum Gradus, nec Coefficientium magnitudines ab iis, quae ad Circulum pertinent, veruntamen Signa solummodo alternatim disposita differunt: tanto iunguntur Circulus, atque Hyperbole inter se sanctissimo foedere.

205. Quae Cosinus demum respiciunt pariter Hyperbolicos omnimoda carent difficultate. Eodem enim ac superius ritu Cosin. C. $x =$

$$\frac{C^{x\sqrt{-1}} + C^{-x\sqrt{-1}}}{2}$$

$$\begin{aligned}
 \text{versam } x &= v^1 + \frac{1^2 \cdot v^3}{2 \cdot 3} + \frac{1^2 \cdot 3^2 \cdot v^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot v^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot v^9}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \\
 &\frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot v^{11}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \&c.; \text{ sic eadem ferente ratione Series } (x\sqrt{-1})^1 - \\
 &\frac{(x\sqrt{-1})^3}{1 \cdot 2 \cdot 3} + \frac{(x\sqrt{-1})^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{(x\sqrt{-1})^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{(x\sqrt{-1})^9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} - \frac{(x\sqrt{-1})^{11}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} \\
 &+ \&c. = z\sqrt{-1} \text{ praebebit inverfam } x\sqrt{-1} = z^1 \sqrt{-1} - \frac{1^2 \cdot z^3 \sqrt{-1}}{2 \cdot 3} \\
 &+ \frac{1^2 \cdot 3^2 \cdot z^5 \sqrt{-1}}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot z^7 \sqrt{-1}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot z^9 \sqrt{-1}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot z^{11} \sqrt{-1}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} \\
 &\&c.; \text{ five, eliminatis Imaginariis, } x = z^1 - \frac{1^2 \cdot z^3}{2 \cdot 3} + \frac{1^2 \cdot 3^2 \cdot z^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot z^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \\
 &+ \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot z^9}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot z^{11}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \&c.; \text{ idemque comprobaretur}
 \end{aligned}$$

adiumento expressionis $x = L(z + \sqrt{z^2 + 1})$ methodum ipsam sequendo in praecitato num^o. 161. fusiore traditam oratione. Exinde nacti sumus geminas Formulas sine Serierum Regressus obliquis certe, quomodocunque vulgatis, canonibus, scilicet

Sin. H.

$$\begin{aligned}
 &+ \frac{(2 \text{ Sect. H.})^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \frac{(2 \text{ Sect. H.})^{11}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \&c.; \\
 &+ \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot (\text{Sin. H.})^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot (\text{Sin. H.})^{11}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \&c.;
 \end{aligned}$$

et inibi

$$\begin{aligned}
 \frac{C^{x\sqrt{-1}} + C^{-x\sqrt{-1}}}{2} &= \frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^\infty + \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^\infty}{2} = 1 - \frac{x^2}{1 \cdot 2} \\
 &+ \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{x^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} - \frac{x^{10}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} + \&c.; \text{ dum} \\
 \text{loco } x \text{ ponatur } -x\sqrt{-1}, \text{ five } \frac{+x}{\sqrt{-1}}, \text{ abit in Cofin. H. } x &= \\
 &\frac{C^x + C^{-x}}{2}
 \end{aligned}$$

$$\frac{C^x + C^{-x}}{2} = \frac{\left(1 + \frac{x}{\infty}\right)^\infty + \left(1 - \frac{x}{\infty}\right)^\infty}{2} = 1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} \\ + \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.: \text{Igitur, sicuti}$$

Series progressu, aut numero terminorum Infinita $v = 1 - \frac{x^2}{1.2} +$

$$\frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} - \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.: \text{alte-}$$

ram praebuit, quemadmodum videre est in num^o. 167., $x = Q -$

$$\left(v^1 + \frac{1^2 \cdot v^3}{2.3} + \frac{1^2 \cdot 3^2 \cdot v^5}{2.3.4.5} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot v^7}{2.3.4.5.6.7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot v^9}{2.3.4.5.6.7.8.9} + \right.$$

$$\left. \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot v^{11}}{2.3.4.5.6.7.8.9.10.11} + \&c.: \right), \text{ five potius } x = (2(1 - v))^{\frac{1}{2}} \times$$

$$\left(1 + \frac{1^2(1-v)^1}{2^2.3} + \frac{1^2.3^2(1-v)^2}{2^3.3.4.5} + \frac{1^2.3^2.5^2(1-v)^3}{2^4.3.4.5.6.7} + \frac{1^2.3^2.5^2.7^2(1-v)^4}{2^5.3.4.5.6.7.8.9} \right. \\ \left. + \frac{1^2.3^2.5^2.7^2.9^2(1-v)^5}{2^6.3.4.5.6.7.8.9.10.11} + \&c.: \right), \text{ consimili quoque genealogia a Serie } z =$$

$$1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} +$$

$$\frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.: \text{ proveniet } -x\sqrt{-1} = \frac{x}{\sqrt{-1}} = (2(1 - z))^{\frac{1}{2}}$$

$$\times \left(1 + \frac{1^2(1-z)^1}{2^2.3} + \frac{1^2.3^2(1-z)^2}{2^3.3.4.5} + \frac{1^2.3^2.5^2(1-z)^3}{2^4.3.4.5.6.7} + \frac{1^2.3^2.5^2.7^2(1-z)^4}{2^5.3.4.5.6.7.8.9} + \right.$$

$$\left. \frac{1^2.3^2.5^2.7^2.9^2(1-z)^5}{2^6.3.4.5.6.7.8.9.10.11} + \&c.: \text{ videlicet, deletis Imaginariis, } x = (2(z - 1))^{\frac{1}{2}}$$

$$\times \left(1 - \frac{1^2(z-1)^1}{2^2.3} + \frac{1^2.3^2(z-1)^2}{2^3.3.4.5} - \frac{1^2.3^2.5^2(z-1)^3}{2^4.3.4.5.6.7} + \frac{1^2.3^2.5^2.7^2(z-1)^4}{2^5.3.4.5.6.7.8.9} - \right.$$

$$\left. \frac{1^2.3^2.5^2.7^2.9^2(z-1)^5}{2^6.3.4.5.6.7.8.9.10.11} + \&c.: \text{ atque ita deinceps absque ullo Limite. Prospe-}$$

ctus hinc duarum enascitur Progressionum

Cosin.

$$\text{Cofin. H.} = 1 + \frac{(2 \text{ Sect. H.})^2}{1.2} + \frac{(2 \text{ Sect. H.})^4}{1.2.3.4} + \frac{(2 \text{ Sect. H.})^6}{1.2.3.4.5.6} + \frac{(2 \text{ Sect. H.})^8}{1.2.3.4.5.6.7.8} + \frac{(2 \text{ Sect. H.})^{10}}{1.2.3.4.5.6.7.8.9.10} + \frac{(2 \text{ Sect. H.})^{12}}{1.2.3.4.5.6.7.8.9.10.11.12} + \&c.$$

$$2 \text{ Sect. H.} = (2 (\text{Cofin. H.} - 1))^{\frac{1}{2}} \left(1 - \frac{1^2 (\text{Cofin. H.} - 1)^1}{2^2.3} + \frac{1^2.3^2 (\text{Cofin. H.} - 1)^2}{2^3.3.4.5} - \frac{1^2.3^2.5^2 (\text{Cofin. H.} - 1)^3}{2^4.3.4.5.6.7} + \frac{1^2.3^2.5^2.7^2 (\text{Cofin. H.} - 1)^4}{2^5.3.4.5.6.7.8.9} - \&c. \right)$$

206. In perficiendam canonicae Trigonometriae Hyperbolico - Circularis theoricae remanent adhuc Tangentium, Secantium, ad ipsasque relatarum Cotangentium, Cofecantium directi, et conversi evolvendi valores.

$$\text{Neminem in primis fugit esse } \frac{C^x - C^{-x}}{C^x + C^{-x}} = \frac{C^{2x} - 1}{C^{2x} + 1} = \frac{\left(1 + \frac{2x}{\infty}\right)^{\infty} - 1}{\left(1 + \frac{2x}{\infty}\right)^{\infty} + 1}$$

$$= \frac{\text{Sin. H. } x}{\text{Cofin. H. } x} = \text{Tang. } x = \frac{\text{Cofin. H. } (2x) + \text{Sin. H. } (2x) - 1}{\text{Cofin. H. } (2x) - \text{Sin. H. } (2x) + 1}. \text{ Istud etiam Ima-}$$

ginariorum ope ab Hyperbola transfertur ad Circulum, ubi revera, ut omnes

$$\text{iam norunt, } \text{Tang. } x = \frac{\text{Sin. C. } x}{\text{Cofin. C. } x} = \frac{C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}}{\sqrt{-1}(C^{x\sqrt{-1}} + C^{-x\sqrt{-1}})}$$

$$= \frac{C^{2x\sqrt{-1}} - 1}{\sqrt{-1}(C^{2x\sqrt{-1}} + 1)}, \text{ vel, si mavis, } = \frac{\text{Cofin. C. } (2x) + (\text{Sin. C. } (2x))\sqrt{-1} - 1}{\sqrt{-1}(\text{Cofin. C. } (2x) + (\text{Sin. C. } (2x))\sqrt{-1} + 1)}$$

$$= \frac{\left(1 + \frac{2x\sqrt{-1}}{\infty}\right)^{\infty} - 1}{\sqrt{-1}\left(\left(1 + \frac{2x\sqrt{-1}}{\infty}\right)^{\infty} + 1\right)}, \text{ dummodo intelligatur fore } x \text{ Circularem}$$

Arcum, videlicet duplum Circularis Sectoris, alteraque in hypothesi x duplum Sectoris Aequilaterae Hyperbolae Apolloniana, nimirum Arcus-Analogi Hyperbolici symbolon, Si ergo Tangens Arcus, Sectorisve dupli x

specie v designetur, a priore Aequatione deducitur $C^{2x} = \frac{1+v}{1-v}$, et idcir-

co $2x = L\left(\frac{1+v}{1-v}\right)$; eodemque pacto secunda Aequatio progignit $C^{2x\sqrt{-1}}$

$= \frac{1+v\sqrt{-1}}{1-v\sqrt{-1}}$, aut $2x\sqrt{-1} = L\left(\frac{1+v\sqrt{-1}}{1-v\sqrt{-1}}\right)$, quae cum Euleriana congruit, tametsi Calculo prolixiore adquisita.

207. His ad instar Lemmatum praesuppositis fiat maioris simplicitatis

caussa in Aequatione $v = \frac{C^{2x} - 1}{C^{2x} + 1}$ Numerator $C^{2x} - 1 = z$, illaque ver-

tetur in formam $v = \frac{z}{2+z} = z(2+z)^{-1} = \frac{z}{2} - \frac{z^2}{4} + \frac{z^3}{8}$

$-\frac{z^4}{16} + \frac{z^5}{32} - \frac{z^6}{64} + \frac{z^7}{128} - \frac{z^8}{256} + \frac{z^9}{512} - \&c:$ ex Neutroniani Bi-

nomii Decrero. Quum autem sit ex hypothesi $z = \frac{(2x)^1}{1} + \frac{(2x)^2}{1.2} +$

$\frac{(2x)^3}{1.2.3} + \frac{(2x)^4}{1.2.3.4} + \frac{(2x)^5}{1.2.3.4.5} + \frac{(2x)^6}{1.2.3.4.5.6} + \frac{(2x)^7}{1.2.3.4.5.6.7} + \frac{(2x)^8}{1.2.3.4.5.6.7.8} +$

$\frac{(2x)^9}{1.2.3.4.5.6.7.8.9} + \frac{(2x)^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c:$, opportunis effectis substitutio-

nibus, obtinebimus Aequalitatem

$v = x + x^2 + \frac{2x^3}{3} + \frac{x^4}{3} + \frac{2x^5}{15} + \frac{2x^6}{45} + \frac{4x^7}{315} + \frac{x^8}{315} + \frac{2x^9}{2835} + \&c:$

$- x^2 - 2x^3 - \frac{7x^4}{3} - 2x^5 - \frac{62x^6}{45} - \frac{4x^7}{5} - \frac{127x^8}{315} - \frac{34x^9}{189} - \&c:$

$+ x^3 + 3x^4 + 5x^5 + 6x^6 + \frac{86x^7}{15} + \frac{23x^8}{5} + \frac{605x^9}{189} + \&c:$

$- x^4 - 4x^5 - \frac{26x^6}{3} - \frac{40x^7}{3} - \frac{81x^8}{5} - \frac{148x^9}{9} - \&c:$

$+ x^5 + 5x^6 + \frac{40x^7}{3} + 25x^8 + \frac{331x^9}{9} + \&c:$

$- x^6 - 6x^7 - 19x^8 - 42x^9 - \frac{1087x^{10}}{15} - \&c:$

$+ x^7 + 7x^8 + \frac{77x^9}{3} + \&c:$

$- x^8 - 8x^9 - \&c:$

$+ x^9 + \&c:$

nempe

nempe $v = \frac{x^1}{1} - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \frac{62x^9}{2835} - \&c.$, terminis

omnibus, quibus insunt Potestates x pari Indice praeditae, destructis, si-

ve potius $v = \frac{x^1}{1^2} - \frac{x^3}{1^2 \cdot 3} + \frac{2x^5}{1^2 \cdot 3 \cdot 5} - \frac{17x^7}{1^2 \cdot 3^2 \cdot 5 \cdot 7} + \frac{62x^9}{1^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 9} -$

$\frac{1382x^{11}}{1^3 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11} + \frac{21844x^{13}}{1^3 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11 \cdot 13} - \frac{929569x^{15}}{1^3 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 9 \cdot 11 \cdot 13 \cdot 15} +$

$\frac{6404582x^{17}}{1^3 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17} - \&c.$, ubi elegantissima progredientium Denomi-

natorum Lex perbelle satis detegitur. Verum, si Coefficientes desideres universos ope communis, a quo pendeant, foederis possidere, scito in Analytico praecedente Triangulo Coefficientes diagonaliter supputatos Series efformare Algebraicas, nimirum singulas vel Numerorum aequalium, vel aequedifferentium, vel quorum Differentiae secundae, tertiae, quartae &c. in Infinitum sint pares. Et re quidem vera diagonalium infima exhibet

1 . 1 . 1 . 1 . 1 . 1 . 1 . 1 . 1 . 1 . &c.; quae proximo supereminet loco tribuit 1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 &c.; deindeque, uti sequens osten-

dit Diagramma,

$$\begin{array}{cccccccccccccccc}
 \frac{2}{3} & \cdot & \frac{7}{3} & \cdot & \frac{15}{3} & \cdot & \frac{26}{3} & \cdot & \frac{40}{3} & \cdot & \frac{57}{3} & \cdot & \frac{77}{3} & \&c; & \frac{1}{3} & \cdot & 2 & \cdot & 6 & \cdot & \frac{40}{3} & \cdot & 25 & \cdot & 42 & \&c; \\
 \frac{5}{3} & \cdot & \frac{8}{3} & \cdot & \frac{11}{3} & \cdot & \frac{14}{3} & \cdot & \frac{17}{3} & \cdot & \frac{20}{3} & & & & & \frac{5}{3} & \cdot & 4 & \cdot & \frac{22}{3} & \cdot & \frac{35}{3} & \cdot & 17 \\
 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & & & & & & & \frac{7}{3} & \cdot & \frac{10}{3} & \cdot & \frac{13}{3} & \cdot & \frac{16}{3} \\
 & & & & & & & & & & & & & & & 1 & \cdot & 1 & \cdot & 1
 \end{array}$$

Ipsa etenim universalis affectio, quam mihi feliciter contigit invenisse, manu veluti ducit ad ceteros Coefficientes ex antea repertis determinandos absque eo, quod valde periculosum Serierum assumptitiarum impetretur adiumentum, aut in re non tam facili necesse sit hariolari. Summae, de qua loquimur, lubricitatis exemplum aliquod adparet in tertio *Liburnensium Opusculorum* nuperrimae editionis *Schediasmate*, Terminus ille $\frac{1087x^{10}}{15}$ ab aliis

quasi feiunctus in Trigono superius ordinato a Differentiarum aequalium Legge profiluit, istudque repertae Legis experimentum inito postea calculo omnigenam confirmationem accepit.

208. Tangentium nunc ad Circulum pertinentium, scilicet vere Trigonometricarum, expressionem elicere dudum traditae analogam, eam, nimirum, quae vulgari methodo dimanat a minus commoda dupliciter Infinitomialis Fractionis evolutione, uti mos extat penes Analyseos Scriptores,

$$\begin{array}{l}
 x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^9}{1.2.3.4.5.6.7.8.9} - \&c; \\
 \hline
 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} - \&c;
 \end{array}$$

erit. Satis enim demonstrant Formulae superexpositae in num°. 206. ab Hyperbolica Tangente ad Circularem transitum fieri statim ac vice x subrogetur $x\sqrt{-1}$, totaque Series ulterius dividatur per $\sqrt{-1}$; quibus effectis, et nuncupata Tangente t , nacti prorsus erimus Aequationem $t =$

$$\frac{x^1}{1}$$

$$\frac{3}{15} \cdot \frac{62}{45} \cdot \frac{86}{15} \cdot \frac{81}{5} \cdot \frac{331}{9} \cdot \frac{1087}{15} \&c.$$

$$\frac{56}{45} \cdot \frac{196}{45} \cdot \frac{471}{45} \cdot \frac{926}{45} \cdot \frac{1606}{45}$$

$$\frac{140}{45} \cdot \frac{275}{45} \cdot \frac{455}{45} \cdot \frac{689}{45}$$

$$\frac{135}{45} \cdot \frac{180}{45} \cdot \frac{225}{45}$$

Ipfa

$$\frac{x^1}{1} + \frac{x^3}{1^2 \cdot 3} + \frac{2x^5}{1^2 \cdot 3 \cdot 5} + \frac{17x^7}{1^2 \cdot 3^2 \cdot 5 \cdot 7} + \frac{62x^9}{1^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 9} + \frac{1382x^{11}}{1^3 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11} +$$

$$\frac{21844x^{13}}{1^3 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11 \cdot 13} + \frac{929569x^{15}}{1^3 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 9 \cdot 11 \cdot 13 \cdot 15} + \frac{6404582x^{17}}{1^3 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17} + \&c., \text{ Si}$$

gnis tantummodo a priore discrepantem, hic omnibus positivis, illic alternis.

209. Series autem praenotatarum inversae mirum est quanta sine Infiniti subsidio ex nostris ipsis principiis, et ex num°. memorato 206. profluant

facilitate. Habetur namque $2x = L\left(\frac{1+v}{1-v}\right)$, five $x = \frac{1}{2} L\left(\frac{1+v}{1-v}\right) = v$

$$+ \frac{v^3}{3} + \frac{v^5}{5} + \frac{v^7}{7} + \frac{v^9}{9} + \&c., \text{ prouti constat si perlegas nume-}$$

rum 102., adeo, ut Area dupli Sectoris Hyperbolici x per Tangentem v

obtineatur facillimae Aequalitatis acquisitione $x = \frac{v^1}{1} + \frac{v^3}{3} + \frac{v^5}{5}$

$$+ \frac{v^7}{7} + \frac{v^9}{9} + \frac{v^{11}}{11} + \frac{v^{13}}{13} + \frac{v^{15}}{15} + \frac{v^{17}}{17} + \&c.: \text{ In hac nihil sim-}$$

plicius optandum, vel Potestatum, aut numericorum Coefficientium processum animadvertas: quin immo non parum mirandum est tantam conversae Seriesi simplicitatem a directa, quae multo magis composita, suboriri. Cumque idem numerus 206. nos edoceat esse pariter Arcum Circularem, five duplum re-

spondentem Sectorem $x = \frac{1}{2\sqrt{-1}} L\left(\frac{1 + v\sqrt{-1}}{1 - v\sqrt{-1}}\right)$, vel maioris distinctionis

gratia $x = \frac{1}{2\sqrt{-1}} L\left(\frac{1 + t\sqrt{-1}}{1 - t\sqrt{-1}}\right)$, exinde consequitur, quod, dummodo

in praecedente Serie pro v ponatur $t\sqrt{-1}$, ac deinceps universa ipsa Series in antecessum prolata per $\sqrt{-1}$ divisionem solitam subeat, petita Series fit

$$\text{Tang. C.} = \frac{(\text{Arc. C.})^1}{1} + \frac{(\text{Arc. C.})^3}{1^2 \cdot 3} + \frac{2(\text{Arc. C.})^5}{1^2 \cdot 3 \cdot 5} +$$

$$2(\text{Sect. Circ.}) = \text{Arc. C.} = \frac{(\text{Tang. C.})^1}{1} + \frac{(\text{Tang. C.})^3}{3} + \frac{(\text{Tang. C.})^5}{5} +$$

nec non

$$\text{Tang. H.} = \frac{(2\text{Sect. H.})^1}{1} + \frac{(2\text{Sect. H.})^3}{1^2 \cdot 3} + \frac{2(2\text{Sect. H.})^5}{1^2 \cdot 3 \cdot 5} +$$

$$\text{Pseudo-Arc. H.} = 2\text{Sect. H.} = \frac{(\text{Tang. H.})^1}{1} + \frac{(\text{Tang. H.})^3}{3} + \frac{(\text{Tang. H.})^5}{5} +$$

210. Ea, quae summopere placuit insignibus viris Leibnitio, atque Hugenio, Apollonianae Hyperbolae Aequilaterae, et Circuli, quin etiam Scalenaе Hyperbolae, ac Ellipseos Sectorum Centralium harmonia, tota sustinetur a superius data gemina Serie $\frac{v^1}{1} \pm \frac{v^3}{3} + \frac{v^5}{5} \pm \frac{v^7}{7} + \frac{v^9}{9}$

$\pm \frac{v^{11}}{11} + \frac{v^{13}}{13} \pm \frac{v^{15}}{15} + \frac{v^{17}}{17} \pm \&c.$, atque idcirco nostris Formulis sese

praebet libentissime, Infinito nequidem salutato. Seriem alternis praeditam Signis, seu ad Circulum relatam, Lagnys successioni Signorum inserviens

ita exposuit $\left(\frac{3t^1 - t^3}{1 \cdot 3}\right) + \left(\frac{7t^5 - 5t^7}{5 \cdot 7}\right) + \left(\frac{11t^9 - 9t^{11}}{9 \cdot 11}\right) + \left(\frac{15t^{13} - 13t^{15}}{13 \cdot 15}\right)$

+

fit emerfura. Profluet itaque eo cafu $x = \frac{t^1}{1} - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7}$
 $+ \frac{t^9}{9} - \frac{t^{11}}{11} + \frac{t^{13}}{13} - \frac{t^{15}}{15} + \frac{t^{17}}{17} - \&c.$, quae, alternatione Signo-
 rum in fucceffionem abeunte, eadem redit ac fupra ad Hyperbolicam Tan-
 gentem relata. Exponi ergo poterunt Series huiufmodi Hyperbolico-Circu-
 lares hac methodo prae omnibus eloquentiore:

Tang. C.

$$\frac{17(\text{Arc. C.})^2}{1^2 \cdot 3^2 \cdot 5 \cdot 7} + \frac{62(\text{Arc. C.})^3}{1^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 9} + \frac{1382(\text{Arc. C.})^{11}}{1^3 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11} + \frac{21844(\text{Arc. C.})^{13}}{1^3 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11 \cdot 13} + \&c.$$

$$\frac{(\text{Tang. C.})^7}{7} + \frac{(\text{Tang. C.})^9}{9} + \frac{(\text{Tang. C.})^{11}}{11} + \frac{(\text{Tang. C.})^{13}}{13} - \&c.,$$

$$\frac{17(2 \text{ Sect. H.})^2}{1^2 \cdot 3^2 \cdot 5 \cdot 7} + \frac{62(2 \text{ Sect. H.})^3}{1^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 9} + \frac{1382(2 \text{ Sect. H.})^{11}}{1^3 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11} + \&c.$$

$$\frac{(\text{Tang. H.})^7}{7} + \frac{(\text{Tang. H.})^9}{9} + \frac{(\text{Tang. H.})^{11}}{11} + \frac{(\text{Tang. H.})^{13}}{13} + \&c.$$

210. Ea

$$+ \left(\frac{19t^{17} - 17t^{19}}{17 \cdot 19} \right) + \left(\frac{23t^{21} - 21t^{23}}{21 \cdot 23} \right) = \&c., \text{ quam forma folummodo,}$$

propter additionem factam binorum, qui fequuntur, terminorum, ab ea
 numeri 209. differre omnes perfpiciunt. Eiusdem infuper Seriei Confecta-
 rium eft altera Formula indemonftrata, cuius Ioannes Bernoullius in Poftthu-
 mis meminit: nuncupato etenim y Sinu-recto Arcus Circularis x habebi-
 tur ipfius Arcus Tangens expreffa per $\frac{y}{(1-y^2)^{\frac{1}{2}}}$; quamobrem $x =$

$$\frac{y^1}{1(1-y^2)^{\frac{1}{2}}} - \frac{y^3}{3(1-y^2)^{\frac{3}{2}}} + \frac{y^5}{5(1-y^2)^{\frac{5}{2}}} - \frac{y^7}{7(1-y^2)^{\frac{7}{2}}} + \frac{y^9}{9(1-y^2)^{\frac{9}{2}}} -$$

$$\frac{y^{11}}{11(1-y^2)^{\frac{11}{2}}} + \&c.$$

$\frac{y^{11}}{11(1-y^2)^{\frac{11}{2}}} + \&c.$, uti prae laudatus Auctor adseruit. Quinimmo paucis

mutatis expressio ipsa Bernoulliana etiam Hyperbolae aptari poterit, in qua

reapse iisdem retentis denominationibus fiet 2 Sect. H. $= \frac{y^1}{1(1+y^2)^{\frac{1}{2}}} +$

$$\frac{y^3}{3(1+y^2)^{\frac{3}{2}}} + \frac{y^5}{5(1+y^2)^{\frac{5}{2}}} + \frac{y^7}{7(1+y^2)^{\frac{7}{2}}} + \frac{y^9}{9(1+y^2)^{\frac{9}{2}}} + \frac{y^{11}}{11(1+y^2)^{\frac{11}{2}}} +$$

&c: sine Limite. Nec abludit subnexa Formula, cuius ope Bernoullius ipse

Arcum Circuli x aequalem esse adfirmavit $\frac{1 \cdot T^1}{1(1+T^2)^1} + \frac{1 \cdot 2 T^3}{1 \cdot 3 (1+T^2)^2} +$

$$\frac{1 \cdot 2 \cdot 4 T^5}{1 \cdot 3 \cdot 5 (1+T^2)^3} + \frac{1 \cdot 2 \cdot 4 \cdot 6 T^7}{1 \cdot 3 \cdot 5 \cdot 7 (1+T^2)^4} + \frac{1 \cdot 2 \cdot 4 \cdot 6 \cdot 8 T^9}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 (1+T^2)^5} +$$

$\frac{1 \cdot 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 T^{11}}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 (1+T^2)^6} + \&c.$, quamvis demonstratione celata elegantissimae

praesertim Numeratorum Progressionis, sed hypothese facta speciei T significantis Tangentem. Tot inter methodos, quae certatim se mihi offerunt, novitatem quamdam praediligens Seriei huiusce processum breviter demonstrepo;

nec data hic loci occasione fileo Fractionem $\frac{T^1}{1^2 + T^2} = \frac{1}{T^1 + \frac{1^2}{T^1}} =$

$\frac{1}{T + Cr} = \Psi^{-1}$, dummodo Nota Cr Cotangentem Arcus exhibeat, ac

Summa Cotangentis, et Tangentis compendii gratia sub elemento Ψ abscondita intelligatur. Procul dubio est Binomii Formulam probe callentibus Fra-

ctionem $\frac{T}{1+T^2}$ in Seriem numero terminorum Infinitam evolvi $T^1 -$

$T^3 + T^5 - T^7 + T^9 - T^{11} + T^{13} - T^{15} + T^{17} - \&c.$, qua cogni-

ta Potestatum etiam $\left(\frac{T}{1+T^2}\right)^2, \left(\frac{T}{1+T^2}\right)^3, \left(\frac{T}{1+T^2}\right)^4, \left(\frac{T}{1+T^2}\right)^5$ &c: Se-

ries respectivae facillimo opere reperientur. Bernoulliana ergo Series rite ordinata primum confestim terminum supputando evadit

x =

$$x = T^1 - T^3 + T^5 - T^7 + T^9 - T^{11} + T^{13} - T^{15} + \&c.,$$

ac subinde secundum
$$+ \frac{2T^3}{3} - \frac{4T^5}{3} + \frac{6T^7}{3} - \frac{8T^9}{3} + \frac{10T^{11}}{3} - \frac{12T^{13}}{3} + \frac{14T^{15}}{3} - \&c.,$$

tertium
$$+ \frac{8T^5}{15} - \frac{24T^7}{15} + \frac{48T^9}{15} - \frac{80T^{11}}{15} + \frac{120T^{13}}{15} - \frac{168T^{15}}{15} + \&c.,$$

quartum
$$+ \frac{48T^7}{105} - \frac{192T^9}{105} + \frac{480T^{11}}{105} - \frac{960T^{13}}{105} + \frac{1680T^{15}}{105} - \&c.,$$

quintum
$$+ \frac{384T^9}{945} - \frac{1920T^{11}}{945} + \frac{5760T^{13}}{945} - \frac{13440T^{15}}{945} + \&c.,$$

sextum
$$+ \frac{3840T^{11}}{10395} - \frac{23040T^{13}}{10395} + \frac{80640T^{15}}{10395} - \&c.,$$

septimum
$$+ \frac{46080T^{13}}{135135} - \frac{276480T^{15}}{135135} + \&c.,$$

octavum &c. &c. addendo redigitur in illam Leibnitii,
$$+ \frac{645120T^{15}}{2027025} - \&c.,$$

nimirum singulis Linearum verticalium terminis in unum collectis peraequat

$$\frac{T^1}{1} - \frac{T^3}{3} + \frac{T^5}{5} - \frac{T^7}{7} + \frac{T^9}{9} - \frac{T^{11}}{11} + \frac{T^{13}}{13} - \frac{T^{15}}{15} + \&c., \text{ ideo}$$

que cum hac postrema alterius formae tam immaniter diversae Series Inf-

nita conveniet
$$\Psi^{-1} + \frac{2T \cdot \Psi^{-2}}{3} + \frac{8T^2 \cdot \Psi^{-3}}{15} + \frac{48T^3 \cdot \Psi^{-4}}{105} + \frac{384T^4 \cdot \Psi^{-5}}{945}$$

$$+ \frac{3840T^5 \cdot \Psi^{-6}}{10395} + \frac{46080T^6 \cdot \Psi^{-7}}{135135} + \frac{645120T^7 \cdot \Psi^{-8}}{2027025} + \&c., \text{ et quasi prae-}$$

ter spem utraeque Series coincident.

211. Ne vero censeatur me in aequipollentiam praedictam forte fortuna incidisse, modum exclusionis assumam ingeniosissimum a Goldbachio, et Eulero usitatum. Video priorem Trigoni illius Algebraici Lineam ho-

rizontalem, sive brevius, evolutam Fractionem $\frac{T}{1+T^2}$ usque a secundo evo-

lutionis termino respectu Series Leibnitianae differre. Ut ideo secundus hic

terminus cum Leibnitiano conveniat addamus oportet $\frac{2}{3} \left(\frac{T}{1+T^2} \right)^2 \times T$,

quia tum addita secunda Linea horizontali duo singuli priores termini analogis a Leibnitio datis ad unguem aequantur. Discrepat autem tertius terminorum, de quibus fit mentio, atque, ut haec differentia evanescat, necesse est adiungamus $\frac{8}{15} \left(\frac{T}{1+T^2} \right)^3 \times T^2$, scilicet tertiam Linearum ho-

izontalium praedicti Trianguli. Consimilis ratiocinatio reliquis etiam extenditur terminis, directaque methodus Bernoullianam Seriem inveniendi simul, et demonstrandi nisi turpiter oscitantem nunc fugere nequit. Clarius quoque cuilibet insipienti observabitur Lex Coefficientium in horizontalibus, verticalibus, ac diagonalibus Lineis posita memorati Trigoni. Nullum de Denominatoribus verbum propterea, quod sint Producta Numerorum imparium 1, 1.3, 1.3.5, 1.3.5.7, 1.3.5.7.9, &c.: Numeratores vero Linearum horizontalium, initio a suprema desumpto, sunt Series Unitatum, aut stylo Oughtredi et Vallisii Monadicorum; Numerorum Naturalium, vel Lateralium, per 2 multiplicatorum; Triangularium per 2.4; Pyramidalium per 2.4.6; Triangulo-triangularium per 2.4.6.8; Pyramido-triangularium per 2.4.6.8.10 &c.; una voce Numerorum latissime Figuratorum, quorum Differentiae primae, secundae, tertiae, quartae, quintae &c., quemadmodum notum est, in nihilum abeunt. Quo ad Lineas diagonaliter consideratas Numeratores adeo progrediuntur, ut ab Unitate, seu Monade incipiant universi, sed prima Linea, ab laeva dextrorsum ceteris ordine computatis, habeat 2, 2.4, 2.4.6, 2.4.6.8, 2.4.6.8.10 &c.; altera 4, 4.6, 4.6.8, 4.6.8.10, 4.6.8.10.12 &c.; tertia 6, 6.8, 6.8.10, 6.8.10.12, 6.8.10.12.14 &c.; quarta 8, 8.10, 8.10.12, 8.10.12.14, 8.10.12.14.16 &c., ac sic deinceps in Infinitum. Linearum denique verticalium progressus sua veluti sponte in aperto locatur: namque ortu ab infimis ducto terminis emergunt Coefficientes ordine subsequente dispositi 2, 1; 4.2, 4.1; 6.4.2, 6.4.1, 6.1; 8.6.4.2, 8.6.4.1, 8.6.1, 8.1; 10.8.6.4.2, 10.8.6.4.1, 10.8.6.1, 10.8.1, 10.1 &c.,

&c.; quibus in Numerorum parium Productis evidentissima ceterarum Linearum continuatio satis luculenter adparet.

212. Sed in tam facili Analytici illius Trianguli anatome prosequenda oleam tempusque perderem, ac lubet potius, quod Bernoullius non fecit, monere posteriorem Seriem non parum iucundam ab ipso traditam Tangentibus quoque extendendam Hyperbolicis. Et re quidem vera Progressio

$$\text{numero terminorum Infinita } \frac{T^1}{1} - \frac{T^3}{3} + \frac{T^5}{5} - \frac{T^7}{7} + \frac{T^9}{9} - \frac{T^{11}}{11} + \frac{T^{13}}{13} - \frac{T^{15}}{15} + \&c. \text{ in aliam vertitur } \frac{T^1}{1} + \frac{T^3}{3} + \frac{T^5}{5} + \frac{T^7}{7} + \frac{T^9}{9} + \frac{T^{11}}{11} + \frac{T^{13}}{13} + \frac{T^{15}}{15} + \&c. \text{ tum, cum vice } T \text{ subeat } T\sqrt{-1},$$

totaque exorta Series divisionis ope ab Imaginaria Magnitudine $\sqrt{-1}$ liberetur. Ista substitutione, et partitione, de quibus superius dictum, effectis Series ad Circulum, aequipollens memoratae Seriei Leibnitianae, evadit

$$\frac{1.T^1}{1(1-T^2)^1} - \frac{1.2.T^3}{1.3(1-T^2)^2} + \frac{1.2.4.T^5}{1.3.5(1-T^2)^3} - \frac{1.2.4.6.T^7}{1.3.5.7(1-T^2)^4} + \frac{1.2.4.6.8.T^9}{1.3.5.7.9(1-T^2)^5} - \frac{1.2.4.6.8.10.T^{11}}{1.3.5.7.9.11(1-T^2)^6} + \&c.; \text{ quae Signis tantum a praecedente distat, et eandem inventionis directam methodum, easdemque antea ostensas affectiones cum prima superius perpenſa communes habet, tametsi haec postrema non amplius Circulum sapiat, veruntamen Hyperbolen Aequilateram Apollonianam. Servato semper mirabilis analogiae iure quoad Formulam } \frac{T}{1-T^2} = \frac{1}{\frac{1}{T} - T} = \Phi^{-1}, \text{ si } \Phi \text{ Differentiam inter Cotangentem, Tangentemque repraesentet Hyperbolici Centralis Sectoris, obtinebitur etiam respectu Hyperbolae 2 Sect. } H = \Phi^{-1} - \frac{2T\Phi^{-2}}{3} + \frac{8T^2\Phi^{-3}}{15} - \frac{48T^3\Phi^{-4}}{105} + \frac{384T^4\Phi^{-5}}{945} - \frac{3840T^5\Phi^{-6}}{10395} + \frac{46080T^6\Phi^{-7}}{135135} - \frac{645120T^7\Phi^{-8}}{2027025} + \&c.; \text{ Sed haec Aequalitas praesupponit Hyperbolicae}$$

cedente distat, et eandem inventionis directam methodum, easdemque antea ostensas affectiones cum prima superius perpenſa communes habet, tametsi haec postrema non amplius Circulum sapiat, veruntamen Hyperbolen Aequilateram Apollonianam. Servato semper mirabilis analogiae iure

$$\text{quoad Formulam } \frac{T}{1-T^2} = \frac{1}{\frac{1}{T} - T} = \Phi^{-1}, \text{ si } \Phi \text{ Differentiam inter}$$

Cotangentem, Tangentemque repraesentet Hyperbolici Centralis Sectoris,

$$\text{obtenebitur etiam respectu Hyperbolae 2 Sect. } H = \Phi^{-1} - \frac{2T\Phi^{-2}}{3} + \frac{8T^2\Phi^{-3}}{15} - \frac{48T^3\Phi^{-4}}{105} + \frac{384T^4\Phi^{-5}}{945} - \frac{3840T^5\Phi^{-6}}{10395} + \frac{46080T^6\Phi^{-7}}{135135} - \frac{645120T^7\Phi^{-8}}{2027025} + \&c.; \text{ Sed haec Aequalitas praesupponit Hyperbolicae}$$

caae

cae Cotangentis notionem, sub cuius nomine Tangentem intelligo Sectoris in opposita Hyperbolae parte adeo supputati, ut duae Secantes, seu Radii Angulum rectum constituent; quod melius legetur in aliquo Schematis Paradigmate.

213. De Cotangentibus, Secantibus, et Cosecantibus pauca quaedam subiungam

$$\text{Cot. } C = \frac{1}{\text{Tang. } C} = \frac{\text{Cofin. } C}{\text{Sin. } C} = \sqrt{-1} \left(\frac{C^{2x\sqrt{-1}} + 1}{C^{2x\sqrt{-1}} - 1} \right) =$$

$$\text{Sec. } C = \frac{1}{\text{Cofin. } C} = \frac{2}{C^{x\sqrt{-1}} + C^{-x\sqrt{-1}}} = \frac{2C^{x\sqrt{-1}}}{C^{2x\sqrt{-1}} + 1} =$$

$$\text{Cofec. } C = \frac{1}{\text{Sin. } C} = \frac{2}{C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}} = \frac{\sqrt{-1} (2C^{x\sqrt{-1}})}{C^{2x\sqrt{-1}} - 1} =$$

$$\text{Vera Cot. } H = \frac{1}{\text{Tang. } H} = \frac{\text{Cofin. } H}{\text{Sin. } H} = \frac{C^{2x} + 1}{C^{2x} - 1} =$$

$$\text{Fictitia Sec. } H = \frac{1}{\text{Cofin. } H} = \frac{2}{C^x + C^{-x}} = \frac{2C^x}{C^{2x} + 1} =$$

$$\text{Fictitia Cofec. } H = \frac{1}{\text{Sin. } H} = \frac{2}{C^x - C^{-x}} = \frac{2C^x}{C^{2x} - 1} =$$

Abque ulterioris eloquii necessitate derivatio, ac similitudo expressionum utriusque

subiungam ex eo, quod ipsa duce superius deducta theoria quisque per se valeat parcius hic dicenda fusiore absolvere explicatione, aut quoquo- modo commentari. Formulas harumce Linearum fundamentales meliori, quem potero, ordine inprimis disponere iuvat.

Cot. =

$$\sqrt{-1} \times \frac{\left(1 + \frac{2x\sqrt{-1}}{\infty}\right)^{\infty} + 1}{\left(1 + \frac{2x\sqrt{-1}}{\infty}\right)^{\infty} - 1} = \sqrt{-1} \left(\frac{\text{Cofin. } C_{2x} + (\text{Sin. } C_{2x})\sqrt{-1} + 1}{\text{Cofin. } C_{2x} + (\text{Sin. } C_{2x})\sqrt{-1} - 1} \right)$$

$$\frac{2 \left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{\left(1 + \frac{2x\sqrt{-1}}{\infty}\right)^{\infty} + 1} = \frac{2(\text{Cofin. } C_x + (\text{Sin. } C_x)\sqrt{-1})}{\text{Cofin. } C_{2x} + (\text{Sin. } C_{2x})\sqrt{-1} + 1}$$

$$\frac{2\sqrt{-1} \left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{\left(1 + \frac{2x\sqrt{-1}}{\infty}\right)^{\infty} - 1} = \frac{2\sqrt{-1}(\text{Cofin. } C_x + (\text{Sin. } C_x)\sqrt{-1})}{\text{Cofin. } C_{2x} + (\text{Sin. } C_{2x})\sqrt{-1} - 1}$$

$$\frac{\left(1 + \frac{2x}{\infty}\right)^{\infty} + 1}{\left(1 + \frac{2x}{\infty}\right)^{\infty} - 1} = \frac{\text{Cofin. } H_{2x} + \text{Sin. } H_{2x} + 1}{\text{Cofin. } H_{2x} + \text{Sin. } H_{2x} - 1}$$

$$\frac{2 \left(1 + \frac{x}{\infty}\right)^{\infty}}{\left(1 + \frac{2x}{\infty}\right)^{\infty} + 1} = 2 \left(\frac{\text{Cofin. } H_x + \text{Sin. } H_x}{\text{Cofin. } H_{2x} + \text{Sin. } H_{2x} + 1} \right)$$

$$\frac{2 \left(1 + \frac{x}{\infty}\right)^{\infty}}{\left(1 + \frac{2x}{\infty}\right)^{\infty} - 1} = 2 \left(\frac{\text{Cofin. } H_x + \text{Sin. } H_x}{\text{Cofin. } H_{2x} + \text{Sin. } H_{2x} - 1} \right) \cdot$$

Absque

utriusque Trigonometriae, circa quam abunde satis versati, in propatulo erit.

214. Cotangens igitur Hyperbolica z Analogum - Arcum suum, nimirum x duplam Aream Sectoris Centralis, hac Aequatione praebebit $x =$

$$\frac{z^{-1}}{1} - \frac{z^{-3}}{3} + \frac{z^{-5}}{5} - \frac{z^{-7}}{7} + \frac{z^{-9}}{9} - \frac{z^{-11}}{11} + \frac{z^{-13}}{13} - \frac{z^{-15}}{15} +$$

&c.; dum ex adverso nuncupata Y Circulari Cotangente habebitur verus

$$\text{Arcus respondens } x = \frac{Y^{-1}}{1} - \frac{Y^{-3}}{3} + \frac{Y^{-5}}{5} - \frac{Y^{-7}}{7} + \frac{Y^{-9}}{9} - \frac{Y^{-11}}{11} +$$

$$+ \frac{Y^{-13}}{13} - \frac{Y^{-15}}{15} + \text{&c. sine ullo Limite. Eadem ratione profluet, ge-}$$

$$\text{minas invertendo variables, } z = \left(\frac{x^1}{1} - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \frac{62x^9}{2835} \right.$$

$$\left. - \frac{1382x^{11}}{155925} + \frac{21844x^{13}}{6081075} - \text{&c.} \right)^{-1} = \frac{x^{-1}}{1} + \frac{x^1}{3} - \frac{x^3}{45} + \frac{2x^5}{945}$$

$$- \frac{x^7}{4725} + \frac{2x^9}{93555} - \frac{1382x^{11}}{638512875} + \frac{4x^{13}}{18243225} - \frac{3617x^{15}}{162820783125} + \text{&c.} =$$

$$\frac{x^{-1}}{1} + \frac{x^1}{1.3^1} - \frac{x^3}{1.3^2.5} + \frac{2x^5}{1.3^3.5.7} - \frac{x^7}{1.3^3.5^2.7} + \frac{2x^9}{1.3^3.5.7.9.11} -$$

$$\frac{1382x^{11}}{1.3^4.5^3.7^2.9.11.13} + \frac{4x^{13}}{1.3^4.5^2.7.9.11.13} - \frac{3617x^{15}}{1.3^4.5^3.7^2.9.11.13.15.17} + \text{&c.}; \text{ nulla-}$$

que ad hanc Seriem adipiscendam, praeter Polynomii, aut Binomii Newtoniani simplicissimam Formulam, alia methodus efflagitatur. Sit nunc loco

$$x \text{ positum } x\sqrt{-1}; \text{ atque emerget } Y = \sqrt{-1} \left(\frac{(x\sqrt{-1})^{-1}}{1} + \frac{(x\sqrt{-1})^1}{3} \right.$$

$$+ \frac{x^2(x\sqrt{-1})^1}{45} + \frac{2x^4(x\sqrt{-1})^1}{945} - \frac{x^6(x\sqrt{-1})^1}{4725} + \frac{2x^8(x\sqrt{-1})^1}{93555} + \frac{1382x^{10}(x\sqrt{-1})^1}{638512875}$$

$$+ \frac{4x^{12}(x\sqrt{-1})^1}{18243225} \left. \right) + \text{&c.} = \frac{x^{-1}}{1} - \frac{x^1}{1.3^1} + \frac{x^3}{1.3^2.5} - \frac{2x^5}{1.3^3.5.7} + \frac{x^7}{1.3^3.5^2.7}$$

$$- \frac{2x^9}{1.3^3.5.7.9.11} + \frac{1382x^{11}}{1.3^4.5^3.7^2.9.11.13} - \frac{4x^{13}}{1.3^4.5^2.7.9.11.13} =$$

$$\frac{3617x^{15}}{1.3^4.5^3.7^2.9.11.13.15.17} - \text{&c.}, \text{ uti passim apud Trigonometras prostat.}$$

215. Nec difficilior de Pseudo-Secante invenienda fit quaestio. Namque,

que, si Hyperbolen verses, et C^{2x} ad instar numⁱ. 207. appelles V, ef-

fluet haec Secans $= \frac{2V^{\frac{1}{2}}}{1+V}$, cuius Formulae evolutio Seriem confestim exhiberet quaesitam. Attamen ista, quae petitur, Series facilius eruetur dum meminerimus, assumpta specie Δ pro Secante Hyperbolica, esse $\Delta =$

$$\left(1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \right. \\ \left. \&c: \right)^{-1} = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \frac{277x^8}{8064} - \frac{50521x^{10}}{3628800} + \\ \frac{540553x^{12}}{95800320} - \&c: = 1 - \frac{x^2}{1.2} + \frac{5x^4}{1.2.3.4} - \frac{61x^6}{1.2.3.4.5.6} + \frac{5(277)x^8}{1.2.3.4.5.6.7.8} - \\ \frac{50521x^{10}}{1.2.3.4.5.6.7.8.9.10} + \frac{5(540553)x^{12}}{1.2.3.4.5.6.7.8.9.10.11.12} - \frac{199360981x^{14}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14} +$$

&c: uti consequimur ab usitata Neutoniana Potentiationis adplicatione. Nulla mora interposita subinde oritur Circularis etiam expressio Secantis Δ , quae ob num^{um}. 213. a mera τx in $x \sqrt{-1}$ translatione dependet. Erit

$$\text{igitur } \Delta = 1 + \frac{x^2}{1.2} + \frac{5x^4}{1.2.3.4} + \frac{61x^6}{1.2.3.4.5.6} + \frac{5.277x^8}{1.2.3.4.5.6.7.8} + \\ \frac{50521x^{10}}{1.2.3.4.5.6.7.8.9.10} + \frac{5(540553)x^{12}}{1.2.3.4.5.6.7.8.9.10.11.12} + \frac{199360981x^{14}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14} + \\ \frac{5.3878302429x^{16}}{1.2.3.4.5.6.7.8.9.10.11.12.13.14.15.16} + \&c: , \text{ permutatione nempe Signorum prio-}$$

ris Seriei, quemadmodum toties supra usuvenit, in successionem conversa. Vix insuper de Seriebus reciprocis promoveri poterit difficultas vel minima, quippe quae a proprietate nativa Secantium aequalium $\frac{1}{\text{Cosin. } \frac{C}{H} \tau x}$

originem simplicissimam ducunt. Hinc aperte obtinemus $x = \left(2 \left(1 - \frac{1}{\Delta} \right) \right)^{\frac{1}{2}} \times$

$$\left(\frac{1 + 1^2 \left(1 - \frac{1}{\Delta} \right)^1}{2^2.3} + \frac{1^2.3^2 \left(1 - \frac{1}{\Delta} \right)^2}{2^3.3.4.5} + \frac{1^2.3^2.5^2 \left(1 - \frac{1}{\Delta} \right)^3}{2^4.3.4.5.6.7} + \right. \\ \left. \frac{1^2.3^2}{1^2.3^2} \right)$$

$1^2.3^2.5^2.7^2 \left(1 - \frac{1}{\Lambda}\right)^4 + \&c: \Bigg)$, vel, si alteram potius eligamus formam,

$$x = \frac{2^5.3.4.5.6.7.8.9}{\left(2 \left(\frac{\Lambda-1}{\Lambda}\right)\right)^{\frac{1}{2}}} \left(1 + \frac{1^2 \left(\frac{\Lambda-1}{\Lambda}\right)^1}{2^2.3} + \frac{1^2.3^2 \left(\frac{\Lambda-1}{\Lambda}\right)^2}{2^3.3.4.5} + \right.$$

$$\left. \frac{1^2.3^2.5^2 \left(\frac{\Lambda-1}{\Lambda}\right)^3}{2^4.3.4.5.6.7} + \frac{1^2.3^2.5^2.7^2 \left(\frac{\Lambda-1}{\Lambda}\right)^4}{2^5.3.4.5.6.7.8.9} + \&c: \right). \text{ Simili ratione}$$

$$x = \frac{2^5.3.4.5.6.7.8.9}{\left(2 \left(\frac{1}{\Delta} - 1\right)\right)^{\frac{1}{2}}} \left(1 - \frac{1^2 \left(\frac{1}{\Delta} - 1\right)^1}{2^2.3} + \frac{1^2.3^2 \left(\frac{1}{\Delta} - 1\right)^2}{2^3.3.4.5} - \right.$$

$$\left. \frac{1^2.3^2.5^2 \left(\frac{1}{\Delta} - 1\right)^3}{2^4.3.4.5.6.7} + \&c: \right), \text{ vel, si mavis, } x = \left(2 \left(\frac{1-\Delta}{\Delta}\right)\right)^{\frac{1}{2}} \times$$

$$\left(1 - \frac{1^2 \left(\frac{1-\Delta}{\Delta}\right)^1}{2^2.3} + \frac{1^2.3^2 \left(\frac{1-\Delta}{\Delta}\right)^2}{2^3.3.4.5} - \frac{1^2.3^2.5^2 \left(\frac{1-\Delta}{\Delta}\right)^3}{2^4.3.4.5.6.7} + \right.$$

$$\left. \frac{1^2.3^2.5^2.7^2 \left(\frac{1-\Delta}{\Delta}\right)^4}{2^5.3.4.5.6.7.8.9} - \&c: \right). \text{ Aequationem offerunt Pseudo-Secanti Hy-}$$

perbolicae accommodatam.

216. Qui iisdem insistens vestigiis Formulas petat Circularium Cofecantium, ac Pseudo-Cofecantium Hyperbolicarum citissime voto suo satisfaciet statim ac senserit esse $\Theta = \text{Cofec. C. } x = \left(\frac{x^1}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^9}{1.2.3.4.5.6.7.8.9} - \frac{x^{11}}{1.2.3.4.5.6.7.8.9.10.11} + \&c: \right)^{-1} =$

$$\frac{x^{-1}}{1} + \frac{x^1}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \frac{127x^7}{604800} + \frac{73x^9}{3421440} + \frac{1414477x^{11}}{653837184000} + \&c: = \frac{x^{-1}}{1^2} + \frac{x^1}{1^2.2.3^1} + \frac{7x^3}{1^2.2.3^2.4.5} + \frac{31x^5}{1^2.2.3^3.4.5.6.7} + \&c:$$

$$\frac{9.127x^7}{1^2.2.3^2.4.5^2.6.7.8.9} + \frac{7.25.73x^9}{1^2.2.3^2.4.5^2.6.7.8.9.10.11} + \frac{2.1414477x^{11}}{1^2.2.3^2.4.5^2.6.7^2.8.9.10.11.12.13} + \frac{7.10.14.15.8191x^{13}}{1^2.2.3^2.4.5^2.6.7^2.8.9.10.11.12.13.14.15} + \&c.,$$

cuius posterioris Membri Denominatores Lege non parum perspicua procedunt. Vicissim habebitur $x =$

$$\frac{\Theta^{-1}}{1} + \frac{1^2.\Theta^{-3}}{1.2.3} + \frac{1^2.3^2.\Theta^{-5}}{1.2.3.4.5} + \frac{1^2.3^2.5^2.\Theta^{-7}}{1.2.3.4.5.6.7} + \frac{1^2.3^2.5^2.7^2.\Theta^{-9}}{1.2.3.4.5.6.7.8.9} + \frac{1^2.3^2.5^2.7^2.9^2.\Theta^{-11}}{1.2.3.4.5.6.7.8.9.10.11} + \&c.,$$

quemadmodum abunde constat si praecedens numerus 161. consulatur. Factis hinc inde iuxta numerum 213. substitutionibus toties recensitis proveniet *fictitia* Cosecans Hyperbolica $\Xi = \frac{x^{-1}}{1^2}$

$$\frac{x^{-1}}{1^2.2.3^2} + \frac{7x^3}{1^2.2.3^2.4.5} + \frac{31x^5}{1^2.2.3^2.4.5.6.7} + \frac{9.127x^7}{1^2.2.3^2.4.5^2.6.7.8.9} + \frac{7.25.73x^9}{1^2.2.3^2.4.5^2.6.7.8.9.10.11} + \frac{2.1414477x^{11}}{1^2.2.3^2.4.5^2.6.7^2.8.9.10.11.12.13} + \&c.,$$

ideoque $x =$

$$\frac{\Xi^{-1}}{1} + \frac{1^2.\Xi^{-3}}{1.2.3} + \frac{1^2.3^2.\Xi^{-5}}{1.2.3.4.5} + \frac{1^2.3^2.5^2.\Xi^{-7}}{1.2.3.4.5.6.7} + \frac{1^2.3^2.5^2.7^2.\Xi^{-9}}{1.2.3.4.5.6.7.8.9} + \frac{1^2.3^2.5^2.7^2.9^2.\Xi^{-11}}{1.2.3.4.5.6.7.8.9.10.11} + \&c. \text{ in Infinitum.}$$

217. Calculi equidem quasi ludum inanem, et infoecundum *fictitiae* illae Secantes, Cosecantisque praeseferunt. Caveant idcirco Tyrones, ne Formulas supra expositas, atque ad Algorithmi Analytici, et uberrimae Harmoniae vires experiundas tantummodo concinnatas, Secantium, Cosecantiumque Hyperbolicarum cum veris confundant. Vera enim Hyperbolae Secans non est reciproca Cosinus, nec ex adverso Cosecans Hyperbolici Sinus reciproca. Haec in Circulo accidit reciprocatio, sed Hyperbolae nequaquam convenit. Expressionis Imaginarietas in causa est, cur ista reciprocitatis affectio recte adhibeatur in Circulo, et respuat Hyperbolen. Id autem ostendere datur perquam facillime. Siquidem tam in Circulo, quam in Hyperbola Se-

cans legitima peraequat $(1 + (\text{Tang.})^2)^{\frac{1}{2}}$, nimirum, respectu habito ad
Dd d Hyperbolen,

Hyperbolen, $\left(1 + \left(\frac{C^{2x} - 1}{C^{2x} + 1}\right)^2\right)^{\frac{1}{2}}$, et ad Circulum

$\left(1 + \left(\frac{C^{2x\sqrt{-1}} - 1}{(\sqrt{-1})(C^{2x\sqrt{-1}} + 1)}\right)^2\right)^{\frac{1}{2}}$. Postrema Formula evadit

$\left(\frac{-4C^{2x\sqrt{-1}}}{-1(C^{2x\sqrt{-1}} + 1)^2}\right)^{\frac{1}{2}} = \frac{2C^{x\sqrt{-1}}}{C^{2x\sqrt{-1}} + 1} = \frac{1}{\text{Cofin. } C.x}$. Veruntamen pri-

mam expressionem tanta non comitatur felicitas, quippe quae solummodo abit in

$\left(\frac{2(C^{4x} + 1)}{(C^{2x} + 1)^2}\right)^{\frac{1}{2}}$, interea dum $\frac{1}{\text{Cofin. } H.x}$ est $\frac{2C^x}{C^{2x} + 1}$ toto coelo in Nu-

meratore distans a Formula praecedente. Eadem dicenda de veris, aut genuinis Hyperbolicis Cofecantibus, propterea quod eas in utraque Curva ge-

neraliter complectatur Formula $(1 + (\text{Cot.})^2)^{\frac{1}{2}}$, scilicet in Circulo

$\left(1 + \left(\frac{(\sqrt{-1})(C^{2x\sqrt{-1}} + 1)}{C^{2x\sqrt{-1}} - 1}\right)^2\right)^{\frac{1}{2}}$, vel $\left(\frac{-1.4C^{2x\sqrt{-1}}}{(C^{2x\sqrt{-1}} - 1)^2}\right)^{\frac{1}{2}}$, seu

$\sqrt{-1} \left(\frac{2C^{x\sqrt{-1}}}{C^{2x\sqrt{-1}} - 1}\right)$, aut denique $\frac{1}{\text{Sin. } C.x}$; quod non ita contingit Hy-

perbolae, quum omnibus pateat esse $\frac{1}{\text{Sin. } H.x} = \frac{2C^x}{C^{2x} - 1}$, sed

$(1 + (\text{Cot.})^2)^{\frac{1}{2}} = \left(\frac{2(C^{4x} + 1)}{(C^{2x} - 1)^2}\right)^{\frac{1}{2}}$; Aequatio ad priorem nullatenus

reducibilis. Formulae nuper inventae $\frac{(2(C^{4x} + 1))^{\frac{1}{2}}}{C^{2x} + 1}$, nec non

(2 (C

$\frac{(2(C^{4x} + 1))^{\frac{1}{2}}}{C^{2x} - 1}$, quae Numeratorem habent communem, et Exponentia-

lem Numeratoris ipsius partem perfectum Quantitatis Exponentialis in De-
 nominatore sitae Quadratum, manu veluti ducerent ad Series numero ter-
 minorum Infinitas directas, conversasve Secantium, et Cosecantium, ac vere
 pertinentes ad Hyperbolam, determinandas. Methodo tamen hoc efficiendi,
 si libeat, praeostensa, eiusdem adplicationem dicabo Lectorum solertiae,
 quibus, dum otium suppetat, ut Trigonometriae etiam oecumenicae Ta-
 bula sive Prospectus ex sparsis superius Formulis coalescens Hyperboli-
 co - Circularibus, ac summatim uno oculi ictu contemplandus aliquando
 curae sit, eos oro, obsecroque. Mihi etenim ad alia properanti sufficiet in
 ordinem digerere una cum geminae Trigonometriae Cotangentibus ea, quae
 Secantes Circulares, ac Cosecantes respiciunt, et usus perquam maximi in
 Trigonometricis disciplinis prae Secantibus, aut Cosecantibus Hyperbolicis,
 neglectis obsoletisque, existimantur.

$$\text{Cot. C.} = \frac{(\text{Arc. C.})^{-1}}{1} - \frac{(\text{Arc. C.})^1}{1 \cdot 3^1} - \frac{(\text{Arc. C.})^3}{1 \cdot 3^2 \cdot 5} - \frac{2(\text{Arc. C.})^5}{1 \cdot 3^3 \cdot 5 \cdot 7} - \frac{(\text{Arc. C.})^7}{1 \cdot 3^3 \cdot 5^2 \cdot 7} - \frac{2(\text{Arc. C.})^9}{1 \cdot 3^3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} - \&c:$$

$$\text{Arc. C} = \frac{(\text{Cot. C})^{-1}}{1} - \frac{(\text{Cot. C})^{-3}}{3} + \frac{(\text{Cot. C})^{-5}}{5} - \frac{(\text{Cot. C})^{-7}}{7} + \frac{(\text{Cot. C})^{-9}}{9} - \frac{(\text{Cot. C})^{-11}}{11} + \frac{(\text{Cot. C})^{-13}}{13} - \&c:$$

$$\text{Cot. H.} = \frac{(2\text{Sec. H})^{-1}}{1} + \frac{(2\text{Sec. H})^1}{1 \cdot 3^1} - \frac{(2\text{Sec. H})^3}{1 \cdot 3^2 \cdot 5} + \frac{2(2\text{Sec. H})^5}{1 \cdot 3^3 \cdot 5 \cdot 7} - \frac{(2\text{Sec. H})^7}{1 \cdot 3^3 \cdot 5^2 \cdot 7} + \frac{2(2\text{Sec. H})^9}{1 \cdot 3^3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} - \&c:$$

$$\text{Sec. H.} = \frac{(\text{Cot. H})^{-1}}{1} + \frac{(\text{Cot. H})^{-3}}{3} + \frac{(\text{Cot. H})^{-5}}{5} + \frac{(\text{Cot. H})^{-7}}{7} + \frac{(\text{Cot. H})^{-9}}{9} + \frac{(\text{Cot. H})^{-11}}{11} + \frac{(\text{Cot. H})^{-13}}{13} + \&c:$$

$$\text{Sec. C} = 1 + \frac{(\text{Arc. C})^2}{1 \cdot 2} + \frac{5(\text{Arc. C})^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{61(\text{Arc. C})^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{1385(\text{Arc. C})^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \frac{50521(\text{Arc. C})^{10}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} + \frac{2702765(\text{Arc. C})^{12}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12} + \&c:$$

$$\text{Arc. C.} = \left[2 \left(\frac{(\text{Sec. C})^{-1}}{\text{Sec. C}} \right) \right]^{\frac{1}{2}} \left[1 + \frac{1^2 \cdot (\frac{\text{Sec. C} - 1}{\text{Sec. C}})^1}{2^2 \cdot 3} + \frac{1^2 \cdot 3^2 \cdot (\frac{\text{Sec. C} - 1}{\text{Sec. C}})^2}{2^3 \cdot 3 \cdot 4 \cdot 5} + \frac{1^2 \cdot 3^3 \cdot 5^2 \cdot (\frac{\text{Sec. C} - 1}{\text{Sec. C}})^3}{2^4 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot (\frac{\text{Sec. C} - 1}{\text{Sec. C}})^4}{2^5 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \&c: \right]$$

$$\text{Cosec. C} = \frac{(\text{Arc. C})^{-1}}{1^2} + \frac{(\text{Arc. C})^1}{1^2 \cdot 2 \cdot 3^1} + \frac{7(\text{Arc. C})^3}{1^2 \cdot 2 \cdot 3^2 \cdot 4 \cdot 5} + \frac{31(\text{Arc. C})^5}{1^2 \cdot 2 \cdot 3^2 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{1143(\text{Arc. C})^7}{1^2 \cdot 2 \cdot 3^3 \cdot 4 \cdot 5^2 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \frac{12776(\text{Arc. C})^9}{1^2 \cdot 2 \cdot 3^2 \cdot 4 \cdot 5^2 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \&c:$$

Arc. C

$$\text{Arc. C} = \frac{(\text{Cofec. C})^{-1}}{1} + \frac{1^2 \cdot (\text{Cofec. C})^{-3}}{1 \cdot 2 \cdot 3} + \frac{1^2 \cdot 3^2 (\text{Cofec. C})^{-5}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{1^2 \cdot 3^2 \cdot 5^2 (\text{Cofec. C})^{-7}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 (\text{Cofec. C})^{-9}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \&c.$$

Harumce Serierum relatio, et Harmonia pene mirabilis nullius, qui praedictorum meminerit, oculos fugient.

218. Sed neglectis iis Formulis, quas in numero praecedente 217. Lectorum vovimus sagacitati, tam magno divitiarum penu potitur Analyfis,

ut Secantes, Cofecantesque Hyperbolicas veras, scilicet $(1 + (T.H)^2)^{\frac{1}{2}}$, et

$(1 + (C.T.H)^2)^{\frac{1}{2}}$, ex nova methodo, tametsi minus eleganti, haurire possimus, vacuumque, quod adhuc restat in doctrina universali Trigonometrica, implere. Quum etenim sit Tang. H (x) =

$$\frac{x^1}{1^2} - \frac{x^3}{1^2 \cdot 3} + \frac{2x^5}{1^2 \cdot 3 \cdot 5} - \frac{17x^7}{1^2 \cdot 3^2 \cdot 5 \cdot 7} + \frac{62x^9}{1^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 9} - \&c., \text{ ex ea profluet Aequatio } (\text{Sec. H}(x))^2 =$$

$$1 + \left(\frac{x^1}{1^2} - \frac{x^3}{1^2 \cdot 3} + \frac{2x^5}{1^2 \cdot 3 \cdot 5} - \frac{17x^7}{1^2 \cdot 3^2 \cdot 5 \cdot 7} + \frac{62x^9}{1^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 9} - \&c. \right)^2, \text{ cuius ope,}$$

praecedentibus regulis exaltationis ad Potestates Binomii, vel Infinitinomii,

$$\text{elicietur Sec. H}(x) = 1 + \frac{x^2}{2} - \frac{11x^4}{24} + \frac{301x^6}{720} - \frac{16631x^8}{40320} + \&c.$$

$$= 1 + \frac{x^2}{1 \cdot 2} - \frac{11x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{301x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \frac{16631x^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \&c., \text{ eloquente ad-}$$

modum in Denominatoribus Lege servata. Eodem pacto, quum ex praecedentibus habeatur x =

$$t + \frac{t^3}{3} + \frac{t^5}{5} + \frac{t^7}{7} + \frac{t^9}{9} + \frac{t^{11}}{11} + \&c.$$

$$= t \left(1 + \frac{t^2}{3} + \frac{t^4}{5} + \frac{t^6}{7} + \frac{t^8}{9} + \frac{t^{10}}{11} + \frac{t^{12}}{13} + \&c. \right), \text{ sitque}$$

Tangens Hyperbolica $t = \sqrt{z^2 - 1}$, qua in Aequatione z Secantem exhibeat pariter Hyperbolicam, emerget post congruas substitutiones Series in-

$$\text{versa } x = (z^2 - 1)^{\frac{1}{2}} \left(1 + \frac{(z^2 - 1)^1}{3} + \frac{(z^2 - 1)^2}{5} + \frac{(z^2 - 1)^3}{7} + \frac{(z^2 - 1)^4}{9} + \frac{(z^2 - 1)^5}{11} + \&c. \right)$$

$+\frac{(z^2-1)^5}{11} + \frac{(z^2-1)^6}{13} + \frac{(z^2-1)^7}{15} + \&c:)$; et huius Formulae omnis

Asymmetria in unicum Factorem reiiicitur. Nec diversus de Cofecantibus

sermo. Cofecans etenim Hyperbolica $v = \left(1 + \left(\frac{x^{-1}}{1} + \frac{x^1}{1.3^1} - \right. \right.$

$$\left. \frac{x^3}{1.3^2.5} + \frac{2x^5}{1.3^3.5.7} - \frac{x^7}{1.3^3.5^2.7} + \frac{2x^9}{1.3^3.5.7.9.11} - \frac{1382x^{11}}{1.3^4.5^3.7^2.9.11.13} + \&c: \right)^{\frac{1}{2}}$$

$$= x^{-1} + \frac{5x^1}{1.2.3^1} - \frac{113x^3}{1.2.3^2.4.5} + \frac{775x^5}{1.2^2.3^3.4.7} - \&c:; \text{ eiusque inversam}$$

Formula tribuit ex num^o. 214. simpliciter derivanda $x = (v^2 - 1)^{-\frac{1}{2}}$

$$\left(1 + \frac{(v^2-1)^{-1}}{3} + \frac{(v^2-1)^{-2}}{5} + \frac{(v^2-1)^{-3}}{7} + \frac{(v^2-1)^{-4}}{9} + \right.$$

$$\left. \frac{(v^2-1)^{-5}}{11} + \&c: \right), \text{ quam in promptu est quomodolibet prosequi.}$$

219. Dum itaque quaestio fieret non de peregrinis, et Harmoniae integritati nimium devotis, sed de veris Secantium, Cofecantiumque Hyperbolicarum directis, ac inversis Aequationibus, eae ita disponi poterunt alias superius expositas aemulantes.

$$\text{Vera Sec. H} = 1 + \frac{(2 \text{ Sect. H})^2}{1.2} - \frac{11(2 \text{ Sect. H})^4}{1.2.3.4} + \frac{301(2 \text{ Sect. H})^6}{1.2.3.4.5.6} - \frac{16631(2 \text{ Sect. H})^8}{1.2.3.4.5.6.7.8} + \&c:$$

$$2 \text{ Sect. H} = ((\text{Sec. H})^2 - 1)^{\frac{1}{2}} \left(1 + \frac{((\text{Sec. H})^2 - 1)^1}{3} + \frac{((\text{Sec. H})^2 - 1)^2}{5} + \frac{((\text{Sec. H})^2 - 1)^3}{7} + \frac{((\text{Sec. H})^2 - 1)^4}{9} + \&c: \right)$$

$$\text{Vera Cofec. H} = \frac{(2 \text{ Sect. H})^{-2}}{1} + \frac{5(2 \text{ Sect. H})^1}{6} - \frac{113(2 \text{ Sect. H})^3}{360} + \frac{775(2 \text{ Sect. H})^5}{3024} - \&c:$$

$$2 \text{ Sect. H} = ((\text{Cofec. H})^2 - 1)^{-\frac{1}{2}} \left(1 + \frac{((\text{Cofec. H})^2 - 1)^{-1}}{3} + \frac{((\text{Cofec. H})^2 - 1)^{-2}}{5} + \frac{((\text{Cofec. H})^2 - 1)^{-3}}{7} + \&c: \right).$$

Quin etiam, si minor elegantia optaretur, diversas quoque ab iis recens traditis in calce numⁱ. 217. Series pro Circulo perhibere datum foret eadem penitus methodo, qua nunc in Hyperbola usi sumus; illaeque Series, utcunque diversimode expositae, aequalem nihilominus, respectu habito ad

E e e

alias

alias numⁱ. praecitati, tribuerent valorem. Radius praeterea, qui semper Unitatem aequat, ideoque manet constans in Circulo, variationem patitur in Hyperbola, novaque huius intuitu in censum venit Linea ad Trigonometriam oecumenicam spectans. Profecto Quadratum Radii Aequilaterae Hyperbolae Apolloniānae, sive $(R.H)^2 = (\text{Cofin. } H)^2 + (\text{Sin. } H)^2 =$

$$\left(\frac{C^x + C^{-x}}{2} \right)^2 + \left(\frac{C^x - C^{-x}}{2} \right)^2 = \frac{C^{2x} + C^{-2x}}{2} = \frac{C^{4x} + 1}{2C^{2x}} = \frac{v^2 + 1}{2v} \\ = \left(\frac{2v}{v^2 + 1} \right)^{-1}, \text{ quibus in Aequationibus } v \text{ peraequet Exponentialem Magnitu-}$$

dinem C^{2x} , et x de more duplum Centralis Hyperbolici Sectoris repraesentet, sive Pseudo-Arcum Hyperbolae Analogum. Hinc methodi ad instar adhibitae in num^o. 207. Radii expressionem inveniremus: sed altera commodior aperitur via huic valori adipiscendo non inidonea. Sinubus quippe, Cofinubusque Hyperbolicis revocatis habebitur $R^2 = \left(1 + \frac{x^2}{1.2} + \right.$

$$\left. \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} \&c: \right)^2 + \left(\frac{x^1}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \&c: \right)^2 = 1 \\ + \frac{2x^2}{1} + \frac{2x^4}{3} + \frac{4x^6}{3^2.5} + \&c:; \text{ quamobrem Radius Hyperbolicus } R = 1 \\ + \frac{x^2}{1} - \frac{x^4}{1.2.3} + \frac{19x^6}{1.2.3^2.5} - \&c:, \text{ cuius Formulae conversa erit } x =$$

$$\left((R.H - 1)^{\frac{1}{2}} \right) \left(1 + \frac{(R.H - 1)^1}{12} - \frac{61(R.H - 1)^2}{720} \&c: \right) \text{ in Infinitum: adeo,}$$

ut, dum in Circulo

$$\text{Rad. } C. = 1$$

Arc. C., sive 2 Sect. C = Magnitudini indefinitae = m . $R = m.1 = m$, sit

$$\text{Rad. } H. = 1 + \frac{(2 \text{ Sect. } H)^2}{1} - \frac{(2 \text{ Sect. } H)^4}{1.2.3^1} + \frac{19(2 \text{ Sect. } H)^6}{1.2.3^2.5} - \&c:$$

$$2 \text{ Sect. } H. = \left((\text{Rad. } H - 1) \right)^{\frac{1}{2}} \left(1 + \frac{(\text{Rad. } H - 1)^1}{1.2^2.3^1} - \frac{61(\text{Rad. } H - 1)^2}{1.2^4.3^2.5} \&c: \right),$$

quas Series parva praeditas utilitate quilibet proprio Marte continuabit.

220. In comperto nunc erunt Radices omnium Infinitinomialium, quae, initio praesertim desumpto a num°. 207., Lineis Trigonometricis adscribantur. Series namque Sinus-rectos, Cofinusque Hyperbolicos comprehendentes, qui Linearum utriusque Trigonometriae, de quibus in praesentiarum loquimur, sunt elementa, non modo Radices habent analogas illis Circularium Serierum, verum etiam eadem ratione in Factores resolvuntur. Hisce repetitis, et primum Tangentium inita argumentatione, monemus, aut

potius memoriae revocamus, Infinitinomialium notissimum $x \left(1 + \frac{x^2}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6.7} + \frac{x^8}{1.2.3.4.5.6.7.8.9} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10.11} + \&c: \right)$
 $= \text{Sin. H. } x = \frac{C^x - C^{-x}}{2}$ tum evanescere, quum $C^x = \frac{1}{C^x}$, cui Aequationi simplicissimae innumeris modis fit satis ponendo $x = \pm nP\sqrt{-1}$

quo facto revera nanciscimur $C^x = C^{\pm nP\sqrt{-1}} = \text{Cofin. Circ. } \pm nP + (\text{Sin. Circ. } \pm nP)\sqrt{-1} = \pm 1$, pariterque $\frac{1}{C^x}$, seu $C^{-x} = C^{\mp nP\sqrt{-1}} =$

$\text{Cofin. Circ. } \mp nP + (\text{Sin. Circ. } \mp nP)\sqrt{-1} = \pm 1$. Quaesitae ergo Radices evadent, prouti superius vidimus parum diversa arguendi methodo, $x = \pm nP\sqrt{-1}$, scilicet, 0, $\pm P\sqrt{-1}$, $\pm 2P\sqrt{-1}$, $\pm 3P\sqrt{-1}$, $\pm 4P\sqrt{-1}$, $\pm 5P\sqrt{-1}$ &c: sine ullo Limite, et idcirco universae, praeter priorem, quod et alias fuit dictum, Imaginariae. Ex quibus inventis emerget Theorema

Radices tribuens numero Infinitas $\pm nP\sqrt{-1}$ Aequationi $x \left(1 + \frac{x^2}{3} + \frac{2x^4}{15} - \frac{17x^6}{315} + \frac{62x^8}{2835} - \frac{1382x^{10}}{155925} + \&c: \right) = 0$, nimirum, Tang. H. x

$= 0$, utpote quae vi numeri 206. annihilatur toties quoties Sin. H. x evanescit, idipsum confirmante Formula $\frac{C^{2x} - 1}{C^{2x} + 1}$, cuius Numerator in Nihilum abit, quando $C^{2x} = 1$, vel $2x = \pm n\pi\sqrt{-1}$, aut $x = \pm nP\sqrt{-1}$.

Eaedem igitur ad amissim erunt Radices Infinitinomialis Aequationis

$x(1$

$$x \left(1 - \frac{x^2}{1^2 \cdot 3} + \frac{2x^4}{1^2 \cdot 3 \cdot 5} - \frac{17x^6}{1^2 \cdot 3^2 \cdot 5 \cdot 7} + \frac{62x^8}{1^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 9} - \frac{1382x^{10}}{1^3 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11} + \frac{21844x^{12}}{1^3 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11 \cdot 13} - \&c. \right) = 0$$

$$\text{ac quae ad alteram pertinent Sinui Hyperbolico recto respondentem } x \left(1 + \frac{x^2}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{x^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \frac{x^{10}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \frac{x^{12}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13} + \&c. \right)$$

$= 0$. Neque haec Radicum aequipollentia in Formulis tam immaniter discrepantibus mirabilis videatur, quum ex praeostensis Infinitinomialium Tangentem Hyperbolicam praebens ab Sinu-recto per Cofinum Hyperbolicum diviso nascatur; unde fit, ut non solummodo congruant Radices iam me-

$$\text{moratae, sed effluat praeterea } x - \frac{x^3}{1^2 \cdot 3} + \frac{2x^5}{1^2 \cdot 3 \cdot 5} - \frac{17x^7}{1^2 \cdot 3^2 \cdot 5 \cdot 7} + \frac{62x^9}{1^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 9} - \frac{1382x^{11}}{1^3 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11} + \frac{21844x^{13}}{1^3 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11 \cdot 13} - \&c. =$$

$$x \left(\frac{\left(1 + \frac{x^2}{1^2 \cdot P^2}\right) \left(1 + \frac{x^2}{2^2 \cdot P^2}\right) \left(1 + \frac{x^2}{3^2 \cdot P^2}\right) \left(1 + \frac{x^2}{4^2 \cdot P^2}\right) \&c.}{\left(1 + \frac{x^2}{1^2 \cdot Q^2}\right) \left(1 + \frac{x^2}{3^2 \cdot Q^2}\right) \left(1 + \frac{x^2}{5^2 \cdot Q^2}\right) \left(1 + \frac{x^2}{7^2 \cdot Q^2}\right) \&c.} \right), \text{ vel potius } =$$

$$x \left(\frac{\left(1 + \frac{x^2}{P^2}\right) \left(1 + \frac{x^2}{4P^2}\right) \left(1 + \frac{x^2}{9P^2}\right) \left(1 + \frac{x^2}{16P^2}\right) \left(1 + \frac{x^2}{25P^2}\right) \left(1 + \frac{x^2}{36P^2}\right) \&c.}{\left(1 + \frac{4x^2}{P^2}\right) \left(1 + \frac{4x^2}{9P^2}\right) \left(1 + \frac{4x^2}{25P^2}\right) \left(1 + \frac{4x^2}{49P^2}\right) \left(1 + \frac{4x^2}{81P^2}\right) \left(1 + \frac{4x^2}{121P^2}\right) \&c.} \right)$$

in Infinitum.

221. Cotangens autem Hyperbolica comprehensa ab Infinitinomio $x^{-1} + \frac{x}{3}$

$$- \frac{x^3}{45} + \frac{2x^5}{945} - \frac{x^7}{4725} + \frac{2x^9}{93555} - \&c.; \text{ eo quod ortum ducat a Formula}$$

$$\frac{C^{2x} + 1}{C^{2x} - 1}, \text{ et, respectu habito ad priorem, istius sit conversa, nullefcit eva-}$$

nesciente Numeratore $C^{2x} + 1$, scilicet, tum quum $2x = \pm \lambda P \sqrt{-1}$, vel $x = \pm \lambda Q \sqrt{-1}$, quemadmodum toties repetita nobis suadent. Data ergo Aequa-

Aequatione $\frac{1}{x} \left(1 + \frac{x^2}{1 \cdot 3^1} - \frac{x^4}{1 \cdot 3^2 \cdot 5} + \frac{2x^6}{1 \cdot 3^3 \cdot 5 \cdot 7} - \frac{x^8}{1 \cdot 3^3 \cdot 5^2 \cdot 7} + \right.$
 $\left. \frac{2x^{10}}{1 \cdot 3^3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} - \frac{1382x^{12}}{1 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 9 \cdot 11 \cdot 13} \&c: \right) = 0$, eius Radices conveniunt cum iis

tam diversae Aequationis a Cosinu deductae Hyperbolico $1 + \frac{x^2}{1 \cdot 2} +$

$\frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{x^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \frac{x^{10}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} + \&c: = 0$,

quae Radices ex num°. 195. sunt $x = \pm Q\sqrt{-1}$, $\pm 3Q\sqrt{-1}$, $\pm 5Q\sqrt{-1}$, $\pm 7Q\sqrt{-1}$, $\pm 9Q\sqrt{-1}$ &c: imparibus Arcuum Quadrantalium Multiplis absque Limite continuatis. Ab eodem fonte dimanat resolutio in Factores, quae praebet

$\frac{1}{x} \left(1 + \frac{x^2}{3} - \frac{x^4}{45} + \frac{2x^6}{945} - \frac{x^8}{4725} + \frac{2x^{10}}{93555} - \frac{1382x^{12}}{638512875} + \&c: \right)$

$= \left(\frac{(1 + \frac{4x^2}{P^2})(1 + \frac{4x^2}{9P^2})(1 + \frac{4x^2}{25P^2})(1 + \frac{4x^2}{49P^2}) \&c:}{(x(1 + \frac{x^2}{P^2})(1 + \frac{x^2}{4P^2})(1 + \frac{x^2}{9P^2})(1 + \frac{x^2}{16P^2}) \&c:)} \right)$, five elegan-

tioris forma selecta in numericis Factorum Coefficientibus enucleandis, aut potius scribendis ad instar systematis alias usitati =

$\left(\frac{(1 + \frac{4x^2}{P^2})(1 + \frac{4x^2}{3^2 \cdot P^2})(1 + \frac{4x^2}{5^2 \cdot P^2})(1 + \frac{4x^2}{7^2 \cdot P^2})(1 + \frac{4x^2}{9^2 \cdot P^2})(1 + \frac{4x^2}{11^2 \cdot P^2}) \&c:}{(x(1 + \frac{x^2}{P^2})(1 + \frac{x^2}{2^2 \cdot P^2})(1 + \frac{x^2}{3^2 \cdot P^2})(1 + \frac{x^2}{4^2 \cdot P^2})(1 + \frac{x^2}{5^2 \cdot P^2})(1 + \frac{x^2}{6^2 \cdot P^2}) \&c:)} \right)$.

Nec dubium promoveatur de neglecta quadam Radice in Factore $\frac{1}{x}$ com-

prehensa. Algebra etenim docet rite ordinandam esse Aequationem in Ra-

dicum investigatione; adeo, ut $\frac{1}{x} \left(1 + \frac{x^2}{1 \cdot 3^1} - \frac{x^4}{1 \cdot 3^2 \cdot 5} + \frac{2x^6}{1 \cdot 3^3 \cdot 5 \cdot 7} - \right.$

$\left. \frac{x^8}{1 \cdot 3^3 \cdot 5^2 \cdot 7} + \&c: \right) = 0$ disponi debeat hoc ordine $1 + \frac{x^2}{1 \cdot 3^1} - \frac{x^4}{1 \cdot 3^2 \cdot 5} +$

$\frac{2x^6}{1 \cdot 3^3 \cdot 5 \cdot 7} - \frac{x^8}{1 \cdot 3^3 \cdot 5^2 \cdot 7} + \frac{2x^{10}}{1 \cdot 3^3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} - \&c: = 0$. $x = 0$. Suspicio quidem for-

tasse oborta Radicis $x = \infty$ ab Aequationis propositae resolutione evidenter excluditur, quod etiam valet de Tangentiali Aequalitate superius tradita,

prouti Formulae ostendunt $\frac{C^{2x} + 1}{C^{2x} - 1} = 0$, ac $\frac{C^{2x} - 1}{C^{2x} + 1} = 0$, quarum

utraq. eo casu evaderet $\frac{C^{2 \cdot \infty}}{C^{2 \cdot \infty}} = 1 = 0$, et absurditatem involveret.

Quinimmo Formulis istis prorsus neglectis Symptomata simpliciora Apolloniae Aequilaterae Hyperbolae ultro demonstrant Sectori Centrali infinite magno non iam Nihilum pro Tangente, aut Cotangente, sed respondere Unitatem.

222. Ab Hyperbola ad Circulum transitu iuxta morem facto detegun-

tur Radices $x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \frac{1382x^{11}}{155925} + \&c: = 0$,

nimirum Aequationi Infinitinomiali Tangentis ad Arcum Circularem x relatae continget evanescencia quando $x = 0$, $\pm P$, $\pm 2P$, $\pm 3P$, $\pm 4P$, $\pm 5P$ &c: , ac generaliter $x = \pm nP$: tota etenim demonstratio in eo consistit, quod, dum in Formula analoga numⁱ. 220. vice x subrogetur $x\sqrt{-1}$,

emergat $x\sqrt{-1} \left(1 + \frac{x^2}{1^2 \cdot 3} + \frac{2x^4}{1^2 \cdot 3 \cdot 5} + \frac{17x^6}{1^2 \cdot 3^2 \cdot 5 \cdot 7} + \frac{62x^8}{1^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 9} + \right.$
 $\left. \frac{1382x^{10}}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11} + \frac{21844x^{12}}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11 \cdot 13} + \&c: \right) = 0$, scilicet, $x^2 + \frac{x^2}{3}$

$+ \frac{2x^4}{15} + \frac{17x^6}{315} + \frac{62x^8}{2835} + \frac{1382x^{10}}{155925} + \&c: = 0$, cuius ideo Aequatio-

nis Radices erunt $x\sqrt{-1} = \pm nP\sqrt{-1}$, vel $x = \pm nP$, uti passim superius innuimus. Exinde profluit resolutio in Factores; namque singulis inventis

Radicibus dubio carebit fore etiam $x + \frac{x^3}{1^2 \cdot 3} + \frac{2x^5}{1^2 \cdot 3 \cdot 5} + \frac{17x^7}{1^2 \cdot 3^2 \cdot 5 \cdot 7} +$

$\frac{62x^9}{1^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 9} + \frac{1382x^{11}}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11} + \frac{21844x^{13}}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 9 \cdot 11 \cdot 13} + \&c: =$

$x \left(\frac{\left(1 - \frac{x^2}{P^2}\right)\left(1 - \frac{x^2}{2^2 \cdot P^2}\right)\left(1 - \frac{x^2}{3^2 \cdot P^2}\right)\left(1 - \frac{x^2}{4^2 \cdot P^2}\right)\left(1 - \frac{x^2}{5^2 \cdot P^2}\right) \&c:}{\left(1 - \frac{4x^2}{P^2}\right)\left(1 - \frac{4x^2}{3^2 \cdot P^2}\right)\left(1 - \frac{4x^2}{5^2 \cdot P^2}\right)\left(1 - \frac{4x^2}{7^2 \cdot P^2}\right)\left(1 - \frac{4x^2}{9^2 \cdot P^2}\right) \&c:} \right),$

aut

aut, rursus aliis realibus Factoribus evolutis, qui in unoquoque secundi Gradus Binomio absconduntur, =

$$x \left(\frac{(1 + \frac{x}{P})(1 - \frac{x}{P})(1 + \frac{x}{2P})(1 - \frac{x}{2P})(1 + \frac{x}{3P})(1 - \frac{x}{3P})(1 + \frac{x}{4P})(1 - \frac{x}{4P}) \&c:}{(1 + \frac{2x}{P})(1 - \frac{2x}{P})(1 + \frac{2x}{3P})(1 - \frac{2x}{3P})(1 + \frac{2x}{5P})(1 - \frac{2x}{5P})(1 + \frac{2x}{7P})(1 - \frac{2x}{7P}) \&c:} \right)$$

in Infinitum. Ratione consimili evolvitur in respectivos Factores alterum

Infinitinomialium, quod Cotangentes Circulares determinat, nempe $x^{-1} -$

$$\frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \frac{x^7}{4725} + \frac{2x^9}{93555} - \frac{1382x^{11}}{638512875} + \&c:, \text{ cui pro-}$$

fecto aequabitur Productum indefinitum

$$\frac{1}{x} \left(\frac{(1 - \frac{2x}{P})(1 + \frac{2x}{P})(1 - \frac{2x}{3P})(1 + \frac{2x}{3P})(1 - \frac{2x}{5P})(1 + \frac{2x}{5P})(1 - \frac{2x}{7P})(1 + \frac{2x}{7P}) \&c:}{(1 - \frac{x}{P})(1 + \frac{x}{P})(1 - \frac{x}{2P})(1 + \frac{x}{2P})(1 - \frac{x}{3P})(1 + \frac{x}{3P})(1 - \frac{x}{4P})(1 + \frac{x}{4P}) \&c:} \right)$$

$$\text{aut } \left(\frac{(1 - \frac{4x^2}{P^2})(1 - \frac{4x^2}{3^2.P^2})(1 - \frac{4x^2}{5^2.P^2})(1 - \frac{4x^2}{7^2.P^2}) \&c:}{x(1 - \frac{x^2}{P^2})(1 - \frac{x^2}{2^2.P^2})(1 - \frac{x^2}{3^2.P^2})(1 - \frac{x^2}{4^2.P^2}) \&c:} \right), \text{ Sunt ete-}$$

nim Aequationis postremo loco propositae $\frac{1}{x} \left(1 - \frac{x^2}{1.3^2} - \frac{x^4}{1.3^2.5^2} - \right.$

$$\left. \frac{2x^6}{1.3^3.5^2.7^2} - \frac{x^8}{1.3^3.5^3.7^2} + \frac{2x^{10}}{1.3^3.5^3.7^2.9^2.11^2} - \frac{1382x^{12}}{1.3^4.5^3.7^2.9^2.11.13} + \&c: \right)$$

= 0 Radices $x\sqrt{-1} = \pm \lambda Q\sqrt{-1} = \pm \frac{\lambda P\sqrt{-1}}{2}$, scilicet, $x = \pm Q$,

$$\pm 3Q, \pm 5Q, \pm 7Q \&c:, \text{ vel } \pm \frac{P}{2}, \pm \frac{3P}{2}, \pm \frac{5P}{2}, \pm \frac{7P}{2}, \pm \frac{9P}{2} \&c:,$$

quomodo superiores Formulae docent, convalidantque plenissimo rationum pondere Trigonometricae Scientiae Elementa.

223. Idem de Secantibus, atque Cosecantibus Arcuum Peripheriae Circularis sermo potest institui, speculatione reiecta fictarum Hyperbolicae

Curvae

Curvae Secantium, et Cofecantium, utpote qui Analogia duce a Circularibus eas facile deducere, nec omnia potius exhaurire abundantius, et uni-

versaliter absoluta conscribere. Porro Fractio $\frac{2C^{x\sqrt{-1}}}{C^{2x\sqrt{-1}} + 1}$, Cofinus Circularis reciproca, Secantem Arcus x compendio prae omnibus elegantiore sup-

peditat, ac, sicuti evidenter nullefcit dum $C^{x\sqrt{-1}}$ in Nihilum, vel

$C^{2x\sqrt{-1}} + 1$ in Infinitum abeat, summa invenientur facilitate Radices Aequa-

tionis notiffimae $1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \frac{277x^8}{8064} + \frac{50521x^{10}}{3628800} + \&c. = 0.$

Nusquam autem evenire potest in casu x Realis, quod Infinita magnitudo Binomii

$C^{2x\sqrt{-1}} + 1$, vel Cofin. $C. 2x + (\text{Sin. } C. 2x) \sqrt{-1} + 1$ petitis Radicibus ansam praebeat, quia haec Denominatoris expressio omnigenam respuit Infinitatem. Dum vero hypothesis fieret x Imaginarij, puta $=$

$\frac{z}{\sqrt{-1}}$, ideoque $x\sqrt{-1} = z$, evidenter adparet non posse $C^{2z} + 1$, vel C^{2z}

in Denominatore Infinitum peraequare, nisi simul Infinite magnus evadat

Numerator $2C^z$, totaque idcirco vertatur Formula in $\frac{2 \cdot \infty}{\infty^2}$, five in $\frac{2}{\infty} =$

0, quae suppositio $x = \frac{z}{\sqrt{-1}}$, nimirum universaliffimi valoris simplicis-Ima-

ginarij, ad nostram rem aliquantulum pertinet, tametsi *fictitiam* directe respiciat

Hyperbolicae Curvae Secantem. Numerator deinde $2C^{x\sqrt{-1}}$, scilicet, 2 (Cofin. $C. x + (\text{Sin. } C. x) \sqrt{-1}$) nequidem annihilari unquam potest exce-

pto casu $x = \frac{z}{\sqrt{-1}} = -\frac{\infty}{\sqrt{-1}}$, videlicet $x\sqrt{-1} = -\infty$. Innumerae

itaque Radices, quae adsignatae Aequationi convenient, Infinite magnum valorem Imaginarium omnes praeseferunt, suntque generaliter comprehensae

ab unica Formula $x = \pm \infty \sqrt{-1} = \frac{\pm \infty}{\sqrt{-1}}$. Hanc ipsam investigationem

confirmat

confirmat resolutio Infinitinomii in Factores, quae a Cofinus Circularis Infinitinomio dudum evoluto statim consequitur, tribuitque $1 + \frac{x^2}{1.2} +$

$$\frac{5x^4}{1.2.3.4} + \frac{61x^6}{1.2.3.4.5.6} + \frac{5.277x^8}{1.2.3.4.5.6.7.8} + \frac{50521x^{10}}{1.2.3.4.5.6.7.8.9.10} +$$

$$\frac{5.540553x^{12}}{1.2.3.4.5.6.7.8.9.10.11.12} + \&c. =$$

$$\frac{1}{(1 - \frac{2x}{P})} \cdot \frac{1}{(1 + \frac{2x}{P})} \cdot \frac{1}{(1 - \frac{2x}{3P})} \cdot \frac{1}{(1 + \frac{2x}{3P})} \cdot \frac{1}{(1 - \frac{2x}{5P})} \cdot \frac{1}{(1 + \frac{2x}{5P})} \cdot \frac{1}{(1 - \frac{2x}{7P})} \cdot \frac{1}{(1 + \frac{2x}{7P})} \&c.$$

aut, si mavis,

$$\frac{1}{(1 - \frac{4x^2}{1^2.P^2})} \cdot \frac{1}{(1 - \frac{4x^2}{3^2.P^2})} \cdot \frac{1}{(1 - \frac{4x^2}{5^2.P^2})} \cdot \frac{1}{(1 - \frac{4x^2}{7^2.P^2})} \cdot \frac{1}{(1 - \frac{4x^2}{9^2.P^2})} \cdot \frac{1}{(1 - \frac{4x^2}{11^2.P^2})} \cdot \frac{1}{(1 - \frac{4x^2}{13^2.P^2})} \&c.$$

Consimiliter obtinetur innumerorum Factorum Series, ex quibus invicem multiplicatis existit alterum Infinitinomium Cofecantem Arcus Circuli x complectens: Sinus etenim recti evolutione in subsidium vocata emergit Aequalitas

$$\text{inter } \frac{x^{-1}}{1^2} + \frac{x^1}{1^2.2.3} + \frac{7x^3}{1^2.2.3^2.4.5} + \frac{31x^5}{1^2.2.3^2.4.5.6.7} + \frac{9.127x^7}{1^2.2.3^2.4.5^2.6.7.8.9} +$$

$$\frac{7.25.73x^9}{1^2.2.3^2.4.5^2.6.7.8.9.10.11} + \&c., \text{ atque}$$

$$x \left(1 - \frac{x}{P}\right) \left(1 + \frac{x}{P}\right) \left(1 - \frac{x}{2P}\right) \left(1 + \frac{x}{2P}\right) \left(1 - \frac{x}{3P}\right) \left(1 + \frac{x}{3P}\right) \left(1 - \frac{x}{4P}\right) \left(1 + \frac{x}{4P}\right) \&c.$$

$$\text{vel } x \left(1 - \frac{x^2}{1^2.P^2}\right) \left(1 - \frac{x^2}{2^2.P^2}\right) \left(1 - \frac{x^2}{3^2.P^2}\right) \left(1 - \frac{x^2}{4^2.P^2}\right) \left(1 - \frac{x^2}{5^2.P^2}\right) \&c.$$

haecque resolutio simul cum Formula in num°. 213 exposita $\sqrt{-1} \left(\frac{2C^{x\sqrt{-1}}}{C^{2x\sqrt{-1}} - 1} \right)$

G g g

fua

sua veluti sponte decernit nullas esse Aequationis $\frac{x^{-1}}{1} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120}$
 $+ \frac{127x^7}{604800} + \frac{73x^9}{3421440} + \frac{1414477x^{11}}{653837184000} + \&c: = 0$ Radices Reales, verunta-

men ex opposito universas Imaginarias, et ab oecumenica expressione $x \sqrt{-1} = \pm \infty$, aut $x = \mp \infty \sqrt{-1}$ omnimode deducendas.

224. Subtilior videtur ea disquisitio, quae Versis-Sinibus, eorumque affectionibus applicatur. Dolosa etenim in eis resolvendis facilitas inest, quae, nisi summo studio emendaretur, contradictionem pareret in re maioris momenti, cui in Capite VI°. navabimus operam. Ab Hyperbolico Si-

nu-verso incipientes monemus ipsius expressionem canonicam $\frac{x^2}{2} \left(1 + \right.$

$$\frac{x^2}{3 \cdot 4} + \frac{x^4}{3 \cdot 4 \cdot 5 \cdot 6} + \frac{x^6}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \frac{x^8}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} + \frac{x^{10}}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12} +$$

$$\frac{x^{12}}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14} + \&c: = 0, \text{ sive finito terminorum numero cir-}$$

cumscriptam $\frac{C^x + C^{-x}}{2} - 1 = 0$ habere Radices $x = \pm 0, \pm \Pi \sqrt{-1},$

$\pm 2 \Pi \sqrt{-1}, \pm 3 \Pi \sqrt{-1}, \&c:$, seu compendio eloquentiore $x = \pm n \Pi \sqrt{-1}$, quum reapse hoc valore in Exponentiali Formula subrogato illa fiat

$$\frac{C^{\pm n \Pi \sqrt{-1}} + C^{\mp n \Pi \sqrt{-1}}}{2} - 1 = 0$$

$$- 1 = \text{Cosin. Circ. } (\pm n \Pi) - 1 = 1 - 1$$

$= 0$. Hisce repertis sine fine Radicibus quisquis sibi statim suaderet fore

$$\frac{x^2}{2} \left(1 + \frac{x^2}{12} + \frac{x^4}{360} + \frac{x^6}{20160} + \frac{x^8}{1814400} + \frac{x^{10}}{239500800} + \&c: = \right.$$

$$\frac{x}{\sqrt{2}} \cdot \frac{x}{\sqrt{2}} \left(1 - \frac{x}{\Pi \sqrt{-1}} \right) \left(1 + \frac{x}{\Pi \sqrt{-1}} \right) \left(1 - \frac{x}{2 \Pi \sqrt{-1}} \right) \left(1 + \frac{x}{2 \Pi \sqrt{-1}} \right)$$

$$\left(1 - \frac{x}{3 \Pi \sqrt{-1}} \right) \left(1 + \frac{x}{3 \Pi \sqrt{-1}} \right) \&c:, \text{ aut } = \frac{x^2}{2} \left(1 + \frac{x^2}{\Pi^2} \right) \left(1 + \frac{x^2}{4 \Pi^2} \right)$$

$$\left(1 + \frac{x^2}{9 \Pi^2} \right) \left(1 + \frac{x^2}{16 \Pi^2} \right) \left(1 + \frac{x^2}{25 \Pi^2} \right) \left(1 + \frac{x^2}{36 \Pi^2} \right) \&c: \text{ in Infinitum;}$$

quae

quae illatio ex supra positis errorem sane maximum continet. Ad detegendum ratiocinationis Analyticae vitium peculiari castigatione opus est, propterea quod iidem inconcussi manere debeant valores Radicum, sed non tantum iidem supra explicati Factores.

225. Formula illa Sinum-versum complectens $\frac{C^x + C^{-x} - 2}{2} = 0$, nimirum, dempto Denominatore, $C^x + C^{-x} - 2 = 0$ redigitur in

$\left(C^{\frac{x}{2}} - C^{-\frac{x}{2}}\right)^2 = 0$, quemadmodum facile est experiri. Hoc periculo facto non iniucundum Theorema nostra methodo enucleatum obiter nacti erimus in Aequatione comprehensum Sin. V. Hyp. $x = \frac{C^x + C^{-x}}{2} - 1 =$

$\frac{\left(C^{\frac{x}{2}} - C^{-\frac{x}{2}}\right)^2}{2} = 2 \left(\frac{C^{\frac{x}{2}} - C^{-\frac{x}{2}}}{2} \right)^2 = 2 \left(\text{Sin. Hyp. } \frac{x}{2} \right)^2$, prouti apud Trigonometras prostat, et aequali ritu una cum ceteris analogis expressionibus Circulo aptatur. Infinitinomium ergo $\frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} +$

$\frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c:$ resolvitur in $2 \left(\frac{x^2}{2^1} + \frac{x^4}{2^3.2.3} + \frac{x^6}{2^5.2.3.4.5} + \frac{x^8}{2^7.2.3.4.5.6.7} + \&c: \right) \left(\frac{x^2}{2^1} + \frac{x^4}{2^3.2.3} + \frac{x^6}{2^5.2.3.4.5} + \frac{x^8}{2^7.2.3.4.5.6.7} + \&c: \right)$, ex quorum Productis evalescit. Infinitinomialium praeterea, quae postremo loco notantur, alterutrum vi numeri 195,

evolvitur in Factores $\frac{x}{2} \left(1 + \frac{x^2}{2^2.P^2} \right) \left(1 + \frac{x^2}{2^2.2^2.P^2} \right) \left(1 + \frac{x^2}{2^2.3^2.P^2} \right) \left(1 + \frac{x^2}{2^2.4^2.P^2} \right) \left(1 + \frac{x^2}{2^2.5^2.P^2} \right) \&c:$, vel simplicius $\frac{x}{2} \left(1 + \frac{x^2}{1^2.\Pi^2} \right) \left(1 + \frac{x^2}{2^2.\Pi^2} \right) \left(1 + \frac{x^2}{3^2.\Pi^2} \right) \left(1 + \frac{x^2}{4^2.\Pi^2} \right) \left(1 + \frac{x^2}{5^2.\Pi^2} \right) \&c:$, quamobrem

erit $\frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c:$

$$\begin{aligned} \&c. = 2 \left(\text{Sin. Hyp. } \frac{x}{2} \right)^2 = \frac{x^2}{2} \left(1 + \frac{x^2}{1^2 \cdot \Pi^2} \right) \left(1 + \frac{x^2}{1^2 \cdot \Pi^2} \right) \\ &\left(1 + \frac{x^2}{2^2 \cdot \Pi^2} \right) \left(1 + \frac{x^2}{2^2 \cdot \Pi^2} \right) \left(1 + \frac{x^2}{3^2 \cdot \Pi^2} \right) \left(1 + \frac{x^2}{3^2 \cdot \Pi^2} \right) \left(1 + \frac{x^2}{4^2 \cdot \Pi^2} \right) \\ &\left(1 + \frac{x^2}{4^2 \cdot \Pi^2} \right) \left(1 + \frac{x^2}{5^2 \cdot \Pi^2} \right) \left(1 + \frac{x^2}{5^2 \cdot \Pi^2} \right) \&c., \text{ ubi observandum congemini} \end{aligned}$$

nari Factores, firmisque manentibus universis earum valoribus numerum etiam congeminari Radicum. En alterum gravissimae curae adhibendae in Factorum Polynomii cuiuscunque derivatione ex prius repertis Radicibus argumentum (*).

226. Prolixius equidem, at eximio miroque consensu eadem Radicum Factorumque iteratio a numeri 197. iam explicita Theorice comprobaretur. Formula namque $C^x - 2 \text{Cofin. Circ. } A + C^{-x}$ in $C^x + C^{-x} - 2$ tum vertitur, quum Arcus $A = 0$. Atqui fuit illic ostensum $C^x - 2 \text{Cofin. Circ. } A + C^{-x} = 2 \left(1 - \text{Cofin. Circ. } A \right) \left(1 + \frac{x^2}{A^2} \right) \left(1 + \frac{x^2}{(\Pi - A)^2} \right)$

$$\begin{aligned} &\left(1 + \frac{x^2}{(\Pi - A)^2} \right) \left(1 + \frac{x^2}{(2\Pi - A)^2} \right) \left(1 + \frac{x^2}{(2\Pi - A)^2} \right) \left(1 + \frac{x^2}{(3\Pi - A)^2} \right) \\ &\left(1 + \frac{x^2}{(3\Pi - A)^2} \right) \&c.; \text{ igitur evanescente } A \text{ profluet } \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} \\ &+ \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c. = \text{Sin. V.} \end{aligned}$$

$$\text{Hyp. } x = \frac{C^x + C^{-x} - 2}{2} = (\text{Sin. V. Circ. } 0) \left(1 + \frac{x^2}{(\text{Arc. C. } 0)^2} \right)$$

$$\begin{aligned} &\left(1 + \frac{x^2}{\Pi^2} \right) \left(1 + \frac{x^2}{\Pi^2} \right) \left(1 + \frac{x^2}{2^2 \cdot \Pi^2} \right) \left(1 + \frac{x^2}{2^2 \cdot \Pi^2} \right) \left(1 + \frac{x^2}{3^2 \cdot \Pi^2} \right) \left(1 + \frac{x^2}{3^2 \cdot \Pi^2} \right) \\ &\&c.; \text{ vel ob notissimas in Circulo ultimas primasque Rationes } = \frac{x^2}{2} \left(1 + \frac{x^2}{\Pi^2} \right) \end{aligned}$$

(*) Non absimilia recensui in numeris 180. 181. 183. 201. summo libranda sedulitatis conatu.

$$\left(1 + \frac{x^2}{\Pi^2}\right)\left(1 + \frac{x^2}{\Pi^2}\right)\left(1 + \frac{x^2}{4\Pi^2}\right)\left(1 + \frac{x^2}{4\Pi^2}\right)\left(1 + \frac{x^2}{9\Pi^2}\right)\left(1 + \frac{x^2}{9\Pi^2}\right)$$

$$\left(1 + \frac{x^2}{16\Pi^2}\right)\left(1 + \frac{x^2}{16\Pi^2}\right)\left(1 + \frac{x^2}{25\Pi^2}\right)\left(1 + \frac{x^2}{25\Pi^2}\right) \&c: \text{ absque ullo}$$

Limite. Nec dubium promoveri Elementaris patitur Geometria circa valo-

$$\text{rem } \sin. V. C. \text{ evan.} \propto \left(1 + \frac{x^2}{(\text{Arc. C. evan.})^2}\right), \text{ nimirum, } \frac{x^2 \cdot \sin. V. C. \text{ evan.}}{(\text{Arc. C. evan.})^2},$$

$$\text{ex eo quod } \sin. V. C. \text{ evan.} = \frac{(\text{Arc. C. evan.})^2}{2}, \text{ et propterea}$$

$$\frac{(\sin. V. C. \text{ evan.}) \cdot x^2}{(\text{Arc. C. evan.})^2} = \frac{x^2}{2}, \text{ in subsidium nequidem vocata Regula Bernoul-}$$

liana pro Fractionis huiusce $\frac{0}{0}$ valore definiendo, qui fuit superius con-

scriptus. Nihilominus haec, utcumque facillima, Infinitesimae Quantitatis tra-
ctatio in caussa fuit, cur praecedentis numeri demonstrationem magis directo
tramite instituerim.

227. Omnia recens allata, mutatis mutandis, sunt Circulo quoque com-
munia. Namque unicum intercedit discrimen, quod Indicem x Realem ab
 $x\sqrt{-1}$ Imaginario secernit, et haudquaquam curanda manet ordinis inversio

$$\frac{x\sqrt{-1}}{2} - C - \frac{-x\sqrt{-1}}{2} = 0, \text{ quum redeat eodem non secus ac ordo di-}$$

$$\text{rectus } 0 = \frac{C - \frac{x\sqrt{-1}}{2} + \frac{-x\sqrt{-1}}{2}}{2}, \text{ veluti vulgaris Algebra docet. Pla-}$$

cuit hic obiter, et unius exempli loco subiungere, quod $\sin. V. \text{ Circ } x =$

$$\frac{x^2}{1.2} - \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} - \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} -$$

$$\frac{x^{12}}{1.2.3.4.5.6.7.8.9.10.11.12} + \&c: \text{, five } \frac{x^2}{2} \left(1 - \frac{x^2}{12} + \frac{x^4}{360} - \frac{x^6}{20160} + \right.$$

$$\left. \frac{x^8}{1814400} - \frac{x^{10}}{239500800} + \&c: = \frac{x}{\sqrt{2}} \cdot \frac{x}{\sqrt{2}} \left(1 - \frac{x}{\Pi}\right) \left(1 + \frac{x}{\Pi}\right)\right.$$

Hhh

(1 —

$$\left(1 - \frac{x}{\Pi}\right)\left(1 + \frac{x}{\Pi}\right)\left(1 - \frac{x}{2\Pi}\right)\left(1 + \frac{x}{2\Pi}\right)\left(1 - \frac{x}{2\Pi}\right)\left(1 + \frac{x}{2\Pi}\right)$$

$$\left(1 - \frac{x}{3\Pi}\right)\left(1 + \frac{x}{3\Pi}\right)\left(1 - \frac{x}{3\Pi}\right)\left(1 + \frac{x}{3\Pi}\right) \&c., \text{ seu Factorum Binariis in-}$$

$$\text{vicem multiplicatis} = \frac{x^2}{2} \left(1 - \frac{x^2}{1^2 \cdot \Pi^2}\right)\left(1 - \frac{x^2}{1^2 \cdot \Pi^2}\right)\left(1 - \frac{x^2}{2^2 \cdot \Pi^2}\right)$$

$$\left(1 - \frac{x^2}{2^2 \cdot \Pi^2}\right)\left(1 - \frac{x^2}{3^2 \cdot \Pi^2}\right)\left(1 - \frac{x^2}{3^2 \cdot \Pi^2}\right)\left(1 - \frac{x^2}{4^2 \cdot \Pi^2}\right)\left(1 - \frac{x^2}{4^2 \cdot \Pi^2}\right)$$

$$\left(1 - \frac{x^2}{5^2 \cdot \Pi^2}\right)\left(1 - \frac{x^2}{5^2 \cdot \Pi^2}\right) \&c.. \text{ Abunde etiam liquet Aequationi Infini-}$$

tinomiae superius traditae, respectu ad Hyperbolen habito, ac nunc Circu-

$$\text{lo applicatae } \frac{x^2}{1.2} - \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} - \frac{x^8}{1.2.3.4.5.6.7.8} +$$

$$\frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} - \&c. = \text{Sin. V. Circ. } x = 0 \text{ respondere Radices}$$

0, 0, Π , Π , $-\Pi$, $-\Pi$, 2Π , 2Π , -2Π , -2Π , 3Π , 3Π , -3Π , -3Π , &c., aut $\pm n\Pi$, $\pm n\Pi$, videlicet ad totam Aequationis vim superandam singulas repetitas.

228. Proderit demum ad consequentium illustrationem superaddere istius diligentissimi Canonis specimen aliud in Circuli Secantibus evolvendis, cuius amplificationem simillimam ad Cosecantes rerum mole perterrefactus libenti animo omitto. Nemo non videt Aequationis veritatem $\text{Sec. Circ. } x = 1$

$$= \frac{1 - \text{Cofin. Circ. } x}{\text{Cofin. C. } x} = \frac{\text{Sin. V. Circ. } x}{\text{Cofin. Circ. } x} = \frac{x^2}{1.2} + \frac{5x^4}{1.2.3.4} + \frac{61x^6}{1.2.3.4.5.6} +$$

$$\frac{5.277x^8}{1.2.3.4.5.6.7.8} + \frac{50521x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c. =$$

$$\frac{\frac{x^2}{2} \left(1 - \frac{x^2}{1^2 \cdot \Pi^2}\right)\left(1 - \frac{x^2}{1^2 \cdot \Pi^2}\right)\left(1 - \frac{x^2}{2^2 \cdot \Pi^2}\right)\left(1 - \frac{x^2}{2^2 \cdot \Pi^2}\right)\left(1 - \frac{x^2}{3^2 \cdot \Pi^2}\right)\left(1 - \frac{x^2}{3^2 \cdot \Pi^2}\right) \&c.}{\left(1 - \frac{x^2}{1^2 \cdot Q^2}\right)\left(1 - \frac{x^2}{3^2 \cdot Q^2}\right)\left(1 - \frac{x^2}{5^2 \cdot Q^2}\right)\left(1 - \frac{x^2}{7^2 \cdot Q^2}\right)\left(1 - \frac{x^2}{9^2 \cdot Q^2}\right)\left(1 - \frac{x^2}{11^2 \cdot Q^2}\right) \&c.}$$

adeo, ut per Infinitos numero Factores huius novae methodi ope Secans Arcus Circularis diversimode ab artic. 223. exprimi possit, nimirum, $\text{Sec. C. } x = 1 +$

$$\frac{x^2}{\Pi^2}$$

$$\frac{x^2}{2} \cdot \frac{\left(1 - \frac{x^2}{1^2 \cdot \Pi^2}\right) \left(1 - \frac{x^2}{1^2 \cdot \Pi^2}\right) \left(1 - \frac{x^2}{2^2 \cdot \Pi^2}\right) \left(1 - \frac{x^2}{2^2 \cdot \Pi^2}\right) \left(1 - \frac{x^2}{3^2 \cdot \Pi^2}\right) \left(1 - \frac{x^2}{3^2 \cdot \Pi^2}\right) \left(1 - \frac{x^2}{4^2 \cdot \Pi^2}\right) \left(1 - \frac{x^2}{4^2 \cdot \Pi^2}\right) \&c.}{\left(1 - \frac{4x^2}{1^2 \cdot P^2}\right) \left(1 - \frac{4x^2}{3^2 \cdot P^2}\right) \left(1 - \frac{4x^2}{5^2 \cdot P^2}\right) \left(1 - \frac{4x^2}{7^2 \cdot P^2}\right) \left(1 - \frac{4x^2}{9^2 \cdot P^2}\right) \left(1 - \frac{4x^2}{11^2 \cdot P^2}\right) \left(1 - \frac{4x^2}{13^2 \cdot P^2}\right) \left(1 - \frac{4x^2}{15^2 \cdot P^2}\right) \&c.}$$

Quod si in simpliciores Factores ipsamet Formula resolvatur proveniet Aequa-

qualitas Sec. C. $x - 1 = \frac{x^2}{2} \times$

$$\frac{\left(1 - \frac{x}{\Pi}\right) \left(1 + \frac{x}{\Pi}\right) \left(1 - \frac{x}{\Pi}\right) \left(1 + \frac{x}{\Pi}\right) \left(1 - \frac{x}{2\Pi}\right) \left(1 + \frac{x}{2\Pi}\right) \left(1 - \frac{x}{2\Pi}\right) \left(1 + \frac{x}{2\Pi}\right) \left(1 - \frac{x}{3\Pi}\right) \left(1 + \frac{x}{3\Pi}\right) \left(1 - \frac{x}{3\Pi}\right) \left(1 + \frac{x}{3\Pi}\right) \&c.}{\left(1 - \frac{2x}{P}\right) \left(1 + \frac{2x}{P}\right) \left(1 - \frac{2x}{3P}\right) \left(1 + \frac{2x}{3P}\right) \left(1 - \frac{2x}{5P}\right) \left(1 + \frac{2x}{5P}\right) \left(1 - \frac{2x}{7P}\right) \left(1 + \frac{2x}{7P}\right) \left(1 - \frac{2x}{9P}\right) \left(1 + \frac{2x}{9P}\right) \left(1 - \frac{2x}{11P}\right) \left(1 + \frac{2x}{11P}\right) \&c.}$$

cuius praesidio pariter obtinentur, praeter Imaginarias hic loci neglectas, $0, 0, \pm \Pi, \pm \Pi, \pm 2\Pi, \pm 2\Pi, \pm 3\Pi, \pm 3\Pi$, vel brevius $\pm n\Pi, \pm n\Pi$ innumerae ac congeminae Radices Reales Aequationis Infinitinomialis $\frac{x^2}{2}$

$$+ \frac{5x^4}{24} + \frac{61x^6}{720} + \frac{277x^8}{8064} + \frac{50521x^{10}}{3628800} + \frac{540553x^{12}}{95800320} + \&c. = 0, \text{ ad}$$

quarum Radicum confirmationem accedunt Trigonometricarum Institutionum evidentiora principia.

229. Auctor est Dalembertius in trigesimoquarto *Opusculorum* Voluminis VI. Schediasmate mysterium quiddam abscondi Trigonometricas inter Series, quo fiat, ut Series directa ope Arcus Circularis Sinum praebens innumeris Arcuum gaudeat valoribus interea dum conversa Arcus per Sinum isti unicum tribuit valorem; idemque Vir acutissimus stimulos addit Analysis pro Paradoxo interpretando excitatis. Haec autem proprietates universis rei Trigonometricae Lineis communis in Sinu recto atque Tangente summam explicari meretur, propterea quod ceterae omnes istis varia dispositis forma coalescant. Mysterium, de quo nunc sermo instituitur, rite translatum Algebraico stylo significat Aequationem $x^1 - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^9}{1.2.3.4.5.6.7.8.9} - \frac{x^{11}}{1.2.3.4.5.6.7.8.9.10.11} + \&c. = 0$ numero

$$x^1 - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^9}{1.2.3.4.5.6.7.8.9} - \frac{x^{11}}{1.2.3.4.5.6.7.8.9.10.11} + \&c. = 0 \text{ numero}$$

Hhh 2

mero Infinitas habere x Reales Radices, veruntamen unicam in altera

$$\text{Aequalitate } v^1 + \frac{v^3}{1.2.3} + \frac{3^2.v^5}{1.2.3.4.5} + \frac{3^2.5^2.v^7}{1.2.3.4.5.6.7} + \frac{3^2.5^2.7^2.v^9}{1.2.3.4.5.6.7.8.9} +$$

&c: — $x = 0$ contineri. Consimiliter, respectu habito ad Tangentem, uni-

$$\text{ca Realis Radix conveniet Aequationi } z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \frac{z^9}{9}$$

$$- \frac{z^{11}}{11} + \text{\&c:} — x = 0, \text{ innumeraeque Reales respondebunt inverfae}$$

$$x + \frac{x^3}{3} + \frac{2x^5}{3.5} + \frac{17x^7}{3^2.5.7} + \frac{62x^9}{3^2.5.7.9} + \frac{1382x^{11}}{3^2.5^2.7.9.11} + \text{\&c:} — z = 0,$$

nuncupatis x, v, z Arcu, Recto-Sinu, et Tangente.

230. Hisce praedefinitis recordemur oportet esse vi numeri 161^{mi} . $C^{x\sqrt{-1}}$

$$= v\sqrt{-1} + \sqrt{1-v^2}, \text{ sive, quod idem sonat, } \left(1 + \frac{x\sqrt{-1}}{\infty}\right)^\infty = v\sqrt{-1}$$

$$+ \sqrt{1-v^2}, \text{ nempe } \frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^\infty}{\sqrt{-1}} = v + \sqrt{v^2-1}, \text{ scilicet, posita } \Phi(x)$$

$$\text{vice Functionis Uniformis } \frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^\infty}{\sqrt{-1}}, \text{ emergere } v = \frac{(\Phi(x))^2 + 1}{2\Phi(x)}$$

primi Ordinis Aequalitatem, unde oritur prior Aequationum superius recensitarum, quae idcirco evidentissime pro Sinu v dat valorem. Ab ista

$$\text{Aequatione directa dimanat inverfa } x = \frac{\infty}{\sqrt{-1}} \left((v\sqrt{-1} + \sqrt{1-v^2})^{\frac{1}{\infty}} - 1 \right),$$

quae, valore τ v determinato, ob Indicem $\frac{1}{\infty}$ innumerorum est capax va-

lorum, ac secundo loco adpositam Aequalitatem progignit. Accessio fit etiam

$$\text{praecedenti demonstrationi si meminerimus } C^{x\sqrt{-1}} = C^{(x \pm n\pi)\sqrt{-1}},$$

$$\text{quae affectionis natura denuo praefert Aequalitatem } C^{x\sqrt{-1}} =$$

$$C^{(x \pm n\pi)\sqrt{-1}}$$

$C^{(x \pm n\pi)\sqrt{-1}} = (v + \sqrt{v^2 - 1})\sqrt{-1}$, pro x Infinitiformem, sed quo ad v Uniformem, et ideo nos docet innumeros Arcus unico Sinui-recto adscribendos, dum Arcui ex adverso tribuit unicum Sinum. Sortiuntur itaque geminae illae Aequationes iisdem suffultum rationibus idem ad amissum symptoma, quod bis saltem vidimus in praecedentibus numeris 59., et 184., mirumque Geometris fuit haec hactenus existimatum. Non abfimili ritu $x = \frac{x^3}{1.3} +$

$$\frac{2x^5}{1.3.5} + \frac{17x^7}{1.3^2.5.7} + \frac{62x^9}{1.3^2.5.7.9} + \&c. = z = 0, \text{ nec non eius inversa } z =$$

$$\frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \frac{z^9}{9} + \&c. = x \text{ aliter expressae per numeros 206. 209. congruunt aliis } C^{2x\sqrt{-1}} = \frac{1 + z\sqrt{-1}}{1 - z\sqrt{-1}}, \text{ vel } \left(1 + \frac{2x\sqrt{-1}}{\infty}\right)^\infty = \frac{1 + z\sqrt{-1}}{1 - z\sqrt{-1}}, \text{ at}$$

$$\text{que } 2x\sqrt{-1} = L\left(\frac{1 + z\sqrt{-1}}{1 - z\sqrt{-1}}\right), \text{ scilicet, } x = \frac{\infty}{2\sqrt{-1}} \left(\left(\frac{1 + z\sqrt{-1}}{1 - z\sqrt{-1}}\right)^\infty - 1\right),$$

$$\text{quarum prima, quum, suffecta } F(x) \text{ loco } \left(1 + \frac{2x\sqrt{-1}}{\infty}\right)^\infty \text{ Uniformis Fun-}$$

$$\text{ctionis, praebeat Aequationem prioris Gradus } z = \frac{F(x) - 1}{\sqrt{-1}(F(x) + 1)}, \text{ satis e-}$$

$$\text{loquenter ostendit unicuique valori } x \text{ propter Radicalium defectum respon-}$$

$$\text{dere tantummodo unum Tangentis } z \text{ valorem, sed vicissim secunda defi-}$$

$$\text{nit innumeros Arcus } x \text{ ob Infinitiformem Potestatem Indicis } \frac{1}{\infty} \text{ datae quo-}$$

$$\text{modolibet Tangenti } z \text{ Analytica adesse necessitate. Qui haec breviter di-}$$

$$\text{cta sedulo, ut par est, perpendere velit nullus dubito, quia ab Exponen-}$$

$$\text{tialium Formularum indole iamdudum nostrae methodo subacta mysterii}$$

$$\text{propositi deductam explicationem, ipsiusque nativum characterem meridia-}$$

$$\text{na luce clarius perspiciat in origine primigenia Aequationum recensita-}$$

$$\text{rum (*).}$$

$$231. \text{ Nec parum afferet voluptatis summa Trigonometricae doctrinae in ipso}$$

$$(*) \text{ Adde quae addidimus in numeris 295. et 296.}$$

ipso fonte res hausta, quae eiusmodi expressionibus Transcendentibus, qua-

les sunt $x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^9}{1.2.3.4.5.6.7.8.9} - \&c.$,

$1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} - \&c.$,

$\frac{x^{10}}{1.2.3.4.5.6.7.8.9.10} + \&c.$, $x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \&c.$,

$\frac{1}{x} - \frac{x}{3} + \frac{x^3}{45} - \frac{2x^5}{945} + \frac{x^7}{4725} - \frac{2x^9}{93555} - \&c.$, $\frac{x^2}{1.2} - \frac{x^4}{1.2.3.4}$

$+ \frac{x^6}{1.2.3.4.5.6} - \frac{x^8}{1.2.3.4.5.6.7.8} + \&c.$, $1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \&c.$

$\frac{277x^8}{8064} + \frac{50521x^{10}}{3628800} + \&c.$, $\frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \frac{127x^7}{604800} + \&c.$,

suo quaeque ordine in praeteritis numeris locatae, aliaeve ex istis compo-
sita, talem praecipit legem, ut idem permaneat earum omnium respectus valor,
tametsi variabili x addatur $n\pi$, vel ab ipsa dematur, aut addatur, dema-
turve $+n\pi$, $-n\pi$, $+n\pi$, $-n\pi$ &c., ac vicissim, iuxta diversam superius la-
tarum Serierum naturam. Fundamenta Theoriae iam posuimus in num°.

193., et procul dubio constabit fore novi speciminis causa $x - \frac{x^3}{1.2.3} +$

$\frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^9}{1.2.3.4.5.6.7.8.9} - \&c. = (\pm n\pi + x) -$

$\frac{(\pm n\pi + x)^3}{1.2.3} + \frac{(\pm n\pi + x)^5}{1.2.3.4.5} - \frac{(\pm n\pi + x)^7}{1.2.3.4.5.6.7} + \frac{(\pm n\pi + x)^9}{1.2.3.4.5.6.7.8.9} -$

$\&c.$, nec non $= (\pm \lambda P - x) - \frac{(\pm \lambda P - x)^3}{1.2.3} + \frac{(\pm \lambda P - x)^5}{1.2.3.4.5} -$

$\frac{(\pm \lambda P - x)^7}{1.2.3.4.5.6.7} + \frac{(\pm \lambda P - x)^9}{1.2.3.4.5.6.7.8.9} - \&c.$, cuius vere mirabilis affectionis in

Exponentialium indole absconditur ratio. Sinus namque rectus Circularis Arcus

x ab illa Serie repraesentatus peraequat $C - \frac{x\sqrt{-1}}{C^{x\sqrt{-1}}} = C - \frac{1}{C^{x\sqrt{-1}}} \cdot 1 - \frac{1}{C^{x\sqrt{-1}}} \cdot 1$

$\frac{1}{2\sqrt{-1}} - \frac{1}{2\sqrt{-1}} = C$

$= C$

$$= C \frac{x\sqrt{-1} \pm n\pi\sqrt{-1}}{C^{x\sqrt{-1} \pm n\pi\sqrt{-1}}} \cdot C \frac{1}{C^{x\sqrt{-1} \pm n\pi\sqrt{-1}}} \\ = C \frac{(\pm n\pi + x)\sqrt{-1}}{C^{(\pm n\pi + x)\sqrt{-1}}}, \text{quemadmodum demonstrare}$$

opus erat. Eadem insuper Formula, facto testo eius valore, hanc quoque

transformationem admittit, $C \frac{x\sqrt{-1}}{C^{x\sqrt{-1}}} = \text{Sin. } C \cdot x$

$$= C \frac{-x\sqrt{-1}}{C^{-x\sqrt{-1}}} \cdot \frac{1}{C^{-x\sqrt{-1}}}$$

$$= C \frac{-x\sqrt{-1} \pm \lambda P\sqrt{-1}}{C^{-x\sqrt{-1} \pm \lambda P\sqrt{-1}}}, \text{videlicet, ob Po-}$$

testatum notissimos canones, $= C \frac{(\pm \lambda P - x)\sqrt{-1}}{C^{(\pm \lambda P - x)\sqrt{-1}}}$

unde plenissima liquet Theorematis veritas, quod maiore etiam compendio

ac dignitate concinnari sic poterit. „ $x \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \frac{x^7}{1.2.3.4.5.6.7} + \dots$

$$+ \frac{x^9}{1.2.3.4.5.6.7.8.9} \dots \&c: \pm m = 0, \text{vel } \left(1 + \frac{x\sqrt{-1}}{\infty}\right)^\infty - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^\infty$$

$\pm m = 0$ Radices habet Reales innumeras $b \pm n\pi, \pm \lambda P - b$, dummodo b minimum Arcuum, quibus Sinus $\mp m$ conveniat, de more significet „
Inita ratiocinatio se quoque ad ceteras Series extendit; neque in arenam descenditur

descensus difficilis ad metam cursus videbitur. Eni valorum sine numero multiplicitas non modo detecta, ut in num^o. praecedente, verum etiam singulorum magnitudo determinata, simulque nova luce perfusae proprietates illae universae, quas in anteaetis numis. passim recensui de Infinitiformibus Exponentialium, aut Logarithmorum Functionibus agens.

232. Si Realibus Imaginaria subrogentur cumulatiores acquirere vires Hyperbolicae Trigonometriae Analogia. Obiter enim subnecto Aequationem praecipuam, cuius simul cum praetermissis in promptu erunt post superius tradita demonstrationes omnibus numeris absolutae. Melliori ordine

servato habebitur itaque $x = \frac{x^1}{1.2.3} + \frac{x^3}{1.2.3.4.5} + \frac{x^5}{1.2.3.4.5.6.7} + \dots$

$$\begin{aligned} & \frac{x^9}{1.2.3.4.5.6.7.8.9} + \&c. = (\pm n\pi \sqrt{-1} + x) + \frac{(\pm n\pi \sqrt{-1} + x)^3}{1.2.3} + \\ & \frac{(\pm n\pi \sqrt{-1} + x)^5}{1.2.3.4.5} + \frac{(\pm n\pi \sqrt{-1} + x)^7}{1.2.3.4.5.6.7} + \frac{(\pm n\pi \sqrt{-1} + x)^9}{1.2.3.4.5.6.7.8.9} + \&c. \\ & = (\pm \lambda P \sqrt{-1} - x) + \frac{(\pm \lambda P \sqrt{-1} - x)^3}{1.2.3} + \frac{(\pm \lambda P \sqrt{-1} - x)^5}{1.2.3.4.5} + \\ & \frac{(\pm \lambda P \sqrt{-1} - x)^7}{1.2.3.4.5.6.7} + \frac{(\pm \lambda P \sqrt{-1} - x)^9}{1.2.3.4.5.6.7.8.9} + \&c. \text{ Exinde consequitur Ae-} \end{aligned}$$

quationis resolutio $x = \frac{x^1}{1.2.3} + \frac{x^3}{1.2.3.4.5} + \frac{x^5}{1.2.3.4.5.6.7} + \frac{x^9}{1.2.3.4.5.6.7.8.9} + \dots$

$$+ \&c. = m = 0, \text{ five, } \frac{\left(1 + \frac{x}{\infty}\right)^{\infty} - \left(1 - \frac{x}{\infty}\right)^{\infty}}{2} = m = 0, \text{ quip-}$$

pe quae, nuncupando b Magnitudinem dupli Centralis Sectoris Aequilaterae Hyperbolae recto Sinui Hyperbolico $\pm m$ respondentem, gaudeat Radicibus b $\pm n\pi \sqrt{-1}$, et $\pm \lambda P \sqrt{-1} - b$, innumeris nempe, ac praeter unicam b Imaginariis. Simpliciores ergo Aequationes

$$\left(1 + \frac{x}{\infty}\right)^{\infty} - \left(1 - \frac{x}{\infty}\right)^{\infty} = 0, \text{ atque } \left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty} = 0$$

Radices gaudent $x = \pm n\pi \sqrt{-1}$, $\pm \lambda P \sqrt{-1}$, et $x = \pm n\pi$, $\pm \lambda P$,

$\pm \lambda P$, vel brevius $x = \pm nP \sqrt{-1}$, ac $x = \pm nP$, prouti in numis. 195. et 200. definiveram.

233. Antequam varia, quae dicenda remanent de Imaginariorum evolutione, adiciantur non negligendam puto ampliationem universalissimam Formularum Sinus, Cofinus &c: Multiplicorum, vel Submultiplicorum Arcuum Circularium in Analyfi Cartesiana exhibentium, absque eo quod ex Infinitesimali penu opem aliquam necesse sit mutuari. Familiaris est enim, et demonstrata superius pro cuiuslibet modi, formaeque Indice expressio

$(\text{Cofin. C. } x + (\text{Sin. C. } x) \sqrt{-1})^z = (C^{x\sqrt{-1}})^z = C^{zx\sqrt{-1}} = \text{Cofin. C. } zx + (\text{Sin. C. } zx) \sqrt{-1}$. His positis regulae exaltationis Binomii nostra methodo universalitati perquammaximae restitutae evincunt, si consulatur praefertim numus. 17., esse omni casu $(\text{Cofin. C. } x + (\text{Sin. C. } x) \sqrt{-1})^z =$

$$(\text{Cofin. C. } x)^z + \frac{z}{1} (\text{Cofin. C. } x)^{z-1} (\text{Sin. C. } x) \sqrt{-1} + \frac{z \cdot z - 1}{1 \cdot 2} (\text{Cofin. C. } x)^{z-2} (\text{Sin. C. } x)^2$$

$$- \frac{z \cdot z - 1 \cdot z - 2}{1 \cdot 2 \cdot 3} (\text{Cofin. C. } x)^{z-3} (\text{Sin. C. } x)^3 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cofin. C. } x)^{z-4} (\text{Sin. C. } x)^4$$

$$- \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Cofin. C. } x)^{z-5} (\text{Sin. C. } x)^5 \sqrt{-1} + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4 \cdot z - 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} (\text{Cofin. C. } x)^{z-6} (\text{Sin. C. } x)^6$$

$$- \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4 \cdot z - 5 \cdot z - 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} (\text{Cofin. C. } x)^{z-7} (\text{Sin. C. } x)^7 \sqrt{-1} + \dots$$

$$\&c: = \text{Cofin. C. } zx + (\text{Sin. C. } zx) \sqrt{-1}$$

$$\text{Binae igitur derivantur necessariae Aequalitates Realia ab Imaginariis secernendo, nimirum, Cofin. C. } zx = (\text{Cofin. C. } x)^z - \frac{z \cdot z - 1}{1 \cdot 2} (\text{Cofin. C. } x)^{z-2} (\text{Sin. C. } x)^2 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cofin. C. } x)^{z-4} (\text{Sin. C. } x)^4 - \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4 \cdot z - 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} (\text{Cofin. C. } x)^{z-6} (\text{Sin. C. } x)^6 + \dots$$

$$\&c:; \text{ praetereaque Sin. C. } zx = \frac{z}{1} (\text{Cofin. C. } x)^{z-1} (\text{Sin. C. } x) - \frac{z \cdot z - 1}{1 \cdot 2} (\text{Cofin. C. } x)^{z-3} (\text{Sin. C. } x)^3 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cofin. C. } x)^{z-5} (\text{Sin. C. } x)^5 - \dots$$

$$\&c:; \text{ praetereaque Sin. C. } zx = \frac{z}{1} (\text{Cofin. C. } x)^{z-1} (\text{Sin. C. } x) - \frac{z \cdot z - 1}{1 \cdot 2} (\text{Cofin. C. } x)^{z-3} (\text{Sin. C. } x)^3 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cofin. C. } x)^{z-5} (\text{Sin. C. } x)^5 - \dots$$

$$\&c:; \text{ praetereaque Sin. C. } zx = \frac{z}{1} (\text{Cofin. C. } x)^{z-1} (\text{Sin. C. } x) - \frac{z \cdot z - 1}{1 \cdot 2} (\text{Cofin. C. } x)^{z-3} (\text{Sin. C. } x)^3 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cofin. C. } x)^{z-5} (\text{Sin. C. } x)^5 - \dots$$

$$\&c:; \text{ praetereaque Sin. C. } zx = \frac{z}{1} (\text{Cofin. C. } x)^{z-1} (\text{Sin. C. } x) - \frac{z \cdot z - 1}{1 \cdot 2} (\text{Cofin. C. } x)^{z-3} (\text{Sin. C. } x)^3 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cofin. C. } x)^{z-5} (\text{Sin. C. } x)^5 - \dots$$

$$\&c:; \text{ praetereaque Sin. C. } zx = \frac{z}{1} (\text{Cofin. C. } x)^{z-1} (\text{Sin. C. } x) - \frac{z \cdot z - 1}{1 \cdot 2} (\text{Cofin. C. } x)^{z-3} (\text{Sin. C. } x)^3 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cofin. C. } x)^{z-5} (\text{Sin. C. } x)^5 - \dots$$

$$\begin{aligned}
 (\text{Sin. C. } x)^1 &= \frac{z \cdot z - 1 \cdot z - 2}{1 \cdot 2 \cdot 3} (\text{Cofin. C. } x)^{z-3} (\text{Sin. C. } x)^3 + \\
 &\frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Cofin. C. } x)^{z-5} (\text{Sin. C. } x)^5 - \\
 &\frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4 \cdot z - 5 \cdot z - 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} (\text{Cofin. C. } x)^{z-7} (\text{Sin. C. } x)^7 + \&c.,
 \end{aligned}$$

quarum Progressionum Lex peculiari illustratione non eget. Hae duae Progressiones sunt summi veluti cardines, quibus tota regitur Angularium *Multifetionum* Theoria a Francisco Vieta inter primos labente saeculo decimosexto nobilitata, et occasione Problematis ab Adriano Romano propositi tot tantisque inventis promota, quorum demonstrationem post fatum Auctoris Alexander Anderfonus adiecit. Nec solummodo Aequationes omnes Cofinus, Sinus, Chordas, Secantes, Cofecantes, Sinus-versos &c: Arcuum Multiplo- rum, aut Submultiplo- rum respicientes ab universali isto fonte dimanant, quin etiam celeberrima pro Tangentibus Series, quam Ioannes Bernoullius dono Rationalitatis insignem Lipsiensibus Actis ad annum MDCCXXII. re- latis inferuit, maiore dignitate, ac generalitate ditata facillimum est nostrae methodi Confectarium. Dum etenim prima Series secundam partiatur emer-

$$\text{get } \frac{\text{Sin. C. } zx}{\text{Cofin. C. } zx} = \text{Tang. C. } zx =$$

$$\begin{aligned}
 \frac{z}{1} (\text{Tang. C. } x) &= \frac{z \cdot z - 1 \cdot z - 2}{1 \cdot 2 \cdot 3} (\text{Tang. C. } x)^3 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Tang. C. } x)^5 - \&c: \\
 1 &= \frac{z \cdot z - 1}{1 \cdot 2} (\text{Tang. C. } x)^2 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Tang. C. } x)^4 - \&c:
 \end{aligned}$$

quicumque sit Multiplicationis, aut Divisionis Index z ; quamobrem obti-

$$\text{nebimus insuper } \frac{\text{Cofin. C. } zx}{\text{Sin. C. } zx} = \text{Cot. C. } zx =$$

$$\begin{aligned}
 1 &= \frac{z \cdot z - 1}{1 \cdot 2} (\text{Cot. C. } x)^{-2} + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cot. C. } x)^{-4} - \&c: \\
 \frac{z}{1} (\text{Cot. C. } x)^{-1} &= \frac{z \cdot z - 1 \cdot z - 2}{1 \cdot 2 \cdot 3} (\text{Cot. C. } x)^{-3} + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Cot. C. } x)^{-5} - \&c:
 \end{aligned}$$

uti evidenter adparet, atque Bernoullianae consonat expressioni.

234. Quoties z sit Numerus integer, ac positivus universae Series illae abrumpuntur: sed vellem, quod Lectoris animo obversaretur mirabilis quidem proprietas Series hae, ut supra, concinnatas, et quandoque terminorum numero gaudentes finito, alterius nihilominus diversissimae formae eodem valore servato capaces esse, quemadmodum occurrit in Formula Neutroniani Binomii. Congeminatio praedictae expressionis sequentem ideo

largitur Aequalitatem $(\text{Cofin. C. } x)^z - \frac{z \cdot z - 1}{1 \cdot 2} (\text{Cofin. C. } x)^{z-2} (\text{Sin. C. } x)^2 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cofin. C. } x)^{z-4} (\text{Sin. C. } x)^4 -$

$\frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4 \cdot z - 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} (\text{Cofin. C. } x)^{z-6} (\text{Sin. C. } x)^6 + \&c: = 1$

$-\frac{z^2 x^2}{1 \cdot 2} + \frac{z^4 x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{z^6 x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{z^8 x^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} - \frac{z^{10} x^{10}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} +$

$\&c:$, nec non alteram $\frac{z}{1} (\text{Cofin. C. } x)^{z-1} (\text{Sin. C. } x) - \frac{z \cdot z - 1 \cdot z - 2}{1 \cdot 2 \cdot 3}$

$(\text{Cofin. C. } x)^{z-3} (\text{Sin. C. } x)^3 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Cofin. C. } x)^{z-5} (\text{Sin. C. } x)^5$

$\text{C. } x)^7 - \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4 \cdot z - 5 \cdot z - 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} (\text{Cofin. C. } x)^{z-7} (\text{Sin. C. } x)^7 + \&c: =$

$\frac{zx}{1} - \frac{z^3 x^3}{1 \cdot 2 \cdot 3} + \frac{z^5 x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{z^7 x^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{z^9 x^9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} \dots$

235. Proveniet deinceps, Cotangentibus neglectis non absimili modo ver-
fandis,

$\frac{z}{1} (\text{Tang. C. } x) - \frac{z \cdot z - 1 \cdot z - 2}{1 \cdot 2 \cdot 3} (\text{Tang. C. } x)^3 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Tang. C. } x)^5 - \&c:$

$1 - \frac{z \cdot z - 1}{1 \cdot 2} (\text{Tang. C. } x)^2 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Tang. C. } x)^4 - \&c:$

$= zx + \frac{z^3 x^3}{3} + \frac{2z^5 x^5}{15} + \frac{17z^7 x^7}{315} + \frac{61z^9 x^9}{2835} + \frac{1382z^{11} x^{11}}{155925} + \&c: \text{ in}$

Infinitum, quamvis prima Aequationum Membra definito terminorum numero aliquando coalescere possint.

236. Haec dicta quoque de Hyperbolica Trigonometria non erit qui sponte

sponte sua non intelligat; ac latius ideo patebit nunquam fatis laudanda, tantoque fulcita argumentorum robore Analogia quum subnexae accedant

Aequalitates. $(\text{Cofin. H. } x + \text{Sin. H. } x)^z = (C^x)^z = C^{zx} = \text{Cofin. H.}$

$zx + \text{Sin. H. } zx = (\text{Cofin. H. } x)^z + \frac{z}{1} (\text{Cofin. H. } x)^{z-1} (\text{Sin. H. } x)$

$+ \frac{z \cdot z - 1}{1 \cdot 2} (\text{Cofin. H. } x)^{z-2} (\text{Sin. H. } x)^2 + \frac{z \cdot z - 1 \cdot z - 2}{1 \cdot 2 \cdot 3} (\text{Cofin. H. } x)^{z-3}$

$(\text{Sin. H. } x)^3 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cofin. H. } x)^{z-4} (\text{Sin. H. } x)^4 +$

$\frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Cofin. H. } x)^{z-5} (\text{Sin. H. } x)^5 +$

$\frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4 \cdot z - 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} (\text{Cofin. H. } x)^{z-6} (\text{Sin. H. } x)^6 +$

$\frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4 \cdot z - 5 \cdot z - 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} (\text{Cofin. H. } x)^{z-7} (\text{Sin. H. } x)^7 + \&c;$

unde dimanant $\text{Cofin. H. } zx = (\text{Cofin. H. } x)^z + \frac{z \cdot z - 1}{1 \cdot 2} (\text{Cofin. H. } x)^{z-2}$

$(\text{Sin. H. } x)^2 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cofin. H. } x)^{z-4} (\text{Sin. H. } x)^4 +$

$\frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4 \cdot z - 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} (\text{Cofin. H. } x)^{z-6} (\text{Sin. H. } x)^6 + \&c;$

$\text{Sin. H. } zx = \frac{z}{1} (\text{Cofin. H. } x)^{z-1} (\text{Sin. H. } x) + \frac{z \cdot z - 1 \cdot z - 2}{1 \cdot 2 \cdot 3} (\text{Cofin. H. } x)^{z-3}$

$(\text{Sin. H. } x)^3 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Cofin. H. } x)^{z-5} (\text{Sin. H. } x)^5 +$

$\frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4 \cdot z - 5 \cdot z - 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} (\text{Cofin. H. } x)^{z-7} (\text{Sin. H. } x)^7 + \&c;$

nec non

$\text{Tang. H. } zx = \frac{z}{1} (\text{Tang. H. } x) + \frac{z \cdot z - 1 \cdot z - 2}{1 \cdot 2 \cdot 3} (\text{Tang. H. } x)^3 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Tang. H. } x)^5 + \&c;$

$1 + \frac{z \cdot z - 1}{1 \cdot 2} (\text{Tang. H. } x)^2 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Tang. H. } x)^4 + \&c;$

$\text{Cot. H. } zx =$

$$\text{Cot. H. } zx = \frac{1 + \frac{z \cdot z - 1}{1 \cdot 2} (\text{Cot. H. } x)^{-2} + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cot. H. } x)^{-4} + \&c:}{\frac{z}{1} (\text{Cot. H. } x)^{-1} + \frac{z \cdot z - 1 \cdot z - 2}{1 \cdot 2 \cdot 3} (\text{Cot. H. } x)^{-3} + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Cot. H. } x)^{-5} + \&c:}$$

Quibus positis nanciscemur praeterea $(\text{Cofin. H. } x)^z + \frac{z \cdot z - 1}{1 \cdot 2} (\text{Cofin. H. } x)^{z-2} (\text{Sin. H. } x)^2 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cofin. H. } x)^{z-4} (\text{Sin. H. } x)^4$

$$+ \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4 \cdot z - 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} (\text{Cofin. H. } x)^{z-6} (\text{Sin. H. } x)^6 + \&c:$$

$$= 1 + \frac{z^2 x^2}{1 \cdot 2} + \frac{z^4 x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{z^6 x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{z^8 x^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \frac{z^{10} x^{10}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} + \&c: , \frac{z}{1} (\text{Sin. H. } x) (\text{Cofin. H. } x)^{z-1} +$$

$$\frac{z \cdot z - 1 \cdot z - 2}{1 \cdot 2 \cdot 3} (\text{Sin. H. } x)^3 (\text{Cofin. H. } x)^{z-3} + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Sin. H. } x)^5 (\text{Cofin. H. } x)^{z-5} + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4 \cdot z - 5 \cdot z - 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} (\text{Sin. H. } x)^7 (\text{Cofin. H. } x)^{z-7} + \&c: = \frac{zx}{1} + \frac{z^3 x^3}{1 \cdot 2 \cdot 3} + \frac{z^5 x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{z^7 x^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{z^9 x^9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \&c: , \text{atque}$$

$$\frac{z}{1} (\text{Tang. H. } x) + \frac{z \cdot z - 1 \cdot z - 2}{1 \cdot 2 \cdot 3} (\text{Tang. H. } x)^3 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3 \cdot z - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Tang. H. } x)^5 + \&c:$$

$$= 1 + \frac{z \cdot z - 1}{1 \cdot 2} (\text{Tang. H. } x)^2 + \frac{z \cdot z - 1 \cdot z - 2 \cdot z - 3}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Tang. H. } x)^4 + \&c:$$

$$= zx - \frac{z^3 x^3}{3} + \frac{2z^5 x^5}{15} - \frac{17z^7 x^7}{315} + \frac{62z^9 x^9}{2835} - \frac{1382z^{11} x^{11}}{155925} + \&c: ;$$

nec aliquis deerit harumce Aequationum in Capite VII°. usus meo iudicio non inelegans.

237. Postquam omnimodae Exponentialium Imaginariorum simplicium formae in universali comprehensae $z^{x\sqrt{-1}}$ ad Sinus, Cofinusque Circulares

fuere superius relatae, nulla difficultas obstabit, quin et magis compositae iisdem subiiciantur principiis. Silentio praetermittam modum Exponentialium adparenter, non substantialiter, discriminatum a priore, et longius,

quam par esset, ita expressum $(v + z)^{x\sqrt{-1}}$, quum loco $v + z$ unicam speciem y in arbitrio sit substituere, quo facto $(v + z)^{x\sqrt{-1}}$ in facilius $y^{x\sqrt{-1}}$ statim vertetur. Vix quoque delibanda manet Formula Poten-

tialis $(z + v\sqrt{-1})^x$, quippe quae abeat in aliam $z^x + \frac{x}{1} z^{x-1} v\sqrt{-1}$

$$\begin{aligned} & - \frac{x \cdot x-1}{1 \cdot 2} z^{x-2} v^2 - \frac{x \cdot x-1 \cdot x-2}{1 \cdot 2 \cdot 3} z^{x-3} v^3 \sqrt{-1} + \\ & \frac{x \cdot x-1 \cdot x-2 \cdot x-3}{1 \cdot 2 \cdot 3 \cdot 4} z^{x-4} v^4 + \frac{x \cdot x-1 \cdot x-2 \cdot x-3 \cdot x-4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} z^{x-5} v^5 \sqrt{-1} - \\ & \frac{x \cdot x-1 \cdot x-2 \cdot x-3 \cdot x-4 \cdot x-5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} z^{x-6} v^6 - \&c.; \text{ initaque cum illa nume-} \end{aligned}$$

ri 233. comparatione gignat Aequalitates $\frac{z}{R} = \frac{\text{Cofin. C. } \Phi}{1}$, et $\frac{v}{R} =$

$\frac{\text{Sin. C. } \Phi}{1}$; unde profluit $R = (z^2 + v^2)^{\frac{1}{2}}$, nec non $(z + v\sqrt{-1})^x = R^x$

$(\text{Cofin. C. } \Phi + (\text{Sin. C. } \Phi) \sqrt{-1})^x = (z^2 + v^2)^{\frac{x}{2}} (\text{Cofin. C. } x\Phi + (\text{Sin. C. } x\Phi) \sqrt{-1})$ iuxta omnium Analystarum inventa, ac praecedentis numeri 194. ab diverso fonte deductam determinationem. Summo igitur iure monuit Dalembertius nihil ob stare possibilitati Aequationum $(z + v\sqrt{-1})^x$

$= (z - v\sqrt{-1})^x$, vel $(z + v\sqrt{-1})^x = (z - v\sqrt{-1})^x$; quae Aequalitates vi praedictorum revera subsistunt tum, quum Sin. C. $x\Phi$, aut

Sin. C. $x\Phi = 0$, et infra maiori lumine perfusae multo magis nitebunt. Minore autem mora in evolvendis Exponentialibus formae

$z^{x+v\sqrt{-1}}$ opus erit, siquidem eadem sint cum $z^x \cdot z^{v\sqrt{-1}}$, nimirum ex antea ostensis, $z^x (\text{Cofin. C. } v(Lz) + (\text{Sin. C. } v(Lz)) \sqrt{-1})$; adeo, ut postrema

strema haec expressio simul cum ceteris haftenus enucleatis eum uniformiter comprobet Analyticum Canonem, qui universa Imaginaria in $T \pm T\sqrt{-1}$ necessario contrahi decernit.

238. Sed, ne singillatim tot minutiores Formulas perlustremus, agendum uno iactu circa universalissimam, ac praeteritas omnes sinu suo complectentem $(z + y\sqrt{-1})^{x + v\sqrt{-1}}$ periculum instituat. Haec autem proposita expressio facilitate perquam maxima decomponitur in $(z + y\sqrt{-1})^x$.

$(z + y\sqrt{-1})^{v\sqrt{-1}}$, scilicet ex antecedentibus, in $(z^2 + y^2)^{\frac{x}{2}}$ (Cofin. C.

$x\Phi + (\text{Sin. C. } x\Phi)\sqrt{-1}$) $(z^2 + y^2)^{\frac{x}{2}}$ (Cofin. C. $v\Phi\sqrt{-1} + (\text{Sin. C. } v\Phi\sqrt{-1})\sqrt{-1}$); quod luculenter constat, quum z , et y in utroque Binomio eiusdem sint Magnitudinis, atque in Formula num. 237. loco $r\sqrt{-1}$ adhiberi tuto queat $v\sqrt{-1}$. Multiplicatione igitur confecta proveniet

$$(z + y\sqrt{-1})^{x + v\sqrt{-1}} = (z^2 + y^2)^{\frac{x}{2}} (z^2 + y^2)^{\frac{v\sqrt{-1}}{2}} (C^{x\Phi\sqrt{-1}})$$

$$(C^{v\Phi\sqrt{-1} \cdot \sqrt{-1}}) = (z^2 + y^2)^{\frac{x}{2}} \cdot C^{-v\Phi} \times C^{x\Phi\sqrt{-1}} \cdot C^{(L\sqrt{y^2 + z^2})v\sqrt{-1}}$$

$$= (z^2 + y^2)^{\frac{x}{2}} \cdot C^{-v\Phi} \cdot C^{(x\Phi + vL\sqrt{z^2 + y^2})\sqrt{-1}} = \left((z^2 + y^2)^{\frac{1}{2}} \right)^x$$

$\cdot C^{-v\Phi} (\text{Cofin. C. } (x\Phi + vL\sqrt{z^2 + y^2}) + (\text{Sin. C. } (x\Phi + vL\sqrt{z^2 + y^2}))\sqrt{-1})$, quae Formula vere oecumenica in hypothese v evanescentis prae-

bet $\left((z^2 + y^2)^{\frac{1}{2}} \right)^x (\text{Cofin. C. } x\Phi + (\text{Sin. C. } x\Phi)\sqrt{-1})$; nulescente y tribuit $z^x (\text{Cofin. C. } vLz + (\text{Sin. C. } vLz)\sqrt{-1})$, eoquod una cum recto-Sinu anni-

hiletur etiam Arcus Φ ; ac demum facto $x = 0$ evadit $(z + y\sqrt{-1})^{v\sqrt{-1}}$

$= C^{-v\Phi} (\text{Cofin. C. } vL\sqrt{z^2 + y^2} + (\text{Sin. C. } vL\sqrt{z^2 + y^2})\sqrt{-1})$. Nec parum admirari fas erit Arcum, vel Angulum Φ in unaquaque Formularum eundem semper existendi modum obtinere, ac generaliter eum esse, cui pro

Sinu-

Sinu-recto $\frac{y}{\sqrt{z^2+y^2}}$, pro Cofinu $\frac{z}{\sqrt{z^2+y^2}}$, et idcirco pro Tangente $\frac{y}{z}$

convenient, sine dependentia ab Indicum valoribus x , atque v , quamvis ii simul cum z , y tum positivos, tum negativos Numeros quoslibuerit significare valeant.

239. Expressio iam concinnata $\left((z^2+y^2)^{\frac{1}{2}}\right)^x \cdot C^{-v\Phi} \left(\text{Cofin. } C(x\Phi \rightarrow vL\sqrt{z^2+y^2}) \rightarrow (\text{Sin. } C(x\Phi \rightarrow vL\sqrt{z^2+y^2}))\sqrt{-1} \right)$, quae ob

$\left(\frac{1}{z^2+y^2}\right)^{\frac{x}{2}} = C^{xL\sqrt{z^2+y^2}}$ hanc etiam formam acquirit

$C^{xL\sqrt{z^2+y^2}-v\Phi} \left(\text{Cofin. } C(x\Phi \rightarrow vL\sqrt{z^2+y^2}) \rightarrow (\text{Sin. } C(x\Phi \rightarrow vL\sqrt{z^2+y^2}))\sqrt{-1} \right)$, ut nonnullis placuit Analyftis, ipsamet est ab Eulero in *Memorabilibus Berolinensis Academiae* anni MDCCXLIX., quum de Aequationum ageret Imaginariis Radicibus, tradita, sed insigni Calculi adparatu, et saepius Infinitesimalis, reperta. Nec novitatem prae nostris habent eae Formulae, quas eximius ille Scriptor subiungit: namque Arcus, vel Angelus Φ hac generali, et compendiosa Sigla exaratus Functio est reapse Infinitiformis, scilicet non tantummodo nunciat Arcum minimum Φ Tangente $\frac{y}{z}$ praeditum, verum etiam intimo, ac tacito sensu $\Phi \pm n\Pi$, vel

$\Phi \pm 2nP$ &c., quod Summarium adferi posset fore universae Trigonometriae Circularis. Difficilior videtur demonstratio perfectae aequipollentiae quo ad superius allatam in num°. 238 Formulae consimiliter universalis a Dalembertio datae tam in *Investigationibus de Calculo Integrali* inter *Memorabilia* praedictae Borussicae Academiae anni MDCCXLVI. evulgatis, quam in *Considerationibus de Caussa Ventorum*, et a collectore postmodum Bougainvillio iisdem fere verbis ad Tyronum usum accommodatae. Formula, de

qua fit sermo, huc reducitur, $(z+y\sqrt{-1})^{x+v\sqrt{-1}} = M \rightarrow N\sqrt{-1}$, et

inibi $M = (z^2+y^2)^{\frac{x}{2}} \cdot C^{-vS\left(\frac{zdy-ydz}{z^2+y^2}\right)} \cdot \text{Cofin. } C\left(vL\sqrt{z^2+y^2} \rightarrow$

$xS\left(\frac{zdy-ydz}{z^2+y^2}\right)\right)$, atque $N = (z^2+y^2)^{\frac{x}{2}} \cdot C^{-vS\left(\frac{zdy-ydz}{z^2+y^2}\right)} \cdot \text{Sin. } C$

$(vL$

$\left(vL\sqrt{z^2 + y^2} + xS\left(\frac{zdy - ydz}{z^2 + y^2}\right) \right)''$. Hanc qui caute contempletur Aequationem Differentialis, et Summatorii Algorithmi subsidio erutam, ac Signo Integralium S ipsam penitus exuere in animo habeat, facili ductu expiscabitur $S\left(\frac{zdy - ydz}{z^2 + y^2}\right) = \Phi$ Arcui Circulari, cui Tangens respondeat $\frac{y}{z}$,

quapropter Dalembertiana evolutio $(z + y\sqrt{-1})^{x + v\sqrt{-1}}$ superius explicata abibit in alteram nequaquam Infinitesimis onustam $(z^2 + y^2)^{\frac{x}{2}} \cdot C^{-v\Phi}$ $(\text{Cofin. } C(vL\sqrt{z^2 + y^2} + x\Phi) + (\text{Sin. } C(vL\sqrt{z^2 + y^2} + x\Phi))\sqrt{-1})$, nempe ad amissim cum praeterita congruentem.

240. Quanta consequentium Theorematum sese offert ex iam praemissis foecunditas! Magnitudines illas Exponentiales, forma Imaginarias, quandoque Realibus adnumerari debere cuncti, ut arbitror, sine duce percipiunt. Sic $(m + r\sqrt{-1})^x$ Realem Quantitatem nuntiabit toties, quoties

$$x \left(\text{Arc. C. Tang. } \frac{r}{m} \right) = \pm nP, \text{ vel } x = \frac{\pm nP}{\text{Arc. C. Tang. } \frac{r}{m}}: \text{ tunc enim}$$

evadet $(m + r\sqrt{-1})^x = \pm \left((m^2 + r^2)^{\frac{1}{2}} \right)^x$, seu verius, pleniusque =

$1^x (\pm m^2 + r^2)^{\frac{x}{2}}$, quae expressio saepius multiformitatem, et aliquando Functionis infinitiformitatem designat. Eodem pacto vertetur in Realem Functionem $\pm 1^x \cdot m^x$ alterum Exponentiale $m^{x + v\sqrt{-1}}$ tum, quum $vLm =$

$\pm nP$, scilicet $v = \pm \frac{nP}{Lm}$; adeo, ut conveniant inter se hae duae tam

diversimode expositae Formulae $\left((m^2 + r^2)^{\frac{1}{2}} \right)^x \pm \frac{nP}{\text{Arc. C. T. } \frac{r}{m}} \pm \frac{nP\sqrt{-1}}{(L(m^2 + r^2)^{\frac{1}{2}})^x}$,

atque $(m + r\sqrt{-1})^{\frac{nP}{\text{Arc. C. T. } \frac{r}{m}}}$, quod vix credendum nisi suprema id evinceret Algebrae auctoritas. Quin etiam universalius $(m + r\sqrt{-1})^x =$

$z^{p+q\sqrt{-1}}$ quando Index $x = \frac{(p+q\sqrt{-1})Lz}{\frac{1}{2}L(m^2+r^2) + \left(\text{Arc. C. T. } \frac{r}{m}\right)\sqrt{-1}}$, atque

idcirco alterutram formam in aliam transferre licebit. Revera $(m+r\sqrt{-1})^x$
 $= (\sqrt{m^2+r^2})^x \left(\text{Cofin. C. x Arc. C. T. } \frac{r}{m} + \left(\text{Sin. C. x Arc. C. T. } \frac{r}{m} \right) \sqrt{-1} \right)$

ex num^o. 237., nimirum $= C^{x \cdot \frac{1}{2}L(m^2+r^2)} \cdot C^{(x \text{ Arc. C. T. } \frac{r}{m})\sqrt{-1}} =$

$C^{x \left(\frac{1}{2}L(m^2+r^2) + \left(\text{Arc. C. T. } \frac{r}{m} \right) \sqrt{-1} \right)}$ dum $z^{p+q\sqrt{-1}} =$

$C^{(p+q\sqrt{-1})Lz}$, quae est superioris Aequationis origo. Cum itaque ha-

beat $(m+r\sqrt{-1})^x = C^{x \left(\frac{1}{2}L(m^2+r^2) + \left(\text{Arc. C. T. } \frac{r}{m} \right) \sqrt{-1} \right)}$, atque
 ex Theoria Exponentialium universos numeros consimiliter afficiente $=$

$C^{xL(m+r\sqrt{-1})}$, hoc effluet utillimum Confectarium, esse videlicet

$L(m+r\sqrt{-1}) = \frac{1}{2}L(m^2+r^2) + \left(\text{Arc. C. T. } \frac{r}{m} \right) \sqrt{-1}$, prouti re-

gulis ab Eulero, ac Valmesleyo traditis consonat, quicumque sit valor r ,
 aut m , positivus, vel negativus.

241. Sed eoquod etiam $(v+t\sqrt{-1})^{y\sqrt{-1}} = C^{-y \cdot \text{Arc. C. T. } \left(\frac{t}{v} \right)}$

$\left(\text{Cofin. C. y L} \sqrt{v^2+t^2} + \left(\text{Sin. C. y L} \sqrt{v^2+t^2} \right) \sqrt{-1} \right)$, ut ad calcem nume-

ri 228., scilicet $= C^{y \left(-\text{Arc. C. T. } \left(\frac{t}{v} \right) + \left(\frac{1}{2}L(v^2+t^2) \right) \sqrt{-1} \right)}$, evadet

Formula ipsa Realis si ponatur $\frac{yL(v^2+t^2)}{2} = \pm nP$, vel $y = \frac{\pm 2nP}{L(v^2+t^2)}$

$= \frac{\pm n\pi}{L(v^2+t^2)}$, et formae erit capax $(v+t\sqrt{-1})^x$ dum y in Numerum

abeat Imaginarium, nec non alterius supra memoratae $m^{z+x\sqrt{-1}}$

$= C^{(z+x\sqrt{-1})Lm}$

$C^{(z+x\sqrt{-1})Lm}$ quando loco z subrogetur $\frac{-y \text{ Arc. C. T. } (\frac{t}{v})}{Lm}$, et loco x

assumatur $\frac{yL(v^2+t^2)}{2Lm}$. Vi generalium harumce Aequationum non ingratam

Lectoribus futuram censeo triplicem metamorphosin, quae sequitur,

$$(v+z\sqrt{-1})^x = y^{\frac{x}{Ly} \left(\frac{1}{2} L(v^2+z^2) + \left(\text{Arc. C. T. } \left(\frac{z}{v} \right) \right) \sqrt{-1} \right)} =$$

$$C^{x \left(\frac{1}{2} L(v^2+z^2) + \left(\text{Arc. C. T. } \left(\frac{z}{v} \right) \right) \sqrt{-1} \right)} = (v+z\sqrt{-1})^{\frac{x\sqrt{-1}}{\sqrt{-1}}} =$$

$$(v+z\sqrt{-1})^{t\sqrt{-1}}, \text{ in qua postrema } \tau \delta t = \frac{x}{\sqrt{-1}}. \text{ Versio unius in alteram}$$

Formulam meridiana luce clarius adparet; quandoquidem secunda nihil aliud

est quam expressio ad modum universalissimum accommodata $y^{\frac{M+N\sqrt{-1}}{2}}$,
 ac praesto sunt Aequationes ad valores M, N speciali quolibet in casu de-
 terminandos.

242. Accedit ad Theoricen paullo nimis neglectam convalidandam mo-
 mentosissima consideratio, quod quavis data expressione $(v+z\sqrt{-1})^x$ sem-
 per ea verti possit in aliam, indeterminata Basi $y^{\frac{M+N\sqrt{-1}}{2}}$,
 sed non vicissim, nisi Aequatio locum habeat *Conditionalis* in peculiari Ba-

$$\text{si } v+z\sqrt{-1}, \text{ quam libuerit assumere, } \frac{2M}{L(v^2+z^2)} = \frac{N}{\text{Arc. C. T. } \left(\frac{z}{v} \right)}. \text{ Ra-}$$

tione consimili Functio generica prae omnibus difficilior $(m+r\sqrt{-1})^{p+q\sqrt{-1}}$

$$= C^{pL\sqrt{m^2+r^2} - q \text{ Arc. Tang. C } \left(\frac{r}{m} \right) \left(\text{Cosin. C } \left(p \text{ Arc. T } \left(\frac{r}{m} \right) + \right. \right. \\ \left. \left. qL\sqrt{m^2+r^2} \right) + \left(\text{Sin. C } \left(p \text{ Arc. T } \left(\frac{r}{m} \right) + qL\sqrt{m^2+r^2} \right) \sqrt{-1} \right) \text{ ex}$$

num°. 239., scilicet, convertendo Factorem, qui Cosinus Sinusque rectos
 complectitur, in Exponentialem,

$$C^{\frac{1}{2}} =$$

$$\frac{1}{C^2} pL(m^2 + r^2) - q \text{Arc. C. T.} \left(\frac{r}{m} \right) + \left(p \text{Arc. C. T.} \left(\frac{r}{m} \right) + \frac{1}{2} qL(m^2 + r^2) \right) \sqrt{-1}$$

aut, si mavis,

$$\frac{1}{x} \left(\frac{1}{2} pL(m^2 + r^2) - q \text{Arc. C. T.} \left(\frac{r}{m} \right) + \left(p \text{Arc. C. T.} \left(\frac{r}{m} \right) + \frac{1}{2} qL(m^2 + r^2) \right) \sqrt{-1} \right),$$

prouti notissima Logarithmico-percurrentium Theoremata docent. Formula

itaque $(m + r\sqrt{-1})^{p+q\sqrt{-1}}$ instar praecedentium contrahetur in simpli-

ciolem ad speciem $x^{z+v\sqrt{-1}}$ pertinentem, sed insuper Realem Magni-

tudinem tribuet statim atque adfit Aequatio *Conditionalis* $p \text{Arc. C. T.} \left(\frac{r}{m} \right)$

$+ \frac{q}{2} L(m^2 + r^2) = \pm nP$: tunc etenim fiet Functio

$$(m + r\sqrt{-1})^{p+q\sqrt{-1}} = \pm C^{\frac{1}{2} pL(m^2 + r^2) - q \text{Arc. C. T.} \left(\frac{r}{m} \right)}, \text{ novorum}$$

Theorematum inexhaustus fons, et origo. At potissimum cunctorum possibilibus Systematum Logarithmicorum Basi quomodocunque Imaginaria-composita, Logarithmisque pariter gaudentium compositis-Imaginariis omni-modi derivatio ex illa generali prius posita Formula pendet, quae tam simplex est, ut haec Systemata ulterius memorantes iacturam temporis facere videremur.

243. Tot sublimis Analyseos divitias maioris ocii cultoribus dicabo, utcunque praevideam iactorum feminum fructus supra spem uberrimae segetis foecundiores. Iis quoque relinquam evolutionem Exponentialium for-

mae oecumenicae $(x + y\sqrt{-1})^{(z+v\sqrt{-1})^{m+r\sqrt{-1}}}$, vel, si libeat, compositioris, cum etiam oscitanter huic satisfieri possit Problemati, viaque iam strata sit ad quamplurima persequenda in Coronidum classe locata. Exem-

plum sit singulare in $\sqrt{-1}^{\sqrt{-1}}$, pro quo duplici Exponentiali evolendo nonnulli recentissimos inter Geometras tanta Calculorum copia defuda-

runt. Gradu etenim ordinatim facto inprimis nanciscimur $\sqrt{-1}^{\sqrt{-1}} = C^{-}$

$$= \bar{C} \text{ Arc. C. T}(\infty) + \left(\frac{1}{2} L_1\right) \sqrt{-1} = C^{-\frac{\lambda P}{2} \mp nP}, \text{ prouti partim a num}^\circ.$$

praecedente 242., partimque a num^{is}. 185. et 240. deducitur. Erit idcirco,

$$\text{synthetico id suadente progressu, } \sqrt{-1}^{\sqrt{-1}} = \sqrt{-1} C^{-\frac{\lambda P}{2} \mp nP}$$

$$= C^{L(\sqrt{-1}) \left(C^{-\frac{\lambda P}{2} \mp nP}\right)} \text{ ex toties visa Aequatione } z^x = C^{(Lz)x}. \text{ Eoquod}$$

autem vi num^{orum}. 189. ac 240. sit $L(\sqrt{-1}) = \pm \frac{\lambda P}{2} \cdot \sqrt{-1}$ obtinebitur

$$C^{L(\sqrt{-1}) \left(C^{-\frac{\lambda P}{2} \mp nP}\right)} = C^{\pm \frac{\lambda P}{2} \left(C^{-\frac{\lambda P}{2} \mp nP}\right) \sqrt{-1}} = \text{Cofin. Cir.}$$

$$\left(\pm \frac{\lambda P}{2} \cdot C^{-\frac{\lambda P}{2} \mp nP}\right) \pm \left(\text{Sin. Cir.} \left(\pm \frac{\lambda P}{2} \cdot C^{-\frac{\lambda P}{2} \mp nP}\right)\right) \sqrt{-1} \text{ ex num}^\circ.$$

157. Huius Aequalitatis specialissimus solummodo casus ille est $\sqrt{-1}^{\sqrt{-1}}$

$$= \text{Cofin. Cir.} \left(\frac{\Pi}{4} \cdot C^{-\frac{\Pi}{4}}\right) - \sqrt{-1} \left(\text{Sin. C} \left(\frac{\Pi}{4} \cdot C^{-\frac{\Pi}{4}}\right)\right) \text{ ab iis, qui}$$

me praecesserunt, ad instar oecumenicae completaeque resolutionis vulgatus.

Interea silendum non puto Functionem Infinitiformem $\sqrt{-1}^{\sqrt{-1}}$ esse semper

Realem $C^{-\lambda Q \mp nP}$ ab Hyperbolae simul, et Circuli Tetragonismo pen-

dentem, sed ex adverso $\sqrt{-1}^{\sqrt{-1}}$ Imaginarium praeseferre valorem pa-
riter Infinities diversum, numerisque Circuli, et Hyperbolae Quadraturam
una exigentibus insignitum.

244. Formulas huiusce speciei $z^{\text{Sin. C. } x}$, $z^{\text{Sin. H. } x}$, vel analogarum

M m m Cofin. C. x
z

Cofin. C. x Cofin. H. x Tang. C. x Cot. C. x Tang. H. x Cot. H. x Sec. C. x
 z , z , z , z , z , z ,

Cofec. C. x
 z

&c: nullam admittere difficultatem omnes arbitror persentire.
 Earum etenim ordine servato memoratae Functiones evadunt, quemadmo-

dum per se patet, z

$$x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.5.6.7} + \frac{x^9}{2.3.4.5.6.7.8.9} - \&c:$$

$$(Lz) \left(x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.5.6.7} + \frac{x^9}{2.3.4.5.6.7.8.9} - \&c: \right)$$

= C

$$= 1 + \frac{(Lz)(\text{Sin. C. x})}{1} + \frac{(Lz)^2(\text{Sin. C. x})^2}{1.2} + \frac{(Lz)^3(\text{Sin. C. x})^3}{1.2.3} + \frac{(Lz)^4(\text{Sin. C. x})^4}{1.2.3.4}$$

$$+ \frac{(Lz)^5(\text{Sin. C. x})^5}{1.2.3.4.5} + \&c:; \text{ deindeque}$$

$$x + \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} + \frac{x^7}{2.3.4.5.6.7} + \frac{x^9}{2.3.4.5.6.7.8.9} + \&c:$$

z

, nimirum

$$1 + \frac{(Lz)(\text{Sin. H. x})}{1} + \frac{(Lz)^2(\text{Sin. H. x})^2}{1.2} + \frac{(Lz)^3(\text{Sin. H. x})^3}{1.2.3} + \frac{(Lz)^4(\text{Sin. H. x})^4}{1.2.3.4}$$

$$+ \frac{(Lz)^5(\text{Sin. H. x})^5}{1.2.3.4.5} + \&c:; z$$

, vel

$$1 + \frac{(Lz)(\text{Cofin. C. x})}{1} + \frac{(Lz)^2(\text{Cofin. C. x})^2}{1.2} + \frac{(Lz)^3(\text{Cofin. C. x})^3}{1.2.3} + \frac{(Lz)^4(\text{Cofin. C. x})^4}{1.2.3.4}$$

$$(Lz) \left(1 + \frac{x^2}{2} + \frac{x^4}{2.3.4} + \frac{x^6}{2.3.4.5.6} \right) + \&c:; C$$

, five $1 + \frac{(Lz)(\text{Cofin. H. x})}{1}$

$$+ \frac{(Lz)^2(\text{Cofin. H. x})^2}{1.2} + \frac{(Lz)^3(\text{Cofin. H. x})^3}{1.2.3} + \frac{(Lz)^4(\text{Cofin. H. x})^4}{1.2.3.4} + \&c:;$$

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \&c:$$

z

, scilicet $1 +$

$$\frac{(Lz)(\text{Tang. C. x})}{1} + \frac{(Lz)^2(\text{Tang. C. x})^2}{1.2} + \frac{(Lz)^3(\text{Tang. C. x})^3}{1.2.3} + \frac{(Lz)^4(\text{Tang. C. x})^4}{1.2.3.4}$$

$$+ \frac{(Lz)^5 (Tang.C.x)^5}{1.2.3.4.5} + \&c.;$$

$$(Lz) \left(x^{-1} - \frac{x^1}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \frac{x^7}{4725} - \&c. \right)$$

C, videlicet $1 -$

$$\frac{(Lz)(Cot.C.x)}{1} + \frac{(Lz)^2(Cot.C.x)^2}{1.2} + \frac{(Lz)^3(Cot.C.x)^3}{1.2.3} + \frac{(Lz)^4(Cot.C.x)^4}{1.2.3.4} + \&c.;$$

nec non quo ad Hyperbolicas Tangentes, Cotangentefque

$$(Lz) \left(x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \&c. \right)$$

C, aut potius $1 -$

$$\frac{(Lz)(Tang.H.x)}{1} + \frac{(Lz)^2(Tang.H.x)^2}{1.2} + \frac{(Lz)^3(Tang.H.x)^3}{1.2.3} + \frac{(Lz)^4(Tang.H.x)^4}{1.2.3.4}$$

$$(Lz) \left(x^{-1} + \frac{x^1}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \frac{x^7}{4725} + \&c. \right)$$

$+ \&c.; C$, seu $1 -$

$$\frac{(Lz)(Cot.H.x)}{1} + \frac{(Lz)^2(Cot.H.x)^2}{1.2} + \frac{(Lz)^3(Cot.H.x)^3}{1.2.3} + \frac{(Lz)^4(Cot.H.x)^4}{1.2.3.4} : \text{ac de-}$$

$$\text{mum evolventur } z \text{ in } z \quad \text{Sec. C. x} \quad 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \frac{277x^8}{8064} + \&c.;$$

$$1 + \frac{(Lz)(Sec.C.x)}{1} + \frac{(Lz)^2(Sec.C.x)^2}{1.2} + \frac{(Lz)^3(Sec.C.x)^3}{1.2.3} + \frac{(Lz)^4(Sec.C.x)^4}{1.2.3.4}$$

$$+ \frac{(Lz)^5(Sec.C.x)^5}{1.2.3.4.5} + \&c.; \text{ et } z \text{ Cofec. C. x} =$$

$$(Lz) \left(x^{-1} + \frac{x^1}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \frac{127x^7}{604800} + \&c. \right)$$

C in $1 -$

$$\frac{(Lz)(Cofec.C.x)}{1} + \frac{(Lz)^2(Cofec.C.x)^2}{1.2} + \frac{(Lz)^3(Cofec.C.x)^3}{1.2.3} + \frac{(Lz)^4(Cofec.C.x)^4}{1.2.3.4}$$

$+ \&c. \text{ fine ullo Limite.}$

245. Nec novam expostulabunt methodum post praecedentium initia operationum tentamina Functiones $y^{z\sqrt{-1}}$, $y^{z\sqrt{-1}x}$, aliaeque confimiles, quippe quum non modo earum evolutio ab expressionibus in num°. 243.

datis

dati dimanet, verum etiam a Formulis Exponentialibus in calce numerorum 63., et 64. Capitis II. promotis. At maiori inserviens elegantiae subnecto illarum facillimam in Series numero terminorum Infinitas resolutionem.

Omnes iam norunt inprimis esse $z^{x\sqrt{-1}} = C^{(Lz)x\sqrt{-1}} = \text{Cofin. C.}(Lz)x + (\text{Sin. C.}(Lz)x)\sqrt{-1}$; quamobrem dubio carebit Aequatio $y^{z^{x\sqrt{-1}}} = \text{Cofin. C.}(Lz)x + (\text{Sin. C.}(Lz)x)\sqrt{-1} = y^{\text{Cofin. C.}(Lz)x} \times y^{(\text{Sin. C.}(Lz)x)\sqrt{-1}}$

$= C^{(Ly)\text{Cofin. C.}(Lz)x} \times C^{(Ly)(\text{Sin. C.}(Lz)x)\sqrt{-1}}$, nimirum Producto simplicium Exponentialium. Pari ratione, quum $(z\sqrt{-1})^x$ peraequet $z^x \cdot \sqrt{-1}^x$,

five $C^{(Lz)x} \cdot C^{(L\sqrt{-1})x}$, vel $C^{(Lz)x} \cdot C^{\pm \frac{\lambda x P \sqrt{-1}}{2}}$, seu

$C^{x(Lz \pm \lambda Q \sqrt{-1})}$, aut etiam, si mavis, $C^{xLz} \times (\text{Cofin. C.} \pm x\lambda Q + (\text{Sin. C.} \pm x\lambda Q)\sqrt{-1})$, habebitur $y^{z^{x\sqrt{-1}}} = y^{C^{x(Lz \pm \lambda Q \sqrt{-1})}}$, videlicet Ex-

pponentiali prioris speciei superius perpenso. Ita ratiocinando sermo institui posset non dispar de Functionibus $(y\sqrt{-1})^{z^x}$, utpote quae, propter $(y\sqrt{-1})^z$

$= C^{zLy} + zL\sqrt{-1} = C^{zLy \pm z\lambda Q \sqrt{-1}}$, vertuntur in $C^{(zLy)^x} \times C^{(\pm z\lambda Q \sqrt{-1})^x}$, vel in expressionem ad secundam classem relatam. Igitur,

Imaginarietate Indicis supremi nihil obstante, Realis erit Formula $y^{z^{x\sqrt{-1}}}$ statim ac xLz , vel $L(z^x) = \pm nP$, fietque y dum n sit Nullitas, aut Numerus par, atque $\frac{1}{y}$ posito n impare, quod infinitis modis contingere posse penes Analystas in confesso est. Sed geminae Functiones, quae remanent, $y^{(z\sqrt{-1})^x}$, $(y\sqrt{-1})^{z^x}$, nulla hypothesis facta mentione earum formam

nam destruentis, Imaginarietatem amittent tum, quum $x = \frac{2n}{\lambda}$ in prio-

re, et altera in Formula pariter $z = \frac{2n}{\lambda}$, Aequatio utroque casu Inf-

nitiformis; eruntque $y^{(z\sqrt{-1})^x} = y^{\pm C^{\frac{2n}{\lambda}(Lz)}}$, ac $(y\sqrt{-1})^z = \pm$

$C^{\frac{2n}{\lambda}Ly}$, ubi valeat Signum $+$ quando $n = 0, 2, 4, 6$, et ceteris Numeris paribus, atque oppositum $-$ dum $n = 1, 3, 5, 7$ &c.; aliisque disparibus.

246. Ii, qui superius expositam probe calleant Theoricen, nequaquam ambigent de Coronidis veritate, quae Algebrae adversari prima fronte videretur, scilicet de Aequatione $(z\sqrt{-1})^0 = +1$, vel, quum $(z\sqrt{-1})^0$ idem sit ac $z^0 \cdot \sqrt{-1}^0$, de altera Aequalitate $\sqrt{-1}^0 = +1$; quarum alterutra non levem lacunam in rerum Analyticarum gaza complebit. Curiosa autem, potius quam utilis, speculatio foret de plus minusve compositis Functionibus Trigonometricis, veluti sunt Sin. C (Sin. C. x), Cofin. C (Sin. C. x), Tang. C (Sin. C. x), Cofin. C (Cofin. C. x), Sin. C (Cofin. C. x), Tang. C (Cofin. C. x), Sin. C (Tang. C. x), Cofin. C (Tang. C. x), Tang. C (Tang. C. x), aliaeque innumerae vario Linearum Trigonometricarum ordine comparatae, nec non earundem omnimodae Potestates. Duo ternave exempla seligam summae facilitatis ad libitum multiplicanda. Functio Sin. C (Sin. C. x) =

$$\frac{C \frac{(\text{Sin. C. } x)\sqrt{-1}}{2\sqrt{-1}} - C \frac{(\text{Sin. C. } x)\sqrt{-1}}{2\sqrt{-1}}}{\left(1 + \frac{(\text{Sin. C. } x)\sqrt{-1}}{\infty}\right)^{\infty} - \left(1 - \frac{(\text{Sin. C. } x)\sqrt{-1}}{\infty}\right)^{\infty}} = \frac{(\text{Sin. C. } x)^1}{1} - \frac{(\text{Sin. C. } x)^3}{1.2.3} + \frac{(\text{Sin. C. } x)^5}{1.2.3.4.5} - \frac{(\text{Sin. C. } x)^7}{1.2.3.4.5.6.7} + \&c. =$$

$$\begin{aligned}
& \frac{x^1}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^9}{1.2.3.4.5.6.7.8.9} \&c: \\
& - \frac{x^3}{1.2.3} + \frac{x^5}{1.2^2.3} - \frac{13x^7}{1.2^2.3^2.4.5} + \&c: - \&c: \\
& + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2^2.3^2.4} + \&c: - \&c: \\
& - \frac{x^7}{1.2.3.4.5.6.7} + \&c: - \&c:
\end{aligned}$$

vel homologis terminis Trigoni in unum reductis ,

$$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{8x^7}{315} + \&c: - \&c: .$$

$$\text{Eodem iure servato Sin. C. (Cofin. C. x)} = \frac{(\text{Cofin. C. x})^1}{1} - \frac{(\text{Cofin. C. x})^3}{1.2.3}$$

$$+ \frac{(\text{Cofin. C. x})^5}{1.2.3.4.5} - \frac{(\text{Cofin. C. x})^7}{1.2.3.4.5.6.7} + \&c: =$$

$$\frac{\left(1 + \frac{(\text{Cofin. C. x})\sqrt{-1}}{\infty}\right)^\infty - \left(1 - \frac{(\text{Cofin. C. x})\sqrt{-1}}{\infty}\right)^\infty}{2\sqrt{-1}}$$

$$= \frac{C^{(\text{Cofin. C. x})\sqrt{-1}} - C^{-(\text{Cofin. C. x})\sqrt{-1}}}{2\sqrt{-1}} =$$

$$1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \&c: - \&c:$$

$$- \frac{1}{6} + \frac{x^2}{4} - \frac{7x^4}{48} + \frac{61x^6}{1440} - \&c: + \&c:$$

$$+ \frac{1}{120} - \frac{x^2}{48} + \frac{13x^4}{576} - \frac{241x^6}{17280} + \&c: - \&c:$$

$$- \frac{1}{5040} + \frac{x^2}{1440} - \frac{19x^4}{17280} + \frac{541x^6}{518400} - \&c: + \&c:$$

$$+ \&c: - \&c: + \&c: - \&c: + \&c: - \&c: ,$$

et inibi, quemadmodum accidit Analyticis Tetragonis, Lineae omnes hori-
zonta-

rizontales, ac verticales sine fine progrediuntur. Monendum subinde non sine aliqua opinor iucunditate Seriem numero terminorum Infinitam, a qua

illud incipit Tetragonum, $\frac{1}{1} - \frac{1}{1.2.3} + \frac{1}{1.2.3.4.5} - \frac{1}{1.2.3.4.5.6.7} + \&c:$

peraequare Sinum-Rectum Circularis Circumferentiae Arcus Radio-Unitati paris, videlicet quamproximum Numerum Decimalem 0,84147098480514 &c:, id comprobante etiam notissima Serie Sinus dati per Arcum. Quod si de

$$\text{Cofin. C. (Tang. C. x)} = \frac{C \frac{(\text{Tang. C. x})\sqrt{-1}}{2} - (\text{Tang. C. x})\sqrt{-1}}{+ C} =$$

$$\frac{\left(1 - \frac{(\text{Tang. C. x})\sqrt{-1}}{\infty}\right)^{\infty} + \left(1 - \frac{(\text{Tang. C. x})\sqrt{-1}}{\infty}\right)^{\infty}}{2} = 1 - \frac{(\text{Tang. C. x})^2}{1.2}$$

$$+ \frac{(\text{Tang. C. x})^4}{1.2.3.4} - \frac{(\text{Tang. C. x})^6}{1.2.3.4.5.6} + \&c: \text{ experimentum sumere libeat, ita}$$

adstruetur opportuni Calculi Paradigma.

$$\begin{aligned} \text{Cofin. C(Tang.C.x)} &= 1 - \frac{x^2}{2} - \frac{x^4}{3} - \frac{17x^6}{90} - \&c: = 1 - \frac{x^2}{2} - \frac{7x^4}{24} - \frac{97x^6}{720} - \&c: \\ &+ \frac{x^4}{24} + \frac{x^6}{18} + \&c: \\ &- \frac{x^6}{720} + \&c: \end{aligned}$$

lisdem insistendo vestigiis universa haec, quae diximus, vel dictis accedere possent, ad Lineas traducerentur Hyperbolicae Trigonometriae, quin etiam ipsarum loco ad Functiones quarumlibet Variabilium, dummodo istae evolutionem in Series numero terminorum Infinitas patiantur. Veruntamen in Serie postrema obiter adnotare conveniet, quod, vice indeterminati Arcus x substituto unoquoque ex innumeris Arcubus illis Δ respectivae Tangenti Aequalibus, et in XXII°. Capite Voluminis secundi celebratissimae *Introductionis* ab Eulero occasione noni Problematis latius consideratis, pro-

$$\text{fluat } 1 - \frac{x^2}{1.2} - \frac{7x^4}{1.2.3.4} - \frac{97x^6}{1.2.3.4.5.6} - \&c: = \text{Cofin. C (Tang. C. x)} \\ = \text{Cofin.}$$

$$= \text{Cofin. C. } x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \&c: \text{ in Infinitum:}$$

Igitur geminae Series sine Limite protractae $1 - \frac{\Delta^2}{1.2} + \frac{\Delta^4}{1.2.3.4} - \frac{\Delta^6}{1.2.3.4.5.6} + \&c:$, atque $1 - \frac{\Delta^2}{1.2} - \frac{7\Delta^4}{1.2.3.4} - \frac{97\Delta^6}{1.2.3.4.5.6} - \&c:$ eundem revera Numerum repraesentabunt.

247. Vix primo in limine sistam extremae, quae me vocat, investigationis circa Logarithmos Linearum Trigonometricarum versantis. Iacobus Gregorius usque ab incunabulis Geometriae recentioris percelebrem reddidit hanc Logisticae-Trigonometriae doctrinam, feliciterque proprio Marte Logarithmum Tangentis, et Secantis ope Circularis Arcus determinavit. Simplex profecto, ac nostrae methodi viribus adeo in certamine impar haec Theoria reperietur, ut methodi ipsius dignitatem minuere prope videatur specimen, seu periculum illud, quod conficere nunc instituimus. Ac primum de $L(\text{Sin. C. } x)$ sermo factus nos ducit ad facillimam

$$\text{Acquationem } L(\text{Sin. C. } x) = \frac{((\text{Sin. C. } x) - 1)^1}{1} - \frac{((\text{Sin. C. } x) - 1)^2}{2} + \frac{((\text{Sin. C. } x) - 1)^3}{3} - \frac{((\text{Sin. C. } x) - 1)^4}{4} + \&c:, \text{ scilicet, propter } (\text{Sin. C. } x) < 1,$$

$$\text{ut per se constat, } L(\text{Sin. C. } x) = \frac{(1 - \text{Sin. C. } x)^1}{1} - \frac{(1 - \text{Sin. C. } x)^2}{2} + \frac{(1 - \text{Sin. C. } x)^3}{3} - \frac{(1 - \text{Sin. C. } x)^4}{4} + \frac{(1 - \text{Sin. C. } x)^5}{5} - \&c:, \text{ quae Series}$$

vertitur facile in aliam Potestatibus Arcus x ordinatim positis conflata, sed nullo elegantiore modo tractabilem. Praeterea $L(\text{Cofin. C. } x) =$

$$\left(\frac{(1 - \text{Cofin. C. } x)^1}{1} + \frac{(1 - \text{Cofin. C. } x)^2}{2} + \frac{(1 - \text{Cofin. C. } x)^3}{3} + \frac{(1 - \text{Cofin. C. } x)^4}{4} + \&c: \right) = \left(\frac{(\text{Sin. V. C. } x)^1}{1} + \frac{(\text{Sin. V. C. } x)^2}{2} + \frac{(\text{Sin. V. C. } x)^3}{3} + \frac{(\text{Sin. V. C. } x)^4}{4} + \frac{(\text{Sin. V. C. } x)^5}{5} + \&c: \right) =$$

$$\left(\frac{x^2}{1.2} \right)$$

$$\left(\frac{x^2}{1.2} - \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} - \frac{x^8}{1.2.3.4.5.6.7.8} + \&c. \right) = -\frac{x^2}{1.2} - \frac{x^4}{1.2^2.3} - \frac{x^6}{1.3^2.5} - \frac{17x^8}{1.2^3.3^2.5.7} - \frac{31x^{10}}{1.3^2.5^2.7.9} \&c.,$$

$$+ \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{640} - \&c.$$

$$+ \frac{x^6}{24} - \frac{x^8}{96} - \&c.$$

$$+ \frac{x^8}{64} - \&c.$$

cuius Progressionis detecta in Denominatoribus indoles prolixiorem non

optat continuationem. Exinde nascitur $L(\text{Sec. C. } x) = L\left(\frac{1}{\text{Cofin. C. } x}\right) =$

$$-L(\text{Cofin. C. } x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14175} + \&c., \text{ quem-}$$

admodum habet Halleyus in relatis ad annos MDCXCV. ac VI. *Philosophicis Transactionibus*.

248. Aeque venusta consequitur Logarithmi τx Tangentis inventio. Praeter oecumenicam etenim ipsum quasi digito monstrandi Formulam

$$\frac{((\text{Tang. C. } x) - 1)^1}{1} - \frac{((\text{Tang. C. } x) - 1)^2}{2} + \frac{((\text{Tang. C. } x) - 1)^3}{3} -$$

$$\frac{((\text{Tang. C. } x) - 1)^4}{4} + \&c. = L(\text{Tang. C. } x) \text{ alia evolutio dimanat a}$$

singulari Theoremate, quod Logarithmos positivos Tangentium Unitate

maiorum a negativis minorum secernit, concluditque $\text{Tang. C. } \left(45^\circ + \frac{x}{2}\right)$

$$= \frac{1 + \text{Tang. C. } \frac{x}{2}}{1 - \text{Tang. C. } \frac{x}{2}}. \text{ Huiusce Theorematis simplicissima demonstratio hoc}$$

modo, si libeat, poterit exornari. Habetur profecto $\text{Tang. C. } (z + v)$

$$= \frac{\text{Sin. C. } (z + v)}{\text{Cofin. C. } (z + v)} = \frac{(\text{Sin. C. } z)(\text{Cofin. C. } v) + (\text{Cofin. C. } z)(\text{Sin. C. } v)}{(\text{Cofin. C. } z)(\text{Cofin. C. } v) - (\text{Sin. C. } z)(\text{Sin. C. } v)} \text{ ex num. } ^\circ. 202.,$$

O o o

five,

five, ut evidens fit, $= \frac{\text{Tang. C. } z + \text{Tang. C. } v}{1 - (\text{Tang. C. } z)(\text{Tang. C. } v)}$: quamobrem erit Tang. C

$$\left(45^\circ \pm \frac{x}{2}\right) = \frac{1 \pm \text{Tang. C. } \frac{x}{2}}{1 - \text{Tang. C. } \frac{x}{2}}, \text{ prouti superius innuimus (*). Quibus}$$

positis nacti erimus Aequationem $L(\text{Tang. C. } z) = L\left(\text{Tang. C. } \left(45^\circ \pm \frac{x}{2}\right)\right)$

$$= L\left(\frac{1 \pm \text{Tang. C. } \frac{x}{2}}{1 - \text{Tang. C. } \frac{x}{2}}\right) = \pm 2 \left(\frac{(\text{Tang. C. } \frac{x}{2})^1}{1} + \frac{(\text{Tang. C. } \frac{x}{2})^3}{3} + \right.$$

$$\left. \frac{(\text{Tang. C. } \frac{x}{2})^5}{5} + \frac{(\text{Tang. C. } \frac{x}{2})^7}{7} + \frac{(\text{Tang. C. } \frac{x}{2})^9}{9} + \&c: \right) \text{ per num}^{\text{um}}. 102.,$$

aut denique $= \pm$

$$\left(x^1 + \frac{x^3}{12} + \frac{x^5}{120} + \frac{17x^7}{20160} + \&c:\right) = \pm \left(\frac{x^1}{1} + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \frac{277x^9}{72576} + \&c:\right)$$

$$+ \frac{x^3}{12} + \frac{x^5}{48} + \frac{11x^7}{2880} + \&c:$$

$$+ \frac{x^5}{80} + \frac{x^7}{192} + \&c:$$

$$+ \frac{x^7}{448} + \&c:$$

post opportunas substitutiones; quod mire consonat Halleianis in Loxodromicam Curvam lucubrationibus. Servata Moduli mensura omnia, quae de Hyperbolicis Logarithmis Circulari Trigonometriae adscriptis affirmantur, de Tabularium, aliorumve Systemate praedicari facile possunt.

249. Harmonica etiam Hyperbolae Trigonometria iisdem gaudet Attributis. Primoribus autem tantummodo labiis spicilegium quiddam delibabo, ut hisce tandem meditationibus optatum finem imponam. Ordinem qualem-
cunque subnexae Series admittunt.

Sin.

(*) Vide Numerum 365.

$$\text{Sin. H (Sin. H. x)} = x^1 + \frac{x^3}{3} + \frac{x^5}{10} + \frac{8x^7}{315} + \&c = \frac{x^1}{1} + \frac{x^3}{1.3} + \frac{x^5}{1.2.5} + \frac{8x^7}{1.3^2.5.7} + \&c:$$

$$\text{L (Cofin. H. x)} = \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45} - \frac{17x^8}{2520} + \&c = \frac{x^2}{1.2} - \frac{x^4}{1.2^2.3} + \frac{x^6}{1.3^2.5} - \frac{17x^8}{1.2^3.3^2.5.7} + \&c:$$

$$\text{Cofin. H(Tang.H.x)} = 1 + \frac{x^2}{2} - \frac{7x^4}{24} + \frac{97x^6}{720} - \&c = \frac{x^0}{1} + \frac{x^2}{1.2} - \frac{7x^4}{1.2.3.4} + \frac{97x^6}{1.2.3.4.5.6} - \&c:$$

Cunctae hae Progressiones simplicem noscunt originem, et frustra illis distinctius enucleandis operam navare censerem.

250. Iudicium laturi de viribus, extensione, ac compendio methodi superius promotae in Trigonometrica doctrina omnibus numeris absolvenda non singulas eius partes minutatim divisas, sed totam simul excutiant. Nihil equidem Trigonometricum, nulla difficilior de Circulo, atque Hyperbole quaestio proponi usquam poterit, quominus aut lucem quammaximam ab ea mutuetur, aut plenior solutionem recipiat. Summa rerum peractarum evidentialia initos semper Calculos ornat, Analystamque confinia Algebrae Cartesiana non praetergressum ita tuto in itinere regit, ut nec a semita deflectat, nec penetralia ipsa Veritati sacra ob oculos sibi ponere pertimescat, ac serena fronte lustrare. Comparatio rite recteque instituta methodi a nobis, quoad sciam, primum excultae cum aliis, quas eximii per Europam Scriptores haecenus prodidere, non omnem fortassis auferet utilitatis eius, pretiique decorem (*).

(*) Rogerus Cotesius veram Trigonometriam in Logometriam Imaginariam, seu in *Modum inexplicabilem*, verti inter primos adseruit loquens de Superficie dimetienda Sphaeroidis oblongae in calce pag^{ae}. 28. *Harmoniae Mensurarum*.

C A P U T VI.

QUAEDAM DE SERIEBUS INFINITIS QUADRATURAM HYPERBOLAE
ET CIRCULI COMPLECTENTIBUS ALIISQUE ARGUMENTI
OCCASIONE MEMORATIS.

251. **N**EC nova moliar inventa de Circuli agens, atque Hyperbolae Tetragonismo, nec antiquis laudem imminuam. Veteribus autem prope obsoletis venustatem dare iucundum semper, et cum dignitate coniunctum existimarunt ingenuarum artium cultores; ita, ut tanto maiore ratione id mihi licere confidam, quia prospectum novo ordine, novisque ornatum coloribus referans adquisitionis facilitatis in celebratissimo argumento virtutem animus est denuo experiundi. Non raro enim contingit vetusta tractando, quod, praeter nitidis meliorisque ordinis accessionem, quaedam eis etiam nova inferantur. Methodum equidem non Inventorum, qui in Veritatis vicinia saepius ad eam consequendam per obliquos serpunt meatus, sed ordinem Inventionis a primis veluti arachnei filii elementis retexere nec usu, nec elegantia caret, praesertim in Mathematicis disciplinis.

252. Circuli, atque Hyperbolae *Quadratrix*, ut aiunt Geometrae, Curva est eiusmodi, quae vires Algebrae Finitorum excedens in Transcendentium familia locatur. Verum non recte Encyclopedistae, de naevo facillime emendando nihil solliciti, Curvarum omnium *Quadratricem* pro universaliter Transcendente definiunt, sibi que ipsis paullo post de Parabola Apolloniana scribentibus adversantur, quippe quum innumerae sint eunctorum generum Curvae Quadraturam Geometricam admittentes, tametsi fuerit Circulo, atque Hyperbolae a Magnitudinum Scientia, perperam tot tantisque adlaborantibus Viris, id absolutissime denegatum. Haec autem Geometrica Hyperbolae, et Circuli Irrationalitas, Tummopere distincta ab Arithmetica Quantitatum Asymmetria, Analystarum adeo torfit ingenia, ut desperatam alterutrius Curvae Quadratricem Geometricam quaerentes Algebrae penum fere divinis inventis certatim locupletaverint. Series porro univer-

sae

fae. numero terminorum Infinitae, formaque varia multiplices, in quibus abunde versantur Capita III.^{um} ac V.^{um}, ope unius Binomii Neutoniāni Formulae Sectores exhibendo Hyperbolae aut Circuli, seu Logarithmos Reales et Imaginarios, per Numeros, aut per Trigonometriae oecumenicae Lineas, Quadraturas Transcendentes praefatas Algebrae Cartesianae nunc primum restituant, latamque ab Infinito ita vindicant iniuriam, ut amplificato Finitorum Calculi imperio dolendum potius sit, quam mirandum, promerita minorem Neutoniano ipsi Binomio haftenus contigisse fortunam.

253. Sed praecipuarum, et in Matheseos Fastis celebriorum Serierum deductioni ad Circulum, vel Hyperbolen pertinentium alia quoque patet via valde discrepans ab ea in V.^o Capite exposita, ac Lectorum sagacitati ibidem dicata. Ad hanc dum accedo novam semitam perambulandam nemo arbitretur universas me velle huiusmodi Series complecti, quum et minus elegantes, et sublimioribus Approximationum Infinitesimalium artificiiis excitatas, et magis, quam par esset, indirectas negligi debere ipsamet rei petat natura. Profecto ea omnia, quae meram Polygonorum Circulo, vel Hyperbolae circumscriptorum, aut inscriptorum redolent Geometriam, tribuuntque Series irrationalibus Numeris perplexas, sedem habere nequeunt aliarum illustrationi hoc in prospectu paratam. Perbella inter illas Vietaea, quin et ceterarum antiquissima, quum Thuano Historico teste fato cesserit Auctor anno MDCIII. (*). Circuli etenim ad circumscriptum Quadratum Proportionem definivit Vieta praesidio Fractionis, cuius Denominator est Productum sine numero terminorum eo magis Radicalium - Radicalibus involutorum, quo longius ii progrediuntur, et eiusmodi formam habentium in ipsius Infiniti Limite inaccessibili, quae ansam postea Leibnitio, senioribusque dederit Bernoulliis hanc Theoricea promovendi ad Aequationum nonnullarum obtinendam resolutionem. Id simul comprobant Ioannis Bernoullii *Opera anecdota*, et *Calculi Integralis Lectiones* Lutetiae Parisiorum in usum Hospitalii annis MDCXCI. XCII. conscriptae, Iacobi Fratris *Positionum Arithmeticarum Pars altera* pene synchrona, ac Leibnitii ad Ioannem Bernoullium Epistola anno MDCCXII. missa. Eadem fere Vietae idea, sed novis usibus

exorna-

(*) Ea Series irrationalis exstat in pagina 400. *Operum Mathematicorum* Editionis MDCXLVI. curante Francisco Schootenio.

exornata in *Cyclometrico* Snellii denuo comparuit, nec multum ab ista Irrationalium methodo distant Ioannis Poleni lucubrationes circa Seriem Analyticam Circuli Tetragonismo inveniendone idoneam ad calcem Dialogi *de Vorticibus Coelestibus* Patavii anno MDCCXII. editi expressam. Potiore autem iure Series asymmetris terminis indefinite compositas Circulo, atque Hyperbolae, et utriusque Curvae Sectoribus accommodavit ingeniosissimus ille Iacobus Gregorius, qui unica Formula Areas Circuli, et Hyperbolae Apolloniana complexus Geometris omnibus praeivit in mutua earum Harmonia detegenda, qua fit, ut, ipsa manente expressionum Analyticarum compositione, Signa tantummodo quarundam differant Magnitudinum.

254. Minus, minusque me cura sollicitat inelegantissimae Circuli Quadraturae a Cluverio propositae, et anno MDCLXXXVI. Lipsiensibus Commentariis insertae. Priores Seriei termini ab isto Archimedei Parabolae Tetragonismi inani detractore vulgatae statuunt

$$\frac{\Pi}{4} = 1 + \frac{2}{5} + \frac{36}{325} + \frac{5488}{160225} + \&c., \text{ ideoque } \Pi = 5 + \frac{3}{5} + \frac{144}{325} + \frac{21952}{160225} + \&c. =$$

$$6 + \frac{14}{325} + \frac{21952}{160225} + \&c., \text{ si geminae post Numerum 5 in unam colli-}$$

gantur Fractiones. Nulla etenim constans Lex afficit Fractionum Numeratores, ac Denominatores; adeo, ut, tametsi Series illa meliore ordine conscri-

$$\text{pta abeat in } 1 + \frac{1.2^1}{1.5^1} + \frac{1.2^2.3^2}{1.5^2.13} + \frac{1.2^4.7^1}{1.5^2.13.493} + \&c. = \frac{\Pi}{4}, \text{ non sup-}$$

petat methodus Progressionis continuandae, nec elegantiae, Legisve defectus ob maioris convergentiae, aut adproximationis scrupulum aliquomodo veniam mereatur, quoniam additis simul terminis in secunda statim Decimali Nota a Ludolphi Numeris immaniter distet Cluverii memorata Proportio. Maiore pretio digna se praebet tot inter allatas non primigeniae originis Series notissima a Lagnyo in *Commentariis Academiae Parisiensis* anno MDCCXIX. exarata, quae, Trianguli Aequilateri circumscripti Perimetro facta = 3,

$$\text{tribuit Circuli Circumferentiae valorem } \frac{16}{9} + \frac{32}{945} + \frac{48}{24057} + \&c., \text{ sive,}$$

dum

dum occulta Lex explicetur, $\frac{8.2}{1.3.3^1} + \frac{8.4}{5.7.3^3} + \frac{8.6}{9.11.3^5} + \frac{8.8}{13.15.3^7} +$
 $\frac{8.10}{17.19.3^9} + \&c:$ in Infinitum. Latus namque Trigoni Aequilateri Unitatem
 aequans Circuli inscripti Radium reddit $\frac{\sqrt{3}}{6}$, et Arcus 30° . Tangentem
 forte fortuna Rationalem $\frac{1}{6}$. Si itaque in Formula universali numeri 209.
 substituantur $\frac{1}{6}$ loco Tangentis t , atque $\frac{\sqrt{3}}{6}$ vice subintelle-
 ctæ Unitatis, seu Radii, emerget $\frac{t^1}{1} - \frac{t^3}{3R^2} + \frac{t^5}{5R^4} - \frac{t^7}{7R^6} + \frac{t^9}{9R^8} -$
 $\frac{t^{11}}{11R^{10}} + \&c:$, vel, uti habet numerus 210., $\frac{3R^2t - R^0t^3}{1 \cdot 3R^2} + \frac{7R^2t^5 - 5R^0t^7}{5 \cdot 7R^6} +$
 $\frac{11R^2t^9 - 9R^0t^{11}}{9 \cdot 11R^{10}} + \frac{15R^2t^{13} - 13R^0t^{15}}{13 \cdot 15R^{14}} + \&c: = \frac{\Pi}{12} = \frac{8}{6.9} + \frac{16}{6.5.7.27} +$
 $\frac{24}{6.9.11.243} + \frac{32}{6.13.15.2187} + \frac{40}{6.17.19.19683} + \&c:$; quamobrem erit $\Pi =$
 $\frac{16}{9} + \frac{32}{945} + \frac{48}{24057} + \frac{64}{426465} + \frac{80}{6357609} + \&c:$, cuius primi Pro-
 gressionis termini ipsimet sunt a praeaudato Auctore in publicam editi uti-
 litatem.

255. Properans vero ad ea, quae maioris intersunt in assumpto argu-
 mento, Lectoribus vovebo libentissime investigationem demonstrationis Se-
 riei, quam Machinius iamdudum protulit edente Ionesio in *Palmariis*, non-
 nullaque obiter tantummodo adiungam de mirabili geminarum Serierum

Leibnitianae $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} -$

$\&c:$, ac Neutoniana $1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + \frac{1}{15} -$

$+\frac{1}{17} - \&c:$ relatione. Prima peraequat Peripheriae Circuli Octantem

in hypothefi facta Radii $= 1$, dum altera, supposito Radio $= \sqrt{\frac{1}{2}}$, vel La-
 tere

$$\begin{aligned}
 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} &\&c.; \text{ fiatque comparatio cum al-} \\
 &+ \frac{2}{5} - \frac{2}{15} + \frac{2}{25} - \frac{2}{35} &c.; \\
 &+ \frac{2}{145} - \frac{2}{455} &c.; \\
 &\&c.;
 \end{aligned}$$

tera Serie Infinita, cuius aequipollentia demonstrationem conceptu facilem

$$\begin{aligned}
 \text{optat, } 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} &\&c.; \text{ videamus esse } 1 \\
 = 1, \text{ ac deinceps } - \frac{1}{3} + \frac{1}{5} &= \frac{2}{15} = \frac{1}{3} - \frac{1}{5}, \text{ nec non } - \frac{1}{7} \\
 &+ \frac{2}{5} - \frac{2}{15}
 \end{aligned}$$

$$+ \frac{1}{9} = - \frac{1}{7} + \frac{1}{9} \&c.; \text{ et haec terminorum aliqua diligentiae accessio-}$$

ne cumulatorum aequalitas perpetuo observetur in Serierum progressu, non amplius de Theoremate dubitandum. Praeterea emerget

$$\begin{aligned}
 \sqrt{2} = \frac{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} &\&c.;}{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} &c.;},
 \end{aligned}$$

quod Confectarium irrationali Numero Platonico praestantiori illustrando aptissimum iucunditatem certe, si non utilitatem, praesefert.

256. Nescio autem quonam fato contigerit, quod, in vulgus edita celeberrima Hyperbolae Quadratura a Brounckero, illic minus foecundam manere passi fuerint posteriores Geometrae, quo praecisa Auctoris verba perduxerunt. Equidem, ni fallor, opinio me detinet singularis elementarem methodum ipsam, qua definivit Brounckerus Aream Hyperbolae Aequilaterae Apollonianae inter Asymptotum, et Ordinatas in Ratione dupla comprehensam exhiberi, dum Potentia sit 1, a Serie numero terminorum Infinita

$$\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \frac{1}{9.10} + \frac{1}{11.12} + \&c., \text{ vel } 1 - \frac{1}{2} + \frac{1}{3}$$

$$- \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \&c., \text{ quae}$$

$$\text{nihil differt ab illa Leibnitii } \frac{1}{8} + \frac{1}{48} + \frac{1}{120} + \frac{1}{224} + \frac{1}{360} + \&c. \text{ ad}$$

Hyperbolicam Potentiam $\frac{1}{4}$ relata, multo plura in se continere. Sen-

tio namque inventum istud, a nativis simplicibusque Rectangulis exortum Hyperbolico Spatio inscriptis, summum fuisse veluti caput, a quo digredi poterant, si animadvertissent Geometrae, omnia etiam elegantiora de Circuli tetragonismo hucusque prolata; idque tam felici successu, ut pauca defumeret a Synthesi, et Analyfi, nec ultra vulgarem captum, Theoremata. Mirum sane erit Seriem Rectangulorum Infinitam a Cartesio repertam, cuius mentio occurrit in *Posthumis* Amstelodami vulgatis anno MDCCIV., nihil ad exprimendam rationalibus Numeris Circuli Quadraturam conferre, illudque emendari sterilitatis vitium ab Hyperbola, prima fronte comparisonem cum Circulo quamlibet respiciente: tantae molis est cognationis, et Harmoniae inter utrasque Curvas sanctissimum foedus. Quanto autem fecundius se ostendet per inferius dicenda Brounckeri primum inventum, de-

$$\text{finitae nimirum Areae Hyperbolicae aequalis } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$$

$$- \frac{1}{6} + \frac{1}{7} - \&c., \text{ eo magis legitimi Auctoris decus imminui non patiar}$$

ab Iacobo Bernoullio, eiusque interprete Cramero illud inventum Leibnitio in *Positionum de Seriebus Infinitis Parte tertia* tribuentibus. Nec *Encyclopediae* me subscribam, ubi vult (*) Mercatorem methodo ab *Arithmetica Infinitorum* Vallisii mutuata Fractionem in Seriem numero terminorum Infinitam evolvisse, et Brounckerianam ideo Quadraturam generaliore redditam indefinito aptasse Spatio Hyperbolico, quum Vallisius ipse, Scriptor certe

philo-

(*) Art. *Quadratura*.

philo-Britannicus, in *Traſſatu de motu* biennio poſt *Logarithmotechniam* publici iuris factam totum Mercatori promeritum decus adſcribat.

257. Sit Hyperbola Aequilatera in Schemate depicta XXIV°. et Brounckeri ſyntheticam ratiocinationem Spatio trapezoidali BCDE applicatam ad Infinite longum in parte oppoſita ſitum Trilineum EBAFGHLE transferamus. Inſcribantur igitur ſine Limite Rectangula Parallelogramma multifariam, ſed aequaliter, in Infinitum ſecando Unitatem AB, et, prouti ex ipſo Schemate adparet, erit Area EBAFGHLE = EBAF + KOFI + NPOM + LRIG

$$+ YQPZ + TXMS + \Phi YR\Theta + \Xi \Delta GA + \&c: = 1 + \frac{1}{2} + \frac{1}{12} + \frac{1}{2}$$

$$+ \frac{1}{56} + \frac{1}{30} + \frac{1}{12} + \frac{1}{2} + \&c:, \text{ quod Conicorum evincunt Elementa, ſcilicet} = 1 + \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{1.2} + \frac{1}{7.8} + \frac{1}{5.6} + \frac{1}{3.4} + \frac{1}{1.2}$$

$$+ \frac{1}{15.16} + \frac{1}{13.14} + \frac{1}{11.12} + \frac{1}{9.10} + \frac{1}{7.8} + \frac{1}{5.6} + \frac{1}{3.4} + \frac{1}{1.2} +$$

$$\&c:, \text{ ſubſequentibus Rectangulis eadem methodo adnumeratis. Inibi redeuntium Numerorum periodus, ac praefertim Rectangula verſus regionem H ſemper ſemperque cum constantibus Areis } \frac{1}{12}, \frac{1}{2} \text{ remeantia iucundiſſimam Seriei}$$

indolem detegunt, et perquammaximam attentionem merentur. Poſtrema autem Seriei forma hoc progreſſu terminorum diſpoſita $1 + \frac{1}{2} + \frac{1}{12}$

$$+ \frac{6}{12} + \frac{1}{56} + \frac{1}{30} + \frac{10}{30} + \frac{14}{56} + \frac{1}{240} + \frac{1}{182} + \frac{1}{132} + \frac{1}{90}$$

$$+ \frac{18}{90} + \frac{22}{132} + \frac{26}{182} + \frac{30}{240} + \&c: \text{ reddit } 1 + \frac{1}{2} + \frac{7}{12} + \frac{11}{30}$$

$$+ \frac{15}{56} + \frac{19}{90} + \frac{23}{132} + \frac{27}{182} + \frac{31}{240} + \&c: = \frac{3}{1.2} + \frac{7}{3.4} +$$

$$\frac{11}{5.6} + \frac{15}{7.8} + \frac{19}{9.10} + \frac{23}{11.12} + \frac{27}{13.14} + \frac{31}{15.16} + \&c:, \text{ Numeratoribus}$$

Arithmetice per Differentiam 4 crescentibus, ac Denominatorum reſpecti-

vos

vos Factores aequantibus simul sumptos, vel simplicius $= 1 + \frac{1}{2} + \frac{1}{3}$

$$+ \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} +$$

$$\frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \&c.; \text{ quemadmodum patet. Nacti ergo erimus eam}$$

ipsam Seriem numero terminorum Infinitam, qua in numeris 109. ac 114. absque Figurarum contemplatione $\tau\delta L(\infty)$, sive $-L(0)$ repraesentavimus. Exinde insperato Syntheseos, Algebraeque consensu evadet EBAFGHLE, vel, ob Magnitudinem Infinitam, EFGHLE, aut, vi Conicorum, Area Sectoris

$$\text{Centralis Infinite producti AELHGFA} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$$

$$\frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} +$$

$$\frac{1}{16} + \&c.; \text{ sive} = 2 \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} \right.$$

$$\left. + \frac{1}{16} + \&c. \right) = 2 \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \right.$$

$$\left. \frac{1}{15} + \frac{1}{17} + \&c. \right), \text{ quod et Calculi ostendit facilitas summa, et alias}$$

fuit memoratum. Sector autem Centralis Infinite magnus AELHGFA duplam abscondit ex Conicis Proportionem respectu habito ad alterum, qui contineatur Radiis AE, et AI per punctum concursus I ducto extremarum Tangentium EI, atque HGIA Quadrantis Perimetri Hyperbolicae, quarum secunda cum Asymptoto congruit; unde huiusce postremi Area Sectoris erit $1 +$

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \&c. \text{ sine ullo Li-}$$

mite. Subtilior est sane, nec in Conicorum Institutionibus, quod sciam, observata legitima haec separatio gemini ex Infinite magnis Sectoribus, qui propter Infinitatem depicto in Schemate secerni nequeunt, at mentis acie finitos Sectores EAD semper duplos quo ad EA, persequente (*) in latebris quoque Infiniti singillatim conspicientur.

(*) Vide proximum numerum 260.

258. Dum vero non Potentia Hyperbolae EBAF, sed ad maiorem computationis simplicitatem sit Semiaxis vel Radius AE = 1, et idcirco Po-

tentia = $\frac{1}{2}$, emerget $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \dots$ = AELHGFA, nimirum = Quadranti Areae totius Asym-

ptoticae; quo posito Octans Spatii Infinite magni Hyperbolici, aut Sector ille Centralis, cui Sinus-Rectus respondeat aequalis Infinites magno Cofinui, Tangensque EI, vel EC = AE Radius = 1, seu tandem Octans, vel Sector a concursu duarum Quadrantis extremarum Tangentium, uti superius, determinatus Aream claudet =

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \dots$. Hifce

in antecessum definitis, quum ex natura Aequationis $y^2 = \pm (x^2 - R^2)$, quae permutato solummodo Signo nunc Hyperbolen praebet, nunc Circulum, universalis tutaque regula sit expressiones Analyticas omnes ad priorem spectantes in alterum transferri posse dummodo Signa, aut cuncta, aut

alterna, variationem patiantur, unica manet in Serie proposita $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \dots$

ordinatio se-

ligenda, nimirum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \dots$

ad Circularis Areae Quadrantem EAΩ dimetiendum huius

Seriei ope conducens. Iure etenim reiicientur ceterae menti obversantes

hypotheses $\pi \approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \dots$

&c.; nec non $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \dots$

&c.; eoquod prima Series Infinitum negativum, scilicet Sector-

rem AEDμUCBA in Hyperbola ipsa suppeditat, et altera, utpote quae eius-

dem sit magnitudinis ac $\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \&c. \right)$, negativum praesefert valorem, quum e-contra

Circularis Sector $EA\Omega$ sit positivus ita, ut Series posterior Sectorem $EA\omega$ potius respiciat. Qua exclusione facta consequitur esse Quadrantalem Aream

$$EA\Omega = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \&c.,$$

ac praeterea Sectoris-Octantis spatium, vel Sectoris a Radio AE , et AF per mutuam intersectionem I Tangentium extremarum Quadrantis Circularis EI , ΩI ducto comprehensi, nimirum $\approx EAF$ Aream, aut Sectoris potius illius habentis Sinum Cosinui parem, Tangentemque aequalem Radio-Unitati, five

$$\frac{EA\Omega FE}{2} = \frac{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \&c.}{2}, \text{ et}$$

idcirco Arcum Octantem Circuli, cuius Radius sit Unitas, Serie Infinita

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \&c. \text{ reprae-}$$

sentari, atque duplo analogiam servare Sectoris, aut Pseudo-Arcui, Aequilaterae Hyperbolae.

259. Universa, quae diximus, congruunt Seriei, ut morem Geometricarum sequar, Leibnitianae; nec ei adipiscendae Arithmetica Infinitorum, Algorithmum Differentialem et Integrale, seu Transformationem Curvarum ad eliminandam asymmetriam, prouti labente saeculo septimo supra decimum Iacobus, Davidque Gregorius, Neutonus, Leibnitius, Ozanamus, Catelanus, Lagnyus, Iacobus Bernoullius, alique adhibuerunt, sed nativam Synthesis Geometricae simplicitatem censui experiendam. Archimedis igitur, atque Brounckeri, qua ad Parabolam, et Hyperbolen Conicam dimetientas continua inscriptione Parallelogrammorum, ac Triangulorum sunt usi, methodum aemulatus deduxi, existente Potentia $= 1$, fore $1 + \frac{1}{2} + \frac{1}{3}$

$$+ \frac{1}{4}$$

$$-\frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14}$$

$$-\frac{1}{15} + \frac{1}{16} - \frac{1}{17} + \&c: = \text{Sect. seu Quad. Hyp. } \infty = \frac{3}{2} - \frac{7}{12}$$

$$-\frac{11}{30} + \frac{15}{56} - \frac{19}{90} + \frac{23}{132} - \frac{27}{182} + \frac{31}{240} + \&c:; \text{ et hac in Serie Nu-}$$

meratores Arithmetice progrediuntur, dum quatuor Unitatibus, uti supra dictum, semper excrescunt. Denominatores vero secundas habent Differentias constantes, nempe octo Unitates, quemadmodum subnexus ostendit Diagramma.

$$\begin{array}{cccccccc} 2 & 12 & 30 & 56 & 90 & 132 & 182 & 240 & \&c: \end{array}$$

$$\begin{array}{cccccccc} 10 & 18 & 26 & 34 & 42 & 50 & 58 & & \\ 8 & 8 & 8 & 8 & 8 & 8 & 8 & & \end{array}$$

. Diversus etiam Series ordo in-

stitui, si libeat, facile poterit, idest Sect. seu Quad. Hyp. $\infty = 2 \left(1 + \frac{1}{3} + \right.$

$$\left. \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \&c: \right) = 2 \left(\frac{1}{2} + \frac{1}{4} + \right.$$

$$\left. \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{16} + \&c: \right): \text{ quapropter evadet, ut}$$

$$\text{per se constat, } \frac{\text{Sect seu Quad. Hyp. } \infty}{2} = \frac{4}{3} + \frac{12}{35} + \frac{20}{99} + \frac{28}{195} + \frac{36}{323}$$

$$+ \&c: = \frac{3}{4} + \frac{7}{24} + \frac{11}{60} + \frac{15}{112} + \frac{19}{180} + \&c:; \text{ istaeque Series adeo}$$

sunt comparatae, ut Numeratores pares in prima, quorum Differentia 8, et in altera impares, Differentia 4 manente, in Arithmetica simul Progressione augeantur, sed Denominatores vicissim impares in prima, ac pares in altera secundis Differentiis gaudeant aequalibus, et respective in dupla Ratione, uti gemini sequentes Typi demonstrant.

$$\begin{array}{cccccccc} 3 & 35 & 99 & 195 & 323 & \&c:, & 4 & 24 & 60 & 112 & 180 & \&c:. \\ 32 & 64 & 96 & 128 & & & 20 & 36 & 52 & 68 & & & \\ 32 & 32 & 32 & & & & 16 & 16 & 16 & & & & \end{array}$$

Quod autem maioris oblectamenti hisce Arithmeticis Meditationibus inesse puto comparatio praestat Brounckerianae cum nostra Series. Deductio enim

facilis

facilis a Serie $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10}$
 $+ \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} - \frac{1}{16} + \frac{1}{17} \&c:$ alterius, quam

Auctor ipse conscripsit, $\frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \frac{1}{56} + \frac{1}{90} + \frac{1}{132} + \frac{1}{182}$
 $+ \frac{1}{240} + \&c: =$ Sect. Hyp. EAD dummodo $EB = 2DC$, atque, ut

in Seriebus Numeri praecedentis, $EBAF = 1$, elargitur $\left(\frac{3}{2} + \frac{7}{12} + \frac{11}{30} + \frac{15}{56} + \frac{19}{90} + \frac{23}{132} + \frac{27}{182} + \frac{31}{240} + \&c:\right) - \left(\frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \frac{1}{56} + \frac{1}{90} + \frac{1}{132} + \frac{1}{182} + \frac{1}{240} + \&c:\right) = DCU_{\mu}D =$

$\infty = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \&c:$, priore nimirum

Harmonica Serie post tot labores in integrum regenerata, quamvis iure non levi videretur aliqua sui parte minuta.

260. Istis, quae hypothese etiam Potentiae Hyperbolicae $= \frac{1}{2}$ ac-

commodantur, facillimo Calculi ductu accedat exemplum Series Rationalis alteram prima fronte diversissimam Rationalem peraequantis, ne, uti saepius solet, desperatum statim existimetur Problema vel eliminationis suspicatae Asymmetriae, vel detegendae latentis Aequalitatis respectu habito ad aliam praecognitam Seriem in data Serie Infinita. Spatium etenim Hyperbolicum EBCD, aut EAD aequale est, inventore Brounckero

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} \&c:$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \&c:, \text{ excluso nimirum}$$

omni impari Denominatore, dum $EA = 1$, eademque in hypothese, sicuti

Tangens

Tangens $E\Pi$ reperitur $= \frac{1}{3}$ vi Proportionis Geometricae $UC - CV : UC ::$

$VD : C\Pi$, scilicet $\frac{3}{2}\sqrt{\frac{1}{2}} : 2\sqrt{\frac{1}{2}} :: \frac{1}{2} : \frac{2}{3} = C\Pi$, oritur iucunda mihi satis

Aequatio $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} \&c.$, aut

$\frac{1}{4} + \frac{1}{24} + \frac{1}{60} + \frac{1}{112} + \frac{1}{180} + \&c. = \frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \frac{1}{9 \cdot 3^9} +$

$\frac{1}{11 \cdot 3^{11}} + \&c.$ ex num^o. 209., cui postremae similem Seriem $\frac{1}{3} - \frac{1}{3 \cdot 3^3} +$

$\frac{1}{5 \cdot 3^5} - \frac{1}{7 \cdot 3^7} + \frac{1}{9 \cdot 3^9} - \frac{1}{11 \cdot 3^{11}} + \&c.$ Analystae pro Circulo, sed obliqua me-

thodo, noverant. Quin etiam, quum ex altera Proportione $AV : VD :: AC :$

Cy , five $\frac{5}{2}\sqrt{\frac{1}{2}} : \frac{1}{2} :: \frac{4}{2}\sqrt{\frac{1}{2}} : \frac{2}{5}$, obtineatur $Ey = \frac{3}{5}$, profluet nova Se-

ries $\frac{3^1}{5^1} + \frac{3^3}{3 \cdot 5^3} + \frac{3^5}{5 \cdot 5^5} + \frac{3^7}{7 \cdot 5^7} + \frac{3^9}{9 \cdot 5^9} + \frac{3^{11}}{11 \cdot 5^{11}} + \&c. = \frac{1}{3^1} + \frac{1}{3 \cdot 3^3} +$

$\frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \frac{1}{9 \cdot 3^9} + \frac{1}{11 \cdot 3^{11}} + \&c. = \frac{1}{4} + \frac{1}{24} + \frac{1}{60} + \frac{1}{112} + \frac{1}{180} + \&c.$,

quarum ambae priores multo magis Brounckeriana Serie convergunt, earumque aequalitas regitur Theoremate Sectoris Centralis $EADvE$ bisecti a Radio $A\Pi$, per concursum extremarum Perimetri Tangentium ducto. Huic theoreticae generali affectioni, tametsi Leibnitius usque ab anno MDCXCI., anteaque Hugenus totam pene Trigonometriam Canonicam tacite ei commendarint, vix in vulgatis Conicorum Synopsis locum praebent plerique ex Institutionum innumerarum Scriptoribus. Hinc erit Tangens Ey ita secta in Π , ut $Ey : E\Pi :: 9 : 5$, vel in Ratione *quadripartiente - quinta*. Series insuper illae, quas Brounckerus, atque Leibnitius Arithmeticis addidere Hyperbolae, et Circuli Quadraturis omnimodam eas ostendendi commoditatem praeseferunt. Nulla mora interiecta methodum sequor fortassis

in tanta Matheoseos luce non novam. Sic $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} -$

Rrr

$\frac{1}{4} +$

$$\frac{1}{4} - \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \&c: = 1 = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}$$

$$+ \frac{1}{5.6} + \frac{1}{6.7} + \&c:; \frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{6} + \frac{1}{6} - \frac{1}{8} + \frac{1}{8} - \frac{1}{10} +$$

$$\frac{1}{10} - \frac{1}{12} + \&c: = \frac{1}{2} = \frac{2}{2.4} + \frac{2}{4.6} + \frac{2}{6.8} + \frac{2}{8.10} + \frac{2}{10.12} +$$

$$\frac{2}{12.14} + \&c:, \text{ quae ex superius posita directe etiam dimanant si termino-}$$

rum omnium dimidia ibidem fumantur; $\frac{1}{2.4} + \frac{1}{4.6} + \frac{1}{6.8} + \frac{1}{8.10} +$

$$\frac{1}{10.12} + \frac{1}{12.14} + \&c: = \frac{1}{4} = \frac{1}{8} + \frac{1}{24} + \frac{1}{48} + \frac{1}{80} + \frac{1}{120} +$$

$$\frac{1}{168} + \&c:; \frac{1}{3} - \frac{1}{3} + \frac{1}{5} - \frac{1}{5} + \frac{1}{7} - \frac{1}{7} + \frac{1}{9} - \frac{1}{9} + \&c:$$

$$= \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \&c: = \frac{1}{2} = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + \&c:;$$

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8} + \&c: = 2 \left(\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \frac{1}{9.11} + \frac{1}{11.13} + \&c: \right) = 1; \frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} +$$

$$\frac{1}{4.6} + \frac{1}{5.7} + \frac{1}{6.8} + \frac{1}{7.9} + \frac{1}{8.10} + \&c: = \frac{3}{4} = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} +$$

$$\frac{1}{24} + \frac{1}{35} + \frac{1}{48} + \frac{1}{63} + \&c: . \text{ Deduci subinde possunt, vel ad in-}$$

vicem comparari aliae Series a Leibnitio repertae, ac summabiles, nimirum, reciproca Trigonalium $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \frac{1}{28} + \&c: =$

$$2 \left(\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \frac{1}{7.8} + \&c: \right) = 2,$$

non vero $1 - \frac{1}{2}$, uti mendose habet Montuclas (*); reciproca Pyramidalium

(*) Pag. 330. Voluminis II. *Historiae Mathematicae* &c:.

$$\text{lium } \frac{1}{1} - \frac{1}{4} + \frac{1}{10} - \frac{1}{20} + \frac{1}{35} - \frac{1}{56} + \frac{1}{84} - \dots = 1 \frac{1}{2} =$$

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{6} - \frac{1}{12} + \frac{1}{15} - \frac{1}{20} + \frac{1}{30} - \frac{1}{35} + \frac{1}{42} - \frac{1}{56} +$$

$$\frac{1}{63} - \dots; \text{ ceteraeque non paucae, quae ex diversis originem ducerent}$$

superiorum Serierum combinationibus.

261. Videant nunc Synthesin peramantes, quam saepenumero contingat in Mathematicis disciplinis nobiliore nimis cultu exornata nativae simplicitati restituere, famamque in antiquis Problematibus sublimi admodum more tractatis acquisitam, si liceret posteris meliori aevo gaudentibus, in nihilum fere redigere. Senferat etenim ante omnes Vallisius in *Sectionum Conicarum novae Methodi* Parte II.^a cognationem Figurae Ellipsium, et Hyperbolarum, communionemque symptomatum a Signi unica variatione pendentem. Felicis huiusce conceptus adiumento in lucem editi anno MDCLV. Brounckerus ipse iam poterat labente anno MDCLVII. transitum non difficilem facere ab Hyperbolae invento Tetragonismo ad Arithmeticam Circuli Quadraturam. Multo autem magis posteriore aetate Leibnitius, Ozanamus, Grandius, ceterique omnes, qui ad integram Circuli mensuram determinandam Curva Asymptotica *Quadratrice* usi sunt, a postremo Scriptore ob Sinus-versos Ordinatis aequales *Versoria* nuncupata, faustioribus auspiciis celeberrimum illud Problema solvere potuissent, aut iamdudum per Series Infinitas resolutum synthetice demonstrare. Nec silentio praetereundum censeo variorum, qui hac de re differuerunt, Curvas praefatas Circuli *Quadratrices* simul adamussim coincidere, et permutatis tantummodo Axibus Coordinatarum convenire cum Linea Ordinis tertii inter Conchoidales Asymptoticas Diametrum habentes a Neutono numeratas, ac signanter Speciei LXIII., et in Schemate septuagesimo-primo sub titulo *tertii Ellipseos Hyperbolismi* depicta. Namque illius Curvae Aequationem Ozanamus ita exhibet $y =$

$$\frac{2ax^2}{a^2 + x^2}, \text{ cui, posita } y = 2a - 2z, \text{ respondet } z = \frac{a^3}{a^2 + x^2}, \text{ quo ducit}$$

Curvae genesis a Grandio prolata, interea dum Curva Newtoni Aequalitatem noscens $xy^2 = -Cx + d$, five, quod idem est, $yx^2 = d - Cy$, distinguitur pariter, versis C in a^2 , et d in a^3 , consimili Aequatione $yx^2 = a^3$

$= a^3 - a^2 y$; adeo, ut nemo de memorata aequipollentia dubitare unquam possit.

262. In Tabula elegantissima *Commentariis Lipsiensibus* anni MDCLXXXII., et *Philosophicis* etiam *Transactiōibus* ad praecedentem annum relatis inserta, quae Prospectum complectitur Serierum Infinitarum, vel summabilium, vel ab Hyperbolae, aut Circuli harmonica Quadratura pendentium, duo vacua occurrunt a Leibnitio neglecta, qualia sunt Series reciproca omnium

$$\begin{aligned} \text{Numerorum naturalium } & \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \\ & + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} \\ & + \frac{1}{19} + \frac{1}{20} + \&c., \text{ et altera univerforum eorundem Quadratorum pa-} \end{aligned}$$

$$\begin{aligned} \text{riter reciproca } & \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \\ & \frac{1}{81} + \frac{1}{100} + \frac{1}{121} + \frac{1}{144} + \frac{1}{169} + \frac{1}{196} + \&c: \text{ ab insequentibus Ana-} \end{aligned}$$

lystis resoluta feliciter. Synthesis nostra Brounckerianae aemulatrix evidenter satis demonstrat absque Logometriae subsidio primam e geminis illis Seriebus Numerum esse Infinitum, ac signanter Aream Hyperbolae inter Asymptotos una ex parte Infinite productam. Quin etiam post inventum Euleri, de quo *Acta Academiae Petropolitanae* anni MDCCXXXVII. con-

suli debent, Series insuper reciproca Numerorum *Primorum* $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{23} + \&c:$ nobilitari admodum poterit Theoremate Exponentiali, quod ipsius Seriei Summam decernit Infinite magnam; namque illud in Aequalitate consistit

$$\begin{aligned} & \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{23} + \&c: \\ \text{C} & \\ & = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} \end{aligned}$$

$\frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \&c.$, nec operae pretium est in huiusce demonstratione immorari diutius. Moneo solummodo a praeterita derivari Aequatione luculentissima Logarithmum Hyperbolicum $\tau\bar{e}$ Infiniti $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \&c.$ non omnium Infinitorum possibilium Minimum, quemadmodum Infinita veluti Finita versantes nonnulli adfirmarunt Geometrae.

263. Alterius deinde Seriei contemplatio, quae alternis, et conformibus Signis disposita $1 \pm \frac{1}{2^2} + \frac{1}{3^2} \pm \frac{1}{4^2} + \frac{1}{5^2} \pm \frac{1}{6^2} + \frac{1}{7^2} \pm \frac{1}{8^2} + \frac{1}{9^2} \pm \&c.$ Analystis primum innotuit sub Calculi Infinitesimalis

initium Aream Curvae Exponentialis $1 + x = C^{xy}$ dimetiri tentantibus, subsidia exigit difficilioris indaginis. Illius equidem Summa exemplum suppeditat periculosa lubricitatis in vaticiniis Problematum, quae Matheseos dignitatem propius respiciant. Ioannes ipse Bernoullius, qui in Epistola ad Leibnitium scripta anno MDCXCVI. fere omnino de Summatione invenienda desperans supposuit iudicio nimis praecipiti Seriem $1 \pm \frac{1}{4} + \frac{1}{9} \pm \frac{1}{16}$

$+ \frac{1}{25} \pm \frac{1}{36} + \frac{1}{49} \pm \frac{1}{64} + \frac{1}{81} \pm \frac{1}{100} + \frac{1}{121} \pm \&c.$ ad aliam

expressionem, praeter Logarithmicam, referri non posse, ostendit postmodum eam Summam non Logarithmos, sive Hyperbolae Quadraturam, sed Circuli ex adverso involventem, ut fidem faciunt *Anecdota* in eius Operum Volumine quarto. Inutile nunc foret Bernoullianam methodum commentari ab Aequatione Infinitinomia Recti-Sinus per Arcum subtiliter deductam posteaquam Euleri *Introductio* eximium hoc argumentum amplificavit, novisque accessionibus auxit, prouti Auctor solet ob mentis aciem incomparabilem nemini secundus Mathematicarum rerum interpret. Verum repertae Summationis analytica veritas sustinetur fundamento non satis probati roboris, nullas, nimirum, in Aequatione Infinitinomiali, quam nos etiam evolvimus num^{is}. 195., ac 200., exhibente Sectores Sinubus-rectis evanescentibus

Circuli, vel Hyperbolae respondentes, $x \mp \frac{x^3}{6} + \frac{x^5}{120} \mp \frac{x^7}{5040} +$

S s s

 $\frac{x^9}{362880}$

$$\frac{x^9}{362880} \mp \frac{x^{11}}{39916800} + \frac{x^{13}}{6227020800} \mp \&c: = 0, \text{ peregrinas Radices admi-}$$

xtas haberi, quae, si revera laterent, tota rueret elaboratissima Theore-
matis demonstratio. Genuinas, et ad rem nostram facientes universas eas
esse Radices probabilius, quam hactenus fuerat, existimo, quia subtilior
Sinuum-verforum Analysis a num^o. 224. ad 228. instituta eandem Seriei

$$\text{Summam } 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2}$$

+ &c: mirum in modum confirmat. Evolutionis enim completae Factorum
Binomialium, illic Euleri, vel Bernoullii methodo renovata, fructus iam

$$\text{fuit } 2 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \&c: \right)$$

$$= \frac{\Pi^2}{12} = \frac{P^2}{3}; \text{ ideoque } 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}$$

$$+ \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \&c: = \frac{P^2}{6}, \text{ qui valor penitus congruit cum}$$

Summatione praecognita. Verum, si aliqua, utut levissima, oscitantis
animi proclivitas, quod facillimum fuerat, dimidiatum Factorum nume-
rum vice omnium ibidem sanxisset, nonne contradictionis vitium obortum,

$$\text{quo nunc Sinus-rectis in computum actis } 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$$

$$+ \frac{1}{36} + \frac{1}{49} + \&c: = \frac{P^2}{6}, \text{ nunc Versis-Sinus consideratis eadem}$$

$$\text{Quadratorum reciprocorum Summa} = \frac{P^2}{3} \text{ nobis obtingeret, suspicionem}$$

fallacis adhibitae methodi quodam iure excitasset? Ad hoc auferendum
periculum, et ne probabilitas in Mathematicum doctrina evidentiae sedem
partiatur, Summationem Seriei, de qua mentio occurrit, Aequationibus In-
finitinomiis sublatis dum tyro adhuc eram, ac hisce summopere delectabar,
inveniendam duxi, atque tum suscepti moliminis fructum, quantuluscun-
que ille fuerit, nunc paucis aperiam.

$$264. \text{ Neminem fugit } 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} +$$

$$\frac{1}{15^2}$$

$\frac{1}{15^2} + \&c: = \frac{3}{4} \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} \right.$
 $\left. + \frac{1}{10^2} - \frac{1}{11^2} + \frac{1}{12^2} - \frac{1}{13^2} + \frac{1}{14^2} - \frac{1}{15^2} + \&c: \right),$ quam Aequa-
 litatem, etsi Iacobus Bernoullius longo ratiocinationis circuitu demonstraverit,
 summam obtinebit quicumque viderit esse $\frac{1}{4} \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} \right.$
 $\left. + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{8^2} - \frac{1}{9^2} + \frac{1}{10^2} - \frac{1}{11^2} + \&c: \right) = \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} -$
 $\frac{1}{8^2} + \frac{1}{10^2} - \frac{1}{12^2} + \frac{1}{14^2} + \&c:$ in Infinitum. Constat praeterea de alterius
 veritate Aequationis $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2}$
 $- \frac{1}{10^2} + \frac{1}{11^2} - \&c: = \frac{1}{2} \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} \right.$
 $\left. + \frac{1}{8^2} - \frac{1}{9^2} + \frac{1}{10^2} - \frac{1}{11^2} + \&c: \right),$ eoquod sit vi praecedentis Aequali-
 tatis $\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \frac{1}{13^2} - \&c: \right) - \left(\frac{1}{2^2} - \frac{1}{4^2} \right.$
 $\left. + \frac{1}{6^2} - \frac{1}{8^2} + \frac{1}{10^2} - \frac{1}{12^2} + \frac{1}{14^2} - \&c: \right) = \left(\frac{3}{4} - \frac{1}{4} \right) \left(1 - \frac{1}{2^2} \right.$
 $\left. + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} - \frac{1}{10^2} + \&c: \right);$
 adeo, ut Summatio omnium memoratarum Serierum ab unica pendeat Sum-
 ma Seriei reciprocae Quadratorum imparium.

265. Quae cum ita sint, et praemissa nos doceant $1 - \frac{1}{3} + \frac{1}{5}$
 $- \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \&c:$ peraequare $\frac{P}{4}$, fiant Magni-
 tudinum earum Quadrata, ac statim Aequationem canonicam nanciscemur

$$\frac{p^2}{16} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{15^2} + \&c:$$

$$- \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \frac{2}{9} - \frac{2}{11} + \frac{2}{13} - \frac{2}{15} + \&c:$$

$$- \frac{2}{3 \cdot 5} + \frac{2}{3 \cdot 7} - \frac{2}{3 \cdot 9} + \frac{2}{3 \cdot 11} - \frac{2}{3 \cdot 13} + \frac{2}{3 \cdot 15} - \&c:$$

continuatis sine Limite
disparibus Numeris

$$- \frac{2}{5 \cdot 7} + \frac{2}{5 \cdot 9} - \frac{2}{5 \cdot 11} + \frac{2}{5 \cdot 13} - \frac{2}{5 \cdot 15} + \&c:$$

$$- \frac{2}{7 \cdot 9} + \frac{2}{7 \cdot 11} - \frac{2}{7 \cdot 13} + \frac{2}{7 \cdot 15} - \&c:$$

$$- \frac{2}{9 \cdot 11} + \frac{2}{9 \cdot 13} - \frac{2}{9 \cdot 15} + \&c:$$

$$- \frac{2}{11 \cdot 13} + \frac{2}{11 \cdot 15} - \&c:$$

$$- \frac{2}{13 \cdot 15} + \&c:$$

Secundum huiusce Aequationis Membrum rite, recteque dispositum exhibet
Aequationem

$$\frac{p^2}{16} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{15^2} + \&c:$$

$$+ \left(\frac{\frac{p}{2} - \frac{2}{1}}{\frac{2}{1} - \frac{2}{3}} \right) + \left(\frac{\frac{2}{1} - \frac{2}{3} - \frac{p}{2}}{\frac{2}{3} - \frac{2}{5}} \right) + \left(\frac{\frac{p}{2} - \frac{2}{1} + \frac{2}{3} - \frac{2}{5}}{\frac{2}{5} - \frac{2}{7}} \right)$$

$$+ \left(\frac{\frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} - \frac{p}{2}}{\frac{2}{7} - \frac{2}{9}} \right) +$$

$$\left(\frac{\frac{p}{2} - \frac{2}{1} + \frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9}}{\frac{2}{9} - \frac{2}{11}} \right) +$$

$$\left(\frac{\frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \frac{2}{9} - \frac{2}{11} - \frac{p}{2}}{\frac{2}{11} - \frac{2}{13}} \right) +$$

$$\left(\frac{p}{2} \right)$$

$$\left(\frac{P}{2} - \frac{2}{1} + \frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \frac{2}{13} \right) + \&c., \text{ scilicet}$$

$$\frac{P^2}{16} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{15^2} + \&c. + \frac{P}{2} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \&c. \right) -$$

$$2 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \&c. \right) + \frac{2}{3} \left(-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \&c. \right) -$$

$$\frac{2}{5} \left(\frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \&c. \right) + \frac{2}{7} \left(-\frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \&c. \right) - \frac{2}{9} \left(\frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \&c. \right) + \frac{2}{11} \left(-\frac{1}{11} + \frac{1}{13} - \&c. \right) - \frac{2}{13} \left(\frac{1}{13} - \&c. \right)$$

+ &c: absque ullo Limite. Haec autem Aequatio simpliciores obtinet for-

$$\text{mam } \frac{P^2}{16} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{15^2} +$$

$$\&c: + \frac{P^2}{8} - \left(2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \frac{2}{9} - \frac{2}{11} + \frac{2}{13} - \&c. \right) \frac{P}{4}$$

$$+ \left(\frac{P}{2} - \frac{2}{1} \right) + \left(\frac{2}{1} - \frac{2}{3} - \frac{P}{2} \right) + \left(\frac{P}{2} - \frac{2}{1} + \frac{2}{3} - \frac{2}{5} \right)$$

$$+ \left(\frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} - \frac{P}{2} \right) +$$

$$\left(\frac{P}{2} - \frac{2}{1} + \frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} \right) +$$

$$\left(\frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \frac{2}{9} - \frac{2}{11} - \frac{P}{2} \right) +$$

$$\left(\frac{P}{2} - \frac{2}{1} + \frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \frac{2}{13} \right) + \&c: =$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \&c: + \frac{P^2}{8} - \frac{P^2}{8} \\ + \frac{P^2}{8} - \frac{P^2}{8} + \frac{P^2}{8} - \frac{P^2}{8} + \frac{P^2}{8} - \frac{P^2}{8} + \frac{P^2}{8} - \frac{P^2}{8} \&c:, \text{quum}$$

eadem semper recurrat in secundo Membro Aequationis terminorum periodus.

$$\text{Est vero Parallela Series } + \frac{P^2}{8} - \frac{P^2}{8} + \frac{P^2}{8} - \frac{P^2}{8} + \frac{P^2}{8} - \frac{P^2}{8} \\ + \frac{P^2}{8} - \frac{P^2}{8} \&c: = \frac{\pm 1}{\frac{8}{P^2} + \frac{8}{P^2}} = \pm \frac{P^2}{16}, \text{ quae Signi congeminationo}$$

Summatoriam Fractionem necessario debet afficere, quia Series Parallelae sunt *convertibiles*; et ideo, si sumatur initium in primo termino $+ \frac{P^2}{8}$ habetur

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \\ \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{12^2} + \frac{1}{14^2} + \frac{1}{16^2} + \\ 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} -$$

adeo, ut in Arithmetica sint Proportione quatuor Series subnotatae, ac numero terminorum Infinitae, $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} \&c:$,

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \&c:, 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} -$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} \&c.: 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} -$$

Exinde consequuntur non spernendae in hac Serierum doctrina Aequalitates

habetur $\frac{1}{\frac{8}{p^2} + \frac{8}{p^2}}$, fin e-contra, quod fieri aequè potest, firmo manente Seriei

ordine, atque valore, in ∞ ^{nesimo} $\frac{p^2}{8}$, emergit $\frac{1}{\frac{8}{p^2} + \frac{8}{p^2}}$. Igi-

tur $\frac{p^2}{16} \pm \frac{p^2}{16} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{15^2} +$

&c: $= \frac{p^2}{8}$, exclusa nempe pseudo-solutione Problematis, seu Nihili abfurditate.

266. Istis antea positis sponte veluti sua a num°. 264. dimanant sequentes Aequalitates

1 +

$$\frac{1}{10^2} + \frac{1}{11^2} + \frac{1}{12^2} + \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + \&c: = \frac{4}{3} \left(\frac{p^2}{8} \right) = \frac{p^2}{6}$$

$$\frac{1}{18^2} + \frac{1}{20^2} + \frac{1}{22^2} + \frac{1}{24^2} + \frac{1}{26^2} + \&c: = \frac{1}{4} \left(\frac{p^2}{6} \right) = \frac{p^2}{24}$$

$$\frac{1}{10^2} + \frac{1}{11^2} - \frac{1}{12^2} + \frac{1}{13^2} - \frac{1}{14^2} + \frac{1}{15^2} - \&c: = \frac{1}{2} \left(\frac{p^2}{6} \right) = \frac{p^2}{12}$$

adeo,

$$- \frac{1}{6^2} \&c:, \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{12^2} \&c:, \text{ et in Proportione}$$

Geometrica ternae Series pariter Infinitae

1 +

$$- \frac{1}{6^2} + \frac{1}{7^2} \&c.: \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{12^2} \&c: \div \dots$$

Exinde

(1 +

$$\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \&c.\right) \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{12^2} + \frac{1}{14^2} + \frac{1}{16^2} + \frac{1}{18^2} + \&c.\right) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{19^2} + \&c.$$

Ab hisce novae profluunt speculationes brevi inedito Opusculo alibi recensitae *De usu Serierum, ut vocant, Parallelarum in nonnullis quaestionibus enodandis*, nimirum, de usu Serierum, quae ob terminorum omnium perfectam aequalitatem nec verum initium, nec medium habent, nec finem, et propter istam singularem affectionem ab universis discrepant Seriebus convergentibus, vel divergentibus, ascendentibus, aut descendenti-
bus. Interim adiungere lubet veritati satis propinquum illarum omnium Serierum valorem ad hoc, ut adlata Theoremata confirmentur.

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{12^2} + \frac{1}{14^2} + \frac{1}{16^2} + \frac{1}{18^2} + \&c. = 0,4112335167120566091181 \&c.,$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} - \frac{1}{10^2} + \&c. = 0,8224670334241132182362 \&c.,$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{19^2} + \&c. = 1,2337005501361698273543 \&c.,$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \&c. = 1,6449340668482264364724 \&c..$$

267. Nec parum afferet voluptatis mutua Serierum comparatio $\tau \tilde{x}$ 1 —

$$\frac{1}{2^2} - \frac{1}{3^3} + \frac{1}{4^4} - \frac{1}{5^5} + \frac{1}{6^6} - \frac{1}{7^7} + \frac{1}{8^8} - \&c. \text{ a Curvae Exponentialis}$$

$$x^x = y \text{ Quadratura, et } \tau \tilde{x} 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} +$$

$\&c.$ a Circuli Tetragonismo enascentis, quippe quae, nihil obstante perinfini earum compositionis differentia, tam modice in Summa differunt, ut prioris in Infinitum continuatae valor a Fractione 0,783430497589 $\&c.$ paulisper discedat. Quod autem mirabilius videbitur in perfecta consistit Aequalitate

$$-\frac{1}{10^2} + \frac{1}{12^2} \&c:) = \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} \&c: \right)^2,$$

$$\frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} \&c: + 3 \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \&c: \right).$$

Ab hisce

qualitate Seriem inter $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{15^2}$

$+ \&c:$, atque alteram ab Ioanne Bernoullio in *Posthumis* sine demon-
stratione conscriptam

$$\frac{1}{1.2} + \frac{2}{1.3.4} + \frac{2.4}{1.3.5.6} + \frac{2.4.6}{1.3.5.7.8} + \frac{2.4.6.8}{1.3.5.7.9.10} +$$

$$\frac{2.4.6.8.10}{1.3.5.7.9.11.12} + \&c:, \text{ ac tam eleganti parium, et imparium Numerorum}$$

dispositione, mutuoque nexu excellentem. Ne amplius Theorematis demon-
stratio in posterum desideretur observandum est a Formulis traditis in nu-
mero 161. deduci

$$\frac{P}{2} = 1 + \frac{1}{6} + \frac{3}{40} + \frac{5}{112} + \frac{35}{1152} + \&c:, \text{ ideoque}$$

$$\left(1 + \frac{1}{6} + \frac{3}{40} + \frac{5}{112} + \frac{35}{1152} + \&c: \right)^2 = 1 + \frac{1}{3} + \frac{8}{45} + \frac{4}{35}$$

$$+ \frac{128}{1575} + \&c:, \text{ prouti constat ex Paradigmathe subnexi Trigoni}$$

$$1 + \frac{1}{6} + \frac{3}{40} + \frac{5}{112} + \frac{35}{1152} + \&c:$$

$$+ \frac{1}{6} + \frac{1}{36} + \frac{1}{80} + \frac{5}{672} + \&c:$$

$$+ \frac{3}{40} + \frac{1}{80} + \frac{9}{1600} + \&c:$$

$$+ \frac{5}{112} + \frac{5}{672} + \&c:$$

$$+ \frac{35}{1152} + \&c:$$

V v v

=

$= \frac{P^2}{4}$; scilicet erit $\frac{1}{2} + \frac{1}{6} + \frac{4}{45} + \frac{2}{35} + \frac{64}{1575} + \&c.$, vel potius,

in meliorem ordinem terminorum Lege disposita ad morem Bernoullianum,

$$\frac{1}{1.2} + \frac{2}{1.3.4} + \frac{2.4}{1.3.5.6} + \frac{2.4.6}{1.3.5.7.8} + \frac{2.4.6.8}{1.3.5.7.9.10} + \&c. = \frac{P^2}{8} = 1$$

$$+ \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \&c. \text{ iuxta duos numeros praecedentes.}$$

Quin etiam, si contemplari diutius Series istas vacaret, non modo se

$$\text{offeret praeter spem Aequalitas } 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13}$$

$$- \frac{1}{15} + \&c. = \frac{1}{1.2^1} + \frac{2}{1.3.2^2} + \frac{2.4}{1.3.5.2^3} + \frac{2.4.6}{1.3.5.7.2^4} +$$

$$\frac{2.4.6.8}{1.3.5.7.9.2^5} + \frac{2.4.6.8.10}{1.3.5.7.9.11.2^6} + \&c., \text{ verum adhaec } \left(\frac{1}{1.2^1} + \frac{2}{1.3.2^2} \right.$$

$$+ \frac{2.4}{1.3.5.2^3} + \frac{2.4.6}{1.3.5.7.2^4} + \frac{2.4.6.8}{1.3.5.7.9.2^5} + \&c. \left. \right) \left(1 + \frac{1}{2.3} + \frac{1.3}{2.4.5} + \right.$$

$$\frac{1.3.5}{2.4.6.7} + \frac{1.3.5.7}{2.4.6.8.9} + \&c. \left. \right) = \frac{P^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} +$$

$$\frac{1}{11^2} + \&c., \text{ quae omnia ab expressionibus datis in praecitato num^o. 161.}$$

ultro colliguntur. Sed nullus finis adesset dum universa singillatim persequi foret in votis.

268. Meditanti mihi multos abhinc annos de Circuli dimensione occurrit Epistola Leibnitii ad Bernoullium, ac signanter XL.^{ma} inter eas, quae *Commercii Philosophici, et Mathematici* comprehenduntur Volumine primo, et valde obstupui, quum ibi perlegerim Seriem, ad instar Continuae Fractionis a Brounckero repertam ante annum MDCLV., videlicet ante epocham editionis *Arithmeticae Infinitorum* Vallisii, ipsi Leibnitio videri minus rei facientem ob lentam terminorum convergentiam, seu quia optabilis maior approximatio in illa deficeret. Et re quidem vera hoc Leibnitii effatum conciliare cum dignitate nesciebam, eoquod Vice-Comitis Angli Series ipsa met sit, sed vario exposita modo, cum notissima Germanici Auctoris $1 -$

$\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \&c.$, cuius inventio tanti a Leibnitio tunc pri-

mam in Mathesi sedem tenente existimata, ut et Ozanamum anno MDCXCVI. stric-
tius, quam par fuisset, ob sui nominis praeteritionem plagii inculaverit, et non
sine mentis aegritudine audierit Britannos Geometras, amicumque Bernoullium
dilectam Seriem illam Infinitam usque ab anno MDCLXX. Iacobo Gregorio prae-
cognitam affirmantes. Vix aequipollentia Serierum, de quibus fit mentio, ul-

terioris eget illustrationis, quum sensim evolvendo Fractionem $\frac{1}{1+i^2}$

evadat

1	$\frac{2}{3}$	$\frac{13}{15}$	$\frac{76}{105}$	$\frac{263}{315}$	$\frac{1}{2+3^2}$	$\frac{1}{2+5^2}$	$\frac{1}{2+7^2}$	$\frac{1}{2+9^2}$	$\frac{1}{2+11^2}$	$\frac{1}{2+\&c.}$
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respondens

1	$\frac{1}{1+i}$	$\frac{1}{1+i^2}$	$\frac{1}{1+i^3}$	$\frac{1}{1+i^4}$	$\frac{1}{1+i^5}$	$\frac{1}{1+i^6}$	$\frac{1}{1+i^7}$	$\frac{1}{1+i^8}$	$\frac{1}{1+i^9}$	$\frac{1}{1+i^{10}}$
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, vel

1	$1 - \frac{1}{3}$	$1 - \frac{1}{3} + \frac{1}{5}$	$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$	$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}$	$\&c.$
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in Infinitum. Haec

pariter derivatur identitas ab universaliore Fractionum Continuarum Theori-
ce, quam in *Petropolitanae Academiae Veteribus Actis*, et postmodum in
Introductione adnotavit Eulerus; sed Circuli Quadraturae doctrinam tanta co-
mitatur facilitas, ut sublimi universalitate neglecta ipsi illustrandae sufficiat
singularis, ac vere nativa Fractionis Continuae in partes integrantes se-
paratio.

Qua facta obtinetur

$$-\frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \&c. = \frac{P}{4}, \text{ praetereaque } \frac{1}{1+i^2} = 1 - \frac{1}{3} + \frac{1}{5}$$

$= \frac{1}{1}$	$\frac{1}{2}$	$\frac{13}{15}$	$\frac{76}{105}$	$\&c.$	$\frac{1}{2+3^2}$	$\frac{1}{2+5^2}$	$\frac{1}{2+7^2}$	$\frac{1}{2+9^2}$	$\frac{1}{2+11^2}$	$\frac{1}{2+\&c.}$
-----------------	---------------	-----------------	------------------	--------	-------------------	-------------------	-------------------	-------------------	--------------------	--------------------

vel

1	$\frac{3}{2}$	$\frac{15}{13}$	$\frac{105}{76}$	$\&c.$	$\frac{1}{2+3^2}$	$\frac{1}{2+5^2}$	$\frac{1}{2+7^2}$	$\frac{1}{2+9^2}$	$\frac{1}{2+11^2}$	$\frac{1}{2+\&c.}$
-----	---------------	-----------------	------------------	--------	-------------------	-------------------	-------------------	-------------------	--------------------	--------------------

five

1	$1 - \frac{1}{3}$	$1 - \frac{1}{3} + \frac{1}{5}$	$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$	$\&c.$	$\frac{1}{2+3^2}$	$\frac{1}{2+5^2}$	$\frac{1}{2+7^2}$	$\frac{1}{2+9^2}$	$\frac{1}{2+11^2}$	$\frac{1}{2+\&c.}$
-----	-------------------	---------------------------------	---	--------	-------------------	-------------------	-------------------	-------------------	--------------------	--------------------

respondens

1	$1 + \frac{1^2}{2}$	$1 + \frac{1^2}{2+3^2}$	$1 + \frac{1^2}{2+3^2+5^2}$	$\&c.$	$\frac{1}{2+3^2}$	$\frac{1}{2+5^2}$	$\frac{1}{2+7^2}$	$\frac{1}{2+9^2}$	$\frac{1}{2+11^2}$	$\frac{1}{2+\&c.}$
-----	---------------------	-------------------------	-----------------------------	--------	-------------------	-------------------	-------------------	-------------------	--------------------	--------------------

, variis partibus ordine computatis.

Ideo

Ideo tota Series a Brounckero data ==

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \&c: = \frac{4}{P}, \text{ aut } =$$

Rationi inter Quadratum Circulo circumscriptum, atque Aream Circuli inscripti iuxta morem Auctoris, scilicet, dum libeat, $= 1 + \frac{1}{3} -$

$$\frac{4}{45} + \frac{44}{945} - \frac{428}{14175} + \&c: =$$

$$1 + \frac{1}{3 + \frac{1}{12}} = \frac{4}{P}, \text{ quemadmodum fas est experiri. Postre-}$$

$$\frac{11 + \frac{1}{132}}{8 + \frac{4494}{145 + \&c:}}$$

ma Fractio Continua, quae terminorum Lege destituitur, oriturque a Serie

$$\text{Infinita } 1 + \frac{1}{3^1} - \frac{4}{3^2 \cdot 5} + \frac{44}{3^3 \cdot 5 \cdot 7} - \frac{428}{3^4 \cdot 5^2 \cdot 7} + \&c:, \text{ praebet, eam inverten-}$$

$$\text{do, } \frac{1}{1 + \frac{1}{3 + \frac{1}{12}}} = \frac{1}{1 + \frac{1}{2 + \frac{9}{2 + \frac{25}{2 + \frac{49}{2 + \&c:}}}}} = \frac{P}{4}; \text{ quod, nisi Al-}$$

gebra evinceret, vix crederent Analystae.

269. Immane itaque discrimen in Fractionum Continuarum forma, quae idoneae sint eidem Circuli Quadraturae repraesentandae, et quasi Infinites variari possent tot ope Serierum Trigonometricarum Capitis V. alternis terminorum Signis gaudentium, meo iudicio decernit unicam illam elegantia praestantior, suique Lege ad continuationem indefinitam facta commendabilem, a Brounckero fuisse repertam eoquod in Circulo viderit aliquid non abfimile Seriei Leibnitianae, a qua procul dubio nativam ducit originem. Auctoris etenim mens erat non Circuli Tetragonismo obviam eundi adiumento divisionis continuae, quaecumque illa fuisset, etiam

exlex,

$$\begin{array}{c}
 \frac{1}{1+1} \\
 \frac{1}{3+1} \\
 \frac{1}{1+1} \\
 \frac{1}{1+1} \\
 \frac{1}{1+1} \\
 \frac{1}{15+1} \\
 \frac{1}{2+1} \\
 \frac{1}{72+1} \\
 \frac{1}{1+1} \\
 \frac{1}{7+1} \&c.
 \end{array}$$

Fractionem aequae Continuae Quadrantale Spatium determinantem, aut Arcum Octantis in Circulo, dummodo Radius sit Unitas. Mihi itaque luculentissimum videtur Seriem reciprocam Numerorum imparium $1 - \frac{1}{3} +$

$$\frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} \&c. \text{ saltem ante annum MDCLV. innotuisse}$$

se Brounckero, ideoque esse aliquantisper retrahendam illius inventionis epocham plusquam ferat non recantanda diutius Historia Mathematicum, quum, permutata tantum forma, sive potius terminorum dispositione, duae Series Leibnitiana, et Brounckeriana sint una, eademque numerorum Progressio. Abscondita ergo Brounckeri methodus, cuius iure merito deperditionem ludent Eulerus, atque Lagrangius (*), fortassis in eo sita est, ut eximius ille Geometra, qui simul cum Vrenno, Nelio, Vallisio in Anglia praeivit inaudita promens de Curvilinearum Analyfi, de ipsorum cum Rectilineis Figuris comparatione, mutuaque Curvarum ex una in aliam metamorphosi, Circulum contemplatus in eandem inciderit Quadraturam Arithmetica, quae postea obviam venit Neutono, Iacobo Gregorio, Leibniz-

tio,

(*) In *Commentariorum Academiae Petropolitanae Novorum* Volumine IX^o., et in Additamentis ad Versionem Gallicam *Institutionum Algebrae* Cl. Euleri.

tio, quin etiam serius et Lagnyo non inter primi ordinis summae Mathe-
seos cultores (*).

270. Nova etenim felici illa aetate florebat Curvarum Geometria, no-
vus rerum Mathematicarum tunc instituebatur ordo ab Italis certatim, Gal-
lis, Anglisque Geometris excitatus; iamque Arithmetica Hyperbolae Qua-
draturam Brounckerus ipse adsecutus multos ante annos, quam in vulgus
ederentur *Philosophicae Transactiones*, quod contigit labente MDCLXV.,
nil mirum, si quoque de Circulo fuerit Theoremata prope synchrona medi-
tatus. Facillimum profecto erat viro ad Mathesin promovendam nato Broun-
ckero Fractionum Continuarum naturam introspicere, et inprimis periodica-
rum, quales suppeditat omnium simplicissimas Magnitudinis etiam cuiuslibet
in extrema ac media Ratione divisio, prouti Leibnitius Bernoullio scribens

specimen praebuit celata demonstratione in Serie $\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}}}$. Ad

$$\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}}}$$

eam demonstrandam monuisse sufficiat evolutionem, seu continuationem Fra-

ctionis $\frac{1}{1+x}$ parere Fractionem illam Continuum, scilicet eam suboriri ab

Aequationis resolutione $\frac{1}{1+x} = x$, aut $x^2 + x - 1 = 0$, Proportio-

nem *divinam* determinantis. Nec in huiusmodi speculationibus Brouncke-
rum fugere poterat Fractionis Continuae in usitatiores Series conversio, ac
vicissim, ceu in superius posito exemplo facilitatis plenissimo Series a Leibnitio

praetermissa $1 - \frac{1}{1.2} + \frac{1}{2.3} - \frac{1}{3.4} + \dots$ revera obtinetur. Fervente

X x x 2

autem

(*) Leibnitianam Seriem maximis extulit laudibus etiam Vivianius noster in
Opusculi Isagoge *de Fornicum constructione ac mensura &c.* Florentiae editi anno
MDCXCII.

autem ingenii vi, medioque quasi suae aetatis circulo emenso (*) Vice-Comes ille, qui summa Artis secreta de Curvarum transformationibus possidebat, nullus dubito, quin Circulum versans huic ratiocinationi simillimum arguendi modum celeriter arripuerit. Ductis a Centro A Quadrantis Circularis ABCD in Schemate XXV°. Radiis ACE, AIF &c: usque ad concursum cum Tangente BG in sublimiore loco posita, nec non EH, FK aequidistantibus Radio BA, constat Aream Rectanguli Parallelogrammi EFKH duplam semper esse Spatii Triangularis EAF, vel, elementa persequendo, duplam Sectoris EAL. Hic Sector, quum ad Sectorem CAI rationem habeat $\tau EA^2 : CA^2$, vel $EA^2 : BA^2$, aut, facta BM ad Secantem normali, et MN Radio AD parallela, $\tau EA : MA$, sive $EH : HN = EFKH : NPKH$, efficiet, ut praecedente constructione usurpata proveniat Quadrilineum $NPKH = \frac{1}{2} CAI$, et idcirco Area $BRNHA = \frac{1}{2} BTCA$, totumque Curvae Asymptoticae Spatium Infinites versus Z protensum in dupla Proportionione sit ad Aream Quadrantis, videlicet Arcum Sectoris Circuli, dum Radius $= 1$, numerice repraesentet. In isto admodum facili mentis humanae conceptu felix Leibnitii inventio, tametsi perplexius ab eo digesta, consistit, quae postea temporis fortuna favente sub initium vertentis saeculi nonnullis Italis Geometris demonstrationem longiore apparatu molitis famam, decusque maximum conciliavit. Idipsum aequae in promptu erat ad Aequilateram Apollonii Hyperbolen non diversa ratione tractandam, illamque stabiliendam Harmoniam, quae Curvas, tanto intervallo quoad perimetri ductum distitas, amice tamen coniungit. Area namque Hyperbolici Sectoris ABTC peraequat dimidium Quadrilinei BRNHA in Curva tribus Asymptotis, ac Diametro praedita, et Speciei LX.^{mae} a Neutono inter Lineas *tertii ordinis* adnumerata sub nomine quarti Hyperbolae Hyperbolismi, atque Aequatione $yx^2 = a^3 + a^2y$ insignita. Punctum N cuiusvis extremum Ordinatum - applicatae invenietur facta AM tertia continua Geometrica proportionali post Semilatus Transversum, seu Radium AB, et Abscissam Centralem AV, ductaque deinde MN Tangenti BG parallela, quemadmodum consulto Schemate XXVI°. ac iisdem Litteris ducibus fusius patet; simulque consequitur Infinitae Magnitudinis Sectorum

(*) Annus nativitatis Brounckeri parum distat a MDCXX., et vita functus est anno MDCLXXXIV. Vir praestantissimus.

Aream ABTDSA parem esse dimidio Asymptotico Spatio praefati Hyperbolismi ex parte Z Infinite producto.

271. Nec a suspicata Brounckerianae methodi coniectura me retrahit argumentum ex eo depromptum, quod Auctor eximius non Seriem, quae

magis nativa, ac facilis videbatur, $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \&c.$

communicaverit, sed potius Continuum Fractionem originariae derivationis caractere emicantem. Inprimis enim Historia nos docet Vallisium Amico suo Brounckero commisisse Series illas Factoribus conflatas, a quorum Interpolatione Quadratura Circuli obtinebatur; adeo, ut, dum in terminis intercalandis adhuc haerebat Vallisius, Amicique consilium, et opem sollicitabat, nihil fide dignius existimandum, quam elegantis antithesis causa rescripsisse Brounckerum a se iam confectum per divisionem continuum, quod ille per continuum multiplicationem moliebatur. Accedit Proportio Quadrati circumscripti ad Circuli Aream 1 a Vallisio signanter petita, quippe

quae non sinebat Seriem rescribi $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \&c.$,

verum eius inversam $\frac{1}{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} \&c.}$ in a-

liam Seriem evolvendam, et omni elegantia, periodoque carentem, prouti num^{us}.

268. abunde satis ostendit, quum e-contra vel directa, vel conversa Fractionis

Continuae aequipollentis forma Legem eandem tueatur iucunditatis, et

analytici eloquii plenissimam. Qui concessa praenotione illius Continuae

Fractionis 1 comitem aequae non adfirmaret scientiam Seriei

$$\frac{1}{1+1}$$

$$\frac{2+9}{2}$$

$$\frac{2+25}{2}$$

$$\&c.$$

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \&c.$; eodem pacto adserere posset Brounckerum Qua-

draturam Hyperbolae hoc peculiari Seriei modo $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8}$

$$+ \frac{1}{9.10}$$

ptor se gessit. Denuo autem errat Mathematicum Historiographus affirmans Eulerum in *Commentariis Petropolitanae Academiae* relatis ad annum

MDCCXXXVII. mutuam aequalitatem Serierum Leibnitii $1 - \frac{1}{3} + \frac{1}{5} -$

$\frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} \&c:$, ac Vallisii

$\frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot 12 \&c:}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot 13 \&c:}$ demonstrasse, vel potius, iuxta ve-

ram Vallisii expressionem, congruentes ostendisse Series utrasque

$\frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot 13 \&c:}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot 12 \&c:}$, aut

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} \&c:$, scilicet $1 - \frac{2}{3} + \frac{2}{5}$

$- \frac{2}{7} + \frac{2}{9} - \frac{2}{11} + \frac{2}{13} \&c:$, atque $\frac{2}{3} + \frac{1}{15} A + \frac{1}{35} B + \frac{1}{63} C$

$+ \frac{1}{99} D + \frac{1}{143} E + \&c:$, nec non $(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}$

$- \frac{1}{11} + \frac{1}{13} \&c:)^{-1}$, ac $1 + \frac{1}{8} A + \frac{1}{24} B + \frac{1}{48} C + \frac{1}{80} D +$

$\frac{1}{120} E + \&c:$, quum Eulerus ipse Seriem praefatam $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$

$+ \frac{1}{9} - \frac{1}{11} + \frac{1}{13} \&c:$ non Vallisiana, sed maximopere diversae Seriei

$\frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \&c:}{4 \cdot 4 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdot 20 \cdot 24 \&c:}$ aequalem, aut, praesupposita veritate

Seriei a Vallisio traditae, parem alteri $\frac{4 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16 \cdot 18 \cdot 20 \&c:}{5 \cdot 7 \cdot 11 \cdot 13 \cdot 13 \cdot 17 \cdot 17 \cdot 19 \&c:}$ concluderit.

273. Nulla potest Analytica methodus coniungere Series a Vallisio, et Leibnitio evulgatas, unamque ex alia deducere praeterquam si directa, atque inversa tractetur Formula Tangentium Circularium, de quibus prae-

sertim

sertim in numeris 208. 209. 222. mentio debita occurrit. Hoc qui fecerit non modo Vallisii, ac Leibnitii, verum etiam Brounckeri, vi praeostensorum, de Circuli Tetragonismo rationalibus Numeris cohibendo reperta ab unico Analyseos Trigonometricae principio obtinebit: nam in unum veluti centrum simul coibunt Series per Infinitos numero Factores, per Divisionem Continuum, atque reciproca Numerorum imparium, quae vix prima fronte aliquam praeferrebant affinitatem. Sed, antequam id aggrediar, observandum puto nihil esse facilius, quam terminos Seriei Leibnitianae tali ordine disponere, ut cum Serie Vallisiana communem Denominatorem recipiant.

$$\text{Et re quidem vera } \frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \frac{2}{13.15} + \frac{2}{17.19} + \frac{2}{21.23}$$

$$+ \&c.; \text{ quo Leibnitii Series reducitur, hanc quoque formam acquirat } \frac{P}{4} =$$

$$\frac{2(1.3.5^2.7^2.9^2.11^2.13^2.15^2.\&c.+1^2.3^2.5.7.9^2.11^2.13^2.15^2.\&c.+1^2.3^2.5^2.7^2.9.11.13^2.15^2.\&c.+1^2.3^2.5^2.7^2.9^2.11^2.13.15.\&c.+ \&c.)}{1^2.3^2.5^2.7^2.9^2.11^2.13^2.15^2.\&c.},$$

progressu in Infinitum continuato, et facta testa Numeratorum sine numero iucunda satis, tametsi ad rem nostram inutili, Lege evidentissime nitescente.

274. Esto T Tangens Arcus Circularis x; habebiturque T =

$$x \left(\frac{(1 - \frac{x}{P})(1 + \frac{x}{P})(1 - \frac{x}{2P})(1 + \frac{x}{2P})(1 - \frac{x}{3P})(1 + \frac{x}{3P})(1 - \frac{x}{4P})(1 + \frac{x}{4P})(1 - \frac{x}{5P})(1 + \frac{x}{5P}) \&c.}{(1 - \frac{2x}{P})(1 + \frac{2x}{P})(1 - \frac{2x}{3P})(1 + \frac{2x}{3P})(1 - \frac{2x}{5P})(1 + \frac{2x}{5P})(1 - \frac{2x}{7P})(1 + \frac{2x}{7P})(1 - \frac{2x}{9P})(1 + \frac{2x}{9P}) \&c.} \right),$$

quos in Factores resolvitur, uti vidimus, Infinitinomialium $x + \frac{x^3}{3} + \frac{2x^5}{3.5}$

$$+ \frac{17x^7}{3^2.5.7} + \frac{62x^9}{3^2.5.7.9} + \frac{1382x^{11}}{3^2.5^2.7.9.11} + \&c.; \text{ et vicissim emergit } x = T -$$

$$\frac{T^3}{3} + \frac{T^5}{5} - \frac{T^7}{7} + \frac{T^9}{9} - \frac{T^{11}}{11} + \frac{T^{13}}{13} - \&c.; \text{ ubi advertendum a}$$

postrema, permutato etiam ordine, viceversa priorem Seriem enasci. Fiat

$$\text{itaque } x = \frac{P}{4}, \text{ et idcirco } T = 1; \text{ eo in casu proveniet } 1 =$$

$$Yyy \quad \frac{P}{4} ((1$$

$$\frac{P}{4} \left(\frac{(1 - \frac{1}{4})(1 + \frac{1}{4})(1 - \frac{1}{8})(1 + \frac{1}{8})(1 - \frac{1}{12})(1 + \frac{1}{12})(1 - \frac{1}{16})(1 + \frac{1}{16})(1 - \frac{1}{20})(1 + \frac{1}{20}) \&c:}{(1 - \frac{1}{2})(1 + \frac{1}{2})(1 - \frac{1}{6})(1 + \frac{1}{6})(1 - \frac{1}{10})(1 + \frac{1}{10})(1 - \frac{1}{14})(1 + \frac{1}{14})(1 - \frac{1}{18})(1 + \frac{1}{18}) \&c:} \right),$$

$$\text{scilicet } \frac{4}{P} = \frac{\frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{9}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{15}{16} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{21}{20} \cdot \&c:}{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{9}{10} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{15}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \&c:} =$$

$$\left(\frac{1}{2} \right) \left(\frac{2 \cdot 6 \cdot 6 \cdot 10 \cdot 10 \cdot 14 \cdot 14 \cdot 18 \cdot 18 \cdot 22 \cdot 22 \cdot \&c:}{4 \cdot 4 \cdot 8 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdot 16 \cdot 20 \cdot 20 \cdot 24 \cdot \&c:} \right) =$$

$$2 \left(\frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot \&c:}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot \&c:} \right) =$$

$$\frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot 13 \cdot \&c:}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot 12 \cdot \&c:}, \text{ quae ad unguem congruit cum ex-}$$

pressionem a Vallisio data, interea dum Series inversa tribuit $\frac{P}{4} = 1 -$

$$\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \&c:, \text{ vel ob praecedentem}$$

$$\text{Seriem, si mavis, } = \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot 12 \cdot \&c:}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot 13 \cdot \&c:}.$$

Eccine vinculum utriusque Seriei, quo sublato frustra aliter investigaretur earum communis fons, et origo. Subinde effluit $Q = \frac{P}{2} =$

$$\frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot 12 \cdot \&c:}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot 13 \cdot \&c:} = 2 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \&c: \right),$$

$$\text{aliaque permulta dimanant, quibus supersedeo singulariter enucleandis.}$$

275. Quod autem ad huius Theorices laudem maximopere conferre cenfeo situm est in ampliacione indefinita Seriei per continuos Factores ad quemcunque Circuli Arcum, Sectoremve, dum illa Vallisii ex adverso Arcum, Ostantem, aut Aream Quadrantis tantummodo respicit; adeo, ut, si-

cuti

cuti indefinitam expressionem designat Aequatio $x = \frac{T^1}{1} - \frac{T^3}{3} + \frac{T^5}{5}$

$-\frac{T^7}{7} + \frac{T^9}{9} - \frac{T^{11}}{11} + \&c.$, aeque intelligi idem debeat conversim de

altera. Facto igitur Arcu $x = \frac{mP}{n}$ nanciscimur canonicam Aequationem

universalissimam $\frac{mP}{nT} =$

$$\frac{\frac{n-2m}{n} \cdot \frac{n+2m}{n} \cdot \frac{3n-2m}{3n} \cdot \frac{3n+2m}{3n} \cdot \frac{5n-2m}{5n} \cdot \frac{5n+2m}{5n} \cdot \frac{7n-2m}{7n} \cdot \frac{7n+2m}{7n} \cdot \frac{9n-2m}{9n} \cdot \frac{9n+2m}{9n} \cdot \&c.}{\frac{n-m}{n} \cdot \frac{n+m}{n} \cdot \frac{2n-m}{2n} \cdot \frac{2n+m}{2n} \cdot \frac{3n-m}{3n} \cdot \frac{3n+m}{3n} \cdot \frac{4n-m}{4n} \cdot \frac{4n+m}{4n} \cdot \frac{5n-m}{5n} \cdot \frac{5n+m}{5n} \cdot \&c.},$$

quae, posita $\frac{m}{n}$ Fractione $= \frac{1}{4}$, denuo reddit peculiarem Vallisii Formu-

lam, et contrahi etiam posset subrogando vice $\frac{mP}{n}$, vel x , expressionem

zP : tunc etenim elicietur $\frac{zP}{T} =$

$$\frac{1 - \frac{2z}{1} \cdot 1 + \frac{2z}{1} \cdot 1 - \frac{2z}{3} \cdot 1 + \frac{2z}{3} \cdot 1 - \frac{2z}{5} \cdot 1 + \frac{2z}{5} \cdot 1 - \frac{2z}{7} \cdot 1 + \frac{2z}{7} \cdot 1 - \frac{2z}{9} \cdot 1 + \frac{2z}{9} \cdot 1 \cdot \&c.}{1 - \frac{z}{1} \cdot 1 + \frac{z}{1} \cdot 1 - \frac{z}{2} \cdot 1 + \frac{z}{2} \cdot 1 - \frac{z}{3} \cdot 1 + \frac{z}{3} \cdot 1 - \frac{z}{4} \cdot 1 + \frac{z}{4} \cdot 1 - \frac{z}{5} \cdot 1 + \frac{z}{5} \cdot 1 \cdot \&c.},$$

conversaue Aequalitas erit $\frac{zP}{T} = 1 - \frac{T^2}{3} + \frac{T^4}{5} - \frac{T^6}{7} + \frac{T^8}{9} -$

$\frac{T^{10}}{11} + \frac{T^{12}}{13} - \&c.$ Formula insuper ipsa Vallisii a Sinubus quoque rectis

dimanat, prouti Eulerus effecit in *Introductionis* Capite XI°. sed nexus ille hoc modo disrumperetur, quem illustrandum suscepi, mutuae scilicet cognationis in unam componentis Analyfin Series celebratissimas Leibnitianam, ac Vallisianam.

276. Sciunt omnes, qui se tradidere Mathesi, Vallisium Quadraturam Circuli suis quaerentem Interpolationibus Areas Curvilinearum Aequatione

(1 —

$(1 - x^2)^m = y$ gaudentium, ideoque et singularem Circuli, pro unico casu $x = 1$ commensurasse. Non adhuc enim Natura Neutronum litterario Orbi donaverat, penes quem gloria manebat Binomii Potestates evolvendi; et is solummodo potuit quorumlibet Segmentorum Curvarum Areas eadem Formula indefinite complecti. Consimili illius aetatis defectu laborant Hyperbolae Tetragonismus a Brounckero pro unico, ac definito Spatio repertus, qui fuit deinde a Mercatore omnibus numeris absolutus, nec non Fractio Continua determinatam totius Circuli exhibens Quadraturam. Quum itaque ostenderim in simplicissima possibilium hypothese Fractionem Brounckeri Seriei Leibnitianae congruere, idipsum universalius moliar, etsi nemo hucusque tentaverit. Profe-

cto Aequatio notissima indefinita $x = \frac{T^1}{1} - \frac{T^3}{3} + \frac{T^5}{5} - \frac{T^7}{7} +$

$\frac{T^9}{9} - \frac{T^{11}}{11} + \frac{T^{13}}{13} - \frac{T^{15}}{15} + \&c$: est revera aequipollens Fractioni Con-

tinuae pariter indefinitae

$$x = \frac{T}{1 + \frac{\frac{1}{1.3} T^2}{1 - \frac{\frac{1}{3} T^2 + \frac{\frac{1}{1.5} T^2}{\frac{1}{3} - \frac{1}{5} T^2 + \frac{\frac{1}{3.7} T^2}{\frac{1}{5} - \frac{1}{7} T^2 + \frac{\frac{1}{5.9} T^2}{\frac{1}{7} - \frac{1}{9} T^2 + \&c}}}}$$

in Infinitum. Fiat $x = \frac{P}{4}$, et idcirco $T = 1$; tunc abibit Aequatio in

$$\frac{P}{4} = \frac{1}{1 + \frac{1}{3}}$$

$$\frac{2}{3} + \frac{1}{5}$$

$$\frac{2}{15} + \frac{1}{21}$$

$$\frac{2}{35} + \frac{1}{45}$$

fine ullo Limite, quam si de more
 $\frac{2}{63} + \&c:$

partiamur praecedentia aemulati eliciemus

$$\frac{1}{1} \left| \frac{1}{1 + \frac{1}{3}} \right| \frac{1}{1 + \frac{1}{3}} \left| \frac{1}{1 + \frac{1}{3}} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

$$\frac{1}{1} \left| \frac{2}{3} \right| \frac{13}{15} \left| \frac{76}{105} \right| \&c:$$

&c:

&c:

&c: . Aliquam tamen exigit subti-

litem postremae Fractionis reductio ad elegantior formam, qua ipsam induit Brounckerus; ac, ne desiderium reconditae aequipollentiae in Lectoribus excitetur, sunt hic synthetico ordine apposita elementa integran-
 tia Seriei Brounckerianae

$$\frac{1}{1} \left| \frac{1}{1 + \frac{1}{2}} \right| \frac{1}{1 + \frac{1}{2}} \left| \frac{1}{1 + \frac{1}{2}} \right| \&c:$$

. Inita comparatione panduntur Aequali-

tates $\frac{1}{1+\frac{1}{\frac{3}{\frac{2}{3}}}}} = \frac{1}{1+\frac{1}{2}}$, ac deinceps $\frac{1}{1+\frac{1}{\frac{3}{\frac{2}{3}+\frac{1}{5}}}}} = \frac{1}{1+\frac{1}{\frac{3}{\frac{2}{3}+\frac{3}{2}}}}} = \frac{1}{1+\frac{1}{\frac{3}{\frac{2+3^2}{2}}}}} = \frac{1}{1+\frac{1}{2}}$

similiterque, iisdem confectis operationibus,

$$\frac{1}{1+\frac{1}{\frac{3}{\frac{2}{3}+\frac{1}{5}}}}} = \frac{1}{1+\frac{1}{\frac{3}{\frac{2}{3}+\frac{1}{5}}}}} = \frac{1}{1+\frac{1}{\frac{3}{\frac{2}{3}+\frac{1}{5}}}}} = \frac{1}{1+\frac{1}{\frac{3}{\frac{2}{3}+\frac{1}{5}}}}} = \frac{1}{1+\frac{1}{\frac{3}{\frac{2}{3}+\frac{1}{5}}}}}$$

et sic aequipollentia sibi semper constans reliquos fideliter terminos comitatur.

277. Amplissimum igitur omnigenarum Serierum Tetragonismo Circuli haftenus adscriptarum prospectum, et Analyticum quasi cippum, unde eae simul digrediuntur veluti eiusdem arboris rami, nempe conformes non tam cognatione, quam originis unitate, periucundum hic fore confido.

$$\text{Arc. C. } zP = \frac{(\text{Tang. } zP)^1}{1} - \frac{(\text{Tang. } zP)^3}{3} + \frac{(\text{Tang. } zP)^5}{5} - \frac{(\text{Tang. } zP)^7}{7} + \frac{(\text{Tang. } zP)^9}{9} - \frac{(\text{Tang. } zP)^{11}}{11} + \frac{(\text{Tang. } zP)^{13}}{13} \&c.$$

$$\text{Arc. C. } zP = \frac{(\text{Tan. } zP)^1}{1 - \frac{1}{1.3} (\text{Tan. } zP)^2}$$

$$1 - \frac{1}{3} (\text{Tan. } zP)^2 + \frac{1}{1.5} (\text{Tan. } zP)^2$$

$$\frac{1}{3} - \frac{1}{5} (\text{Tan. } zP)^2 + \frac{1}{3.7} (\text{Tan. } zP)^2$$

$$\frac{1}{5} - \frac{1}{7} (\text{Tan. } zP)^2 + \frac{1}{5.9} (\text{Tan. } zP)^2$$

$$\frac{1}{7} - \frac{1}{9} (\text{Tan. } zP)^2 + \&c.$$

Arc.

Arc. C. zP =

$$(\text{Tan. } zP) \left(\frac{1 - \frac{2z}{1} \cdot 1 + \frac{2z}{1} \cdot 1 - \frac{2z}{3} \cdot 1 + \frac{2z}{3} \cdot 1 - \frac{2z}{5} \cdot 1 + \frac{2z}{5} \cdot 1 - \frac{2z}{7} \cdot 1 + \frac{2z}{7} \cdot 1 - \frac{2z}{9} \cdot 1 + \frac{2z}{9} \cdot 1 \cdot \&c:}{1 - \frac{z}{1} \cdot 1 + \frac{z}{1} \cdot 1 - \frac{z}{2} \cdot 1 + \frac{z}{2} \cdot 1 - \frac{z}{3} \cdot 1 + \frac{z}{3} \cdot 1 - \frac{z}{4} \cdot 1 + \frac{z}{4} \cdot 1 - \frac{z}{5} \cdot 1 + \frac{z}{5} \cdot 1 \cdot \&c:} \right).$$

In postrema Formula occurrit notandum, quod nullam difficultatem praeferebat nec simul, nec discriminatim eventura Arcus atque Tangentis evanescentia. Hypothesi namque selecta $\tau z = 0$ proveniet Aequatio $\frac{0}{0} =$

$$\frac{1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot \&c:}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot \&c:}, \text{ quae contradictionem non implicat;}$$

ex altero casu $\tau z = nP$ fluit Aequalitas $nP =$

$$o \left(\frac{1 - \frac{2n}{1} \cdot 1 + \frac{2n}{1} \cdot 1 - \frac{2n}{3} \cdot 1 + \frac{2n}{3} \cdot 1 - \frac{2n}{5} \cdot 1 + \frac{2n}{5} \cdot 1 - \frac{2n}{7} \cdot 1 + \frac{2n}{7} \cdot 1 - \frac{2n}{9} \cdot 1 + \frac{2n}{9} \cdot 1 \cdot \&c:}{1 - \frac{n}{1} \cdot 1 + \frac{n}{1} \cdot 1 - \frac{n}{2} \cdot 1 + \frac{n}{2} \cdot 1 - \frac{n}{3} \cdot 1 + \frac{n}{3} \cdot 1 - \frac{n}{4} \cdot 1 + \frac{n}{4} \cdot 1 - \frac{n}{5} \cdot 1 + \frac{n}{5} \cdot 1 \cdot \&c:} \right),$$

quae reapse locum habet quoties n sit Numerus integer positivus, vel negativus, quia aliquis tum necessario nullefcet ex innumeris Denominatoris

Factoribus, ideoque fiet $nP = \frac{0}{0} \cdot \frac{A}{B}$. Si placeat in exemplo $\tau z = n = 1$

periculum facere nullus dubito, quin per se pateat esse $P =$

$$\frac{o \cdot -1 \cdot 3 \cdot \frac{1}{3} \cdot \frac{5}{3} \cdot \frac{3}{5} \cdot \frac{7}{5} \cdot \frac{5}{7} \cdot \frac{9}{7} \cdot \frac{7}{9} \cdot \frac{11}{9} \cdot \&c:}{o \cdot 2 \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \&c:} =$$

$$\frac{o \cdot -1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot \&c:}{o \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot \&c:} = \frac{-o}{o} = \frac{-Pd z}{dz} = P \text{ ex Regula Ber-}$$

noulliana; quae conclusio cum veritate consentit. Iuvabit denique Infinitae Tangentis eventum aliquem obiter etiam sub oculos ponere, quod tum fa-

cilius continget, quum $n = \frac{1}{2}$. Eo casu se contemplandam praebet for-

ma Aequationis $Q =$

$$\infty \cdot \frac{0}{1} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{8}{9} \cdot \frac{10}{11} \cdot \frac{10}{11} \cdot \&c: \\ \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{7}{8} \cdot \frac{9}{8} \cdot \frac{9}{10} \cdot \frac{11}{10} \cdot \&c:$$

, scilicet vi nume-

$$\text{ri 274.}, \text{ qui recte consuli debet, } Q = \frac{\infty \cdot 0 \cdot Q}{\frac{1}{Q}} = \frac{1}{-PdZ} \cdot -2dz \cdot Q$$

$= Q$, substitutis, uti superius, loco ∞ , et 0 , eorum minimis symbolis, aut elementis.

278. Tot uberrimas proprietates ab unica Tangentium Circularium Formula directa, et inversa consecutis, obvium, ac facile dictu est ampliorē messem ceteras omnes expressiones Trigonometricas polliceri. Parcius autem de reliquis generatim acturi enarrabimus tantummodo ab Aequatione Versos-Sinus respiciente, quam num°. 227. in Factores resolvimus,

$$\text{Sin. V. C. } x = \frac{x^2}{2} \left(1 - \frac{x^2}{\Pi^2} \right) \left(1 - \frac{x^2}{\Pi^2} \right) \left(1 - \frac{x^2}{4\Pi^2} \right) \left(1 - \frac{x^2}{4\Pi^2} \right)$$

$$\left(1 - \frac{x^2}{9\Pi^2} \right) \left(1 - \frac{x^2}{9\Pi^2} \right) \left(1 - \frac{x^2}{16\Pi^2} \right) \left(1 - \frac{x^2}{16\Pi^2} \right) \left(1 - \frac{x^2}{25\Pi^2} \right)$$

$$\left(1 - \frac{x^2}{25\Pi^2} \right) \&c: \text{ profluere, dum } x = \frac{\Pi}{4}, \text{ Aequalitatem } 1 = \frac{\Pi^2}{32}$$

$$\left(\frac{15}{16} \cdot \frac{15}{16} \cdot \frac{63}{64} \cdot \frac{63}{64} \cdot \frac{143}{144} \cdot \frac{143}{144} \cdot \frac{255}{256} \cdot \frac{255}{256} \cdot \&c: \right). \text{ Exinde erit } \frac{\Pi}{4\sqrt{2}}$$

$$= \frac{Q}{\sqrt{2}} = \frac{16}{15} \cdot \frac{64}{63} \cdot \frac{144}{143} \cdot \frac{256}{255} \cdot \frac{400}{399} \cdot \&c: = \frac{4 \cdot 4 \cdot 8 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdot 16 \cdot 20 \cdot 20 \cdot \&c:}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \cdot 19 \cdot 21 \cdot \&c:}, \text{ et}$$

favente fortuna, si haec ultima expressio dividat aliam in numero 274.

$$\text{prolatam, } Q = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot 12 \cdot 14 \cdot 14 \cdot 16 \cdot 16 \cdot 18 \cdot 18 \cdot 20 \cdot \&c:}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot 13 \cdot 13 \cdot 15 \cdot 15 \cdot 17 \cdot 17 \cdot 19 \cdot 19 \cdot \&c:}, \text{ eliminabitur}$$

$$Q, \text{ orieturque Platonica Irrationalis Magnitudo } \sqrt{2} =$$

$$\frac{2 \cdot 2 \cdot 6 \cdot 6 \cdot 10 \cdot 10 \cdot 14 \cdot 14 \cdot 18 \cdot 18 \cdot 22 \cdot 22 \cdot \&c:}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \cdot 19 \cdot 21 \cdot 23 \cdot \&c:}, \text{ quemadmodum } \textit{Introductionis} \text{ praefato in Ca-$$

pite XI°. , sed valde diversa semita, invenit Eulerus. Quis autem sibi uspiam, Analyfi profundius inconsulta, suaderet ternas Series triplicis formae, nimirum

$$x \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \frac{x^5}{6} + \frac{x^6}{7} - \frac{x^7}{8} + \&c: \right)$$

$$= \frac{x}{1 + \frac{1}{2}x}$$

$$1 - \frac{1}{2}x + 1 \cdot \frac{1}{3}x$$

$$\frac{1}{2} - \frac{1}{3}x + \frac{1}{2} \cdot \frac{1}{4}x$$

$$\frac{1}{3} - \frac{1}{4}x + \frac{1}{3} \cdot \frac{1}{5}x$$

$$\frac{1}{4} - \frac{1}{5}x + \&c:$$

adeo, ut $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} \&c:$, five Series a

Brounckero data $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \frac{1}{9.10} + \&c:$, vel de more re-

centiorum Geometrarum L 2, $= \frac{1}{1 + \frac{1}{2}}$

$$\frac{1}{2} + \frac{1}{3}$$

$$\frac{1}{6} + \frac{1}{8}$$

$$\frac{1}{12} + \frac{1}{15}$$

$$\frac{1}{20} + \&c: . \text{Ista Fra-}$$

ctio non solum singulas supputando partes ad amissim cum Serie proposita convenit, verum etiam in eandem redigitur formam, quam obiter numero 271. nunciavi. Satis eloquens erit meo iudicio subiectum Paradigma.

$$1 - \frac{1}{2}$$

$$1 - \frac{1}{2} \quad 1 - \frac{1}{2} + \frac{1}{3} \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c:$$

$$\frac{1}{2}$$

$$\frac{5}{6}$$

$$\frac{7}{12}$$

$$\frac{47}{60}$$

$$\frac{1}{1 + \frac{1}{2}}$$

$$\frac{1}{1 + \frac{1}{2}}$$

$$\frac{1}{1 + \frac{1}{2}}$$

$$\frac{1}{1 + \frac{1}{2}}$$

$$\frac{1}{2}$$

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&c:

&c: &c:.

280. Per-

280. Perbelle renovatur in indefinita Fractione Continua idem acci-
dens, quod ipsamet usu ob enascentem divergentiam destituatur toties, quo-
ties x superet 1, vel Logarithmus quaeratur Numeri supra Binarium. Quin
et singulare quiddam occurrit in hypothese $\tau x = 2$, vel $\tau L 3$: tunc enim
Fractio Continua abit in 2

$$\begin{array}{r} \frac{1}{1+1} \\ \frac{0+\frac{2}{3}}{\frac{3}{3}} \\ \frac{-\frac{1}{6}+\frac{1}{4}}{\frac{6}{4}} \\ \frac{-\frac{1}{6}+\frac{2}{15}}{\frac{6}{15}} \\ \frac{-\frac{3}{20}+\&c:}{\frac{20}{20}} \end{array} \text{ ubi Denominator}$$

o novum fallaciae detegit argumentum. Quod vero utilius, aptiusque ad
rem nostram promovendam videtur est omnimoda illius Fractionis Conti-
nuae universalitas quando etiam x negativus evadat. Revera — ($x +$

$$\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} + \frac{x^{10}}{10} + \&c:),$$

$$\text{scilicet } L(1-x) = \frac{-x}{1-\frac{1}{2}x}$$

$$1 + \frac{1}{2}x - 1 \cdot \frac{1}{3}x$$

$$\frac{1}{2} + \frac{1}{3}x - \frac{1}{2} \cdot \frac{1}{4}x$$

$$\frac{1}{3} + \frac{1}{4}x - \frac{1}{3} \cdot \frac{1}{5}x$$

$$\frac{1}{4} + \frac{1}{5}x - \&c: ; \text{ atque}$$

hoc pacto eadem forma Continuae Fractionis utriusque Hyperbolici $L(1 \pm x)$
repraesentandi capax Logometricam doctrinam locupletabit. Specimen il-
lius non parum mirabilis aequipollentiae tentandum fit in periculosa ma-
gis, errorique obnoxia hypothese $L 0$, qui Logarithmus negativus habetur

Magni-

— $\frac{1}{9}$ — &c: sine ullo Limite. Partium ordine dispositarum subnexus Ty-

pus perpulchram veritatem demonstrat.

$\frac{-1}{1}$	$\frac{-1}{1 - \frac{1}{2}}$	$\frac{-1}{1 - \frac{1}{2}}$	$\frac{-1}{1 - \frac{1}{2}}$	; quamobrem $\frac{-1}{1 - \frac{1}{2}}$
	$\frac{1 + \frac{1}{2}}{2}$	$\frac{1 + \frac{1}{2} - \frac{1}{3}}{2}$	$\frac{1 + \frac{1}{2} - \frac{1}{3}}{2}$	$\frac{1 + \frac{1}{2} - 1 \cdot \frac{1}{3}}{2}$
		$\frac{\frac{1}{2} + \frac{1}{3}}{2}$	$\frac{\frac{1}{2} + \frac{1}{3} - \frac{1}{8}}{2}$	$\frac{\frac{1}{2} + \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{4}}{2}$
			$\frac{7}{12}$	&c:
$\frac{-1}{1 - \frac{1}{3}}$	$\frac{-1}{1 - \frac{1}{2}}$	$\frac{-1}{1 - \frac{1}{2}}$		
	$\frac{1 + \frac{1}{2} - \frac{2}{5}}{5}$	$\frac{1 + \frac{1}{2} - \frac{1}{3}}{3}$		
$-\frac{3}{2}$	$-\frac{11}{6}$	$-\frac{25}{12}$	$\frac{1}{2} + \frac{1}{3} - \frac{3}{14}$	
$-1 - 1 - \frac{1}{2}$	$-1 - \frac{1}{2} - \frac{1}{3}$	$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4}$		

$= \infty$ negativo, nimirum

$$\frac{1}{4} + \frac{1}{5} - \&c:$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c., \text{ auf Name-}$$

B b b b

ro Infinite

ro Infinite magno, sed positivo, vel Asymptotico Hyperbolae Apollonianae Spatio per Potentiam diviso.

281. Subinde colligitur esse $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} +$

$$\frac{1}{15} + \&c: = \frac{1}{2} \left(\frac{1}{1 - \frac{1}{2}} \right. \\ \left. \frac{1}{1 + \frac{1}{2} - 1 \cdot \frac{1}{3}} \right. \\ \left. \frac{1}{\frac{1}{2} + \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{4}} \right. \\ \left. \frac{1}{\frac{1}{3} + \frac{1}{4} - \frac{1}{3} \cdot \frac{1}{5}} \right. \\ \left. \frac{1}{\frac{1}{4} + \frac{1}{5} - \&c:} \right),$$

propterea quod ex iam dictis $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \&c: =$

$$\frac{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c:}{2}. \text{ Et, sicuti Area}$$

cuiuslibet Sectoris Aequilaterae Hyperbolae per $\frac{1}{2}$ divisa, sive Logarithmus Arcui Circulari analogiam servans, ope respectivae Tangentis exhibetur hac Aequatione

$$\frac{T^1}{1} + \frac{T^3}{3} + \frac{T^5}{5} + \frac{T^7}{7} + \frac{T^9}{9} + \frac{T^{11}}{11} + \frac{T^{13}}{13} \\ + \&c: = \frac{\text{Sect. H.}}{\frac{1}{2}} = 2 \text{ Sect. H} = \frac{1}{2} L \left(\frac{1+T}{1-T} \right), \text{ non difficile erit ipsam}$$

in Fractionem Continuum vertere indefinitum valorem comprehendentem hoc modo,

$$2 \text{ Sect. H} = \frac{(\text{Tang. H})^1}{1} + \frac{(\text{Tang. H})^3}{3} + \frac{(\text{Tang. H})^5}{5} + \frac{(\text{Tang. H})^7}{7} + \frac{(\text{Tang. H})^9}{9} \&c:$$

$$= (\text{Tan. H})^1$$

$$1 - \frac{1}{1.3} (\text{Tan. H})^2$$

$$1 + \frac{1}{3} (\text{Tan. H})^2 + \frac{1}{1.5} (\text{Tan. H})^2$$

$$- \frac{1}{3} - \frac{1}{5} (\text{Tan. H})^2 + \frac{1}{3.7} (\text{Tan. H})^2$$

$$\frac{1}{5} + \frac{1}{7} (\text{Tan. H})^2 + \frac{1}{5} \cdot \frac{1}{9} (\text{Tan. H})^2$$

$$- \frac{1}{7} - \frac{1}{9} (\text{Tan. H})^2 \&c:.$$

282. Habent nunc Analystae alterum insignis exemplum moliminis in Tangente Hyperbolica = 1, vel peraequante Radium, aut Transversum Semi-Latus: namque

$$\begin{array}{l} 1 - \frac{1}{1.3} \\ 1 + \frac{1}{3} + \frac{1}{1.5} \\ - \frac{1}{3} - \frac{1}{5} + \frac{1}{3.7} \\ \frac{1}{5} + \frac{1}{7} + \frac{1}{5.9} \\ - \frac{1}{7} - \frac{1}{9} \&c: \end{array} = 1 - \frac{1}{1.3} = 1 + \frac{1}{3} = \frac{4}{3} \left| \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} \right. = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} = 1 + \frac{1}{3} + \frac{1}{5} = \frac{23}{15} \&c:$$

+ &c: , prouti incoepta, e regione, partium separatio gradatim confecta procul dubio confirmabit. Comparatio autem Serierum nuperrime evolutarum Confectarium non leve progignit, perfectae nimirum aequalitatis duas inter Fractiones Continuas sine Limite productas

$$\frac{1}{2} \left(\right.$$

$$\begin{aligned}
 & \frac{1}{2} \left(\frac{1}{1 - \frac{1}{2}} \right) \\
 & \quad \frac{1}{1 + \frac{1}{3} - \frac{1.1}{3}} \\
 & \quad \quad \frac{1}{\frac{1}{2} + \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{4}} \\
 & \quad \quad \quad \frac{1}{\frac{1}{3} + \frac{1}{4} - \frac{1}{3} \cdot \frac{1}{5}} \\
 & \quad \quad \quad \quad \frac{1}{\frac{1}{4} + \frac{1}{5} \&c;} \\
 & = \frac{1}{1 - \frac{1.1}{3}} \\
 & \quad \frac{1}{1 + \frac{1}{3} - \frac{1.1}{5}} \\
 & \quad \quad \frac{1}{-\frac{1}{3} - \frac{1}{5} + \frac{1}{3} \cdot \frac{1}{7}} \\
 & \quad \quad \quad \frac{1}{\frac{1}{5} + \frac{1}{7} - \frac{1}{5} \cdot \frac{1}{9}} \\
 & \quad \quad \quad \quad - \frac{1}{7} - \frac{1}{9} \&c:
 \end{aligned}$$

quamvis universi Numeri Naturales in prima locum habeant, ac solummodo Impares in secunda.

283. Sunt itaque Continuae Fractiones adplicabiles etiam Seriebus Infinitis non alternantium terminorum conditione praeditis, quod repugnare videtur sententiae doctissimi Euleri (*). Veruntamen, quidquid de hoc post superius adlata sit iudicandum, Tetragonismus Hyperbolae adiumento Tangentis Seriem respuit Infinitam per Factores innumeros ad instar Valisianae pro Circulo determinandam. Est enim vi numⁱ. 220. veritati universaliter consentanea Aequatio subsequens Tang. H. (2 Sect. H) = 2 Sect. H ×

$$\left(\frac{1 + \frac{(2 \text{ Sect. H})^2}{p^2}}{1 + 4 \frac{(2 \text{ Sect. H})^2}{p^2}} \right) \left(\frac{1 + \frac{(2 \text{ Sect. H})^2}{2^2 p^2}}{1 + 4 \frac{(2 \text{ Sect. H})^2}{3^2 p^2}} \right) \left(\frac{1 + \frac{(2 \text{ Sect. H})^2}{3^2 p^2}}{1 + 4 \frac{(2 \text{ Sect. H})^2}{5^2 p^2}} \right) \left(\frac{1 + \frac{(2 \text{ Sect. H})^2}{4^2 p^2}}{1 + 4 \frac{(2 \text{ Sect. H})^2}{7^2 p^2}} \right) \&c:$$

a qua, dum hypothesis fieret Tan. H = 1, ut in num^o. 274., derivaretur

$$\frac{1}{\infty} = \frac{\infty^2 \cdot \infty^2 \cdot \infty^2 \cdot \infty^2 \cdot \infty^2 \cdot \infty^2 \cdot \infty^2 \&c:}{(4 \cdot \infty^2)(4 \cdot \infty^2)(4 \cdot \infty^2)(4 \cdot \infty^2)(4 \cdot \infty^2)(4 \cdot \infty^2)(4 \cdot \infty^2) \&c:}, \text{ vel}$$

$$0 = \frac{1}{4^\infty}; \text{ quod nihil constituit Hyperbolicis Areis finitis accommodandum.}$$

Eo minus

(*) Profecto numeri 365. 369. 370. 371. 372. 373., ac totum Caput XVIII. primi Voluminis celeberrimae *Introductionis* &c: una cum pereximia Disquisitione de *Continuis Fractionibus* in *Commentariis Academiae Petropolitanae* anni MDCCXXXVII. nullam offerunt Seriem *successivis* terminorum Signis gaudentem.

Eo minus Hyperbolae Quadratura eruetur ope usitatae substitutionis
 $(2 \text{ Sect. H})^2 = P^2 z$, quae ducit ad Aequationem $\text{Tang. H.} = 2 \text{ Sect. H.} \times$

$$\left(\frac{1 + \frac{z}{1} \cdot 1 + \frac{z}{4} \cdot 1 + \frac{z}{9} \cdot 1 + \frac{z}{16} \cdot 1 + \frac{z}{25} \cdot 1 + \frac{z}{36} \cdot 1 + \frac{z}{49} \cdot \&c:}{1 + \frac{4z}{1} \cdot 1 + \frac{4z}{9} \cdot 1 + \frac{4z}{25} \cdot 1 + \frac{4z}{49} \cdot 1 + \frac{4z}{81} \cdot 1 + \frac{4z}{121} \cdot 1 + \frac{4z}{169} \cdot \&c:} \right), \text{ scilicet}$$

$$\frac{2 \text{ Sect. H}}{\text{Tang. H}} = \frac{1 + \frac{4z}{1} \cdot 1 + \frac{4z}{9} \cdot 1 + \frac{4z}{25} \cdot 1 + \frac{4z}{49} \cdot 1 + \frac{4z}{81} \cdot 1 + \frac{4z}{121} \cdot 1 + \frac{4z}{169} \cdot \&c:}{1 + \frac{z}{1} \cdot 1 + \frac{z}{4} \cdot 1 + \frac{z}{9} \cdot 1 + \frac{z}{16} \cdot 1 + \frac{z}{25} \cdot 1 + \frac{z}{36} \cdot 1 + \frac{z}{49} \cdot \&c:} : \text{fiqui-}$$

dem $z = \frac{(2 \text{ Sect. H})^2}{P^2}$ comprehendit indefinitam simul Hyperbolae, quae

unica petebatur, ac definitam Circuli Quadraturam, neque aliter, veluti
 occurrit feliciter in casu praefati numeri 274., hae Transcendentes Magni-
 tudines eliminantur. Periucundum nihilo tamen minus Geometrarum plerisque
 videbitur quod hypothefi facta $2 \text{ Sect. H.} = P$, five Areae Sectoris Hyper-
 bolici aequalis Circuli Semissi proveniat Tangens respectiva Sectoris illius

$$= P \left(\frac{\frac{2}{1} \cdot \frac{5}{4} \cdot \frac{10}{9} \cdot \frac{17}{16} \cdot \frac{26}{25} \cdot \frac{37}{36} \cdot \frac{50}{49} \cdot \&c:}{\frac{5}{1} \cdot \frac{13}{9} \cdot \frac{29}{25} \cdot \frac{53}{49} \cdot \frac{85}{81} \cdot \frac{125}{121} \cdot \frac{173}{169} \cdot \&c:} \right); \text{ ubi adnotandum super-}$$

venit Numeratorem affinitatem quamdam habere cum Serie $\frac{3}{4} \cdot \frac{8}{9} \cdot \frac{15}{16}$,

$\frac{24}{25} \cdot \frac{35}{36} \cdot \frac{48}{49} \cdot \&c:$ ab Eulero in Schediasmate toties citato demonstrata

$= \frac{1}{2}$. Alios Numeri z valores persequendos Lectoribus linquens silentio

praetermittere nequeo affinitatem recensitam in eo positam esse, quod Eu-

leri Series ita exprimatur $\frac{1}{2} = \left(\frac{2^2-1}{2^2} \right) \left(\frac{3^2-1}{3^2} \right) \left(\frac{4^2-1}{4^2} \right) \left(\frac{5^2-1}{5^2} \right) \left(\frac{6^2-1}{6^2} \right) \left(\frac{7^2-1}{7^2} \right)$

$\&c:$, nostraque fit $\frac{1}{2} \cdot \frac{\text{Tang. } (2 \text{ Sect. H} = P)}{P} =$

Cccc

$$\left(\frac{2^2+1}{2^2} \right)$$

$$\left(\frac{\left(\frac{2^2+1}{2^2}\right)\left(\frac{3^2+1}{3^2}\right)\left(\frac{4^2+1}{4^2}\right)\left(\frac{5^2+1}{5^2}\right)\left(\frac{6^2+1}{6^2}\right)\left(\frac{7^2+1}{7^2}\right) \&c:}{\left(\frac{1^2+4}{1^2}\right)\left(\frac{3^2+4}{3^2}\right)\left(\frac{5^2+4}{5^2}\right)\left(\frac{7^2+4}{7^2}\right)\left(\frac{9^2+4}{9^2}\right)\left(\frac{11^2+4}{11^2}\right) \&c:} \right). \text{ Singulari igitur arte,}$$

ac summopere diffita ab Hyperbolico-Circularis Trigonometriae regulis Eulerus ostendit in supra memoratis *Petropolitanæ Academiae Commentariis*

$$\begin{aligned} \tau \delta L^\infty &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \\ &\frac{1}{11} + \&c: = 2 \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \&c: \right) = \\ &2 \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \&c: \right), \text{ redigi posse in} \end{aligned}$$

$$\text{Factum } \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot \&c:}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdot 12 \cdot 16 \cdot 18 \cdot 22 \cdot \&c:}, \text{ ideoque alteram sine nume-}$$

ro Factorum Seriem, quæ Numeros universos, præter primos, complectitur

$$\frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot \&c:}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot \&c:}, \text{ five potius}$$

$$2 \cdot \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \frac{6}{5} \cdot \frac{7}{6} \cdot \frac{8}{7} \cdot \frac{9}{8} \cdot \frac{10}{9} \cdot \&c:, \text{ vel denuo } 2 \cdot 2 \cdot \frac{2}{1}$$

$$\frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \frac{6}{5} \cdot \frac{7}{6} \cdot \frac{8}{7} \cdot \frac{9}{8} \cdot \frac{10}{9} \cdot \&c:, \text{ rursusque } 2 \cdot 2 \cdot 2 \cdot \frac{2}{1} \cdot \frac{3}{2}$$

$$\frac{4}{3} \cdot \frac{5}{4} \cdot \frac{6}{5} \cdot \frac{7}{6} \cdot \frac{8}{7} \cdot \frac{9}{8} \cdot \frac{10}{9} \cdot \&c: \text{ absque fine, aut } 2 + \frac{1}{2} A + \frac{1}{3} B$$

$$+ \frac{1}{4} C + \frac{1}{5} D + \frac{1}{6} E + \frac{1}{7} F + \frac{1}{8} G + \frac{1}{9} H + \&c: \text{ esse valoris pariter In-}$$

finiti, prouti ex ipsius Seriei regeneratione ab Eulero, Goldbachioque neglecta, et Seriei plusquam-Harmonicae indole diversimode comprobatur. Quam caute vero fidendum sit huic methodo minus directæ Series istas in continuos Factores transformandi exemplo constat Brounckeriani Hyperbolæ Tetragonismi, de quo ipsa Euleri Diatriba flet. Seriei enim

$$1^I - \frac{1^{II}}{2} + \frac{1^{IV}}{3} - \frac{1^{III}}{4} + \frac{1^{VI}}{5} - \frac{1^{VII}}{6} + \frac{1^{VIII}}{7} - \frac{1^V}{8} + \frac{1^I}{9} - \frac{1^X}{10} + \frac{1^I}{11} - \frac{1^{VIII}}{12} + \frac{1^I}{13} - \frac{1^{XI}}{14} + \&c:$$

fui medietas

$$\text{addatur } \frac{1^{II}}{2} - \frac{1^{III}}{4} + \frac{1^{VII}}{6} - \frac{1^V}{8} + \frac{1^X}{10} - \frac{1^{VIII}}{12} + \frac{1^{XI}}{14} - \&c: ,$$

fietque singulis terminis rite

collectis $\frac{3}{2} - \frac{3}{4} + \frac{3}{6} - \frac{3}{8} + \frac{3}{10} - \frac{3}{12} + \frac{3}{14} - \&c: ,$ Summa equi-
dem veritati congruens, attamen alio modo iis simul iunctis, uti ordo super-
positorum pandit Commato-Indicum,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \&c: \text{ evaderet, Summa}$$

enormiter fallax, quippe quae nihil minus concluderet, quam $1 = 1 \frac{1}{2}$. Hu-

iusce regenerationis, et Infinitatis male ominatae vitium eadem in Serie
renovabitur, dum illius dimidium assumendi quaestio esset: namque singulis
ordine bisectis terminis

$$\text{enasceretur } \frac{1^{II}}{2} - \frac{1^{III}}{4} + \frac{1^{VI}}{6} - \frac{1^I}{8} + \frac{1^I}{10} - \frac{1^I}{12} + \frac{1^I}{14} - \&c: , \text{ recte}$$

quidem; sed admodum fallaciter si hac medietate dem-

$$\text{pta ab } 1^I - \frac{1^{II}}{2} + \frac{1^{IV}}{3} - \frac{1^{III}}{4} + \frac{1^V}{5} - \frac{1^{VI}}{6} + \frac{1^{VII}}{7} - \&c: \text{ tueremur}$$

mendosi evanescantiam

$$\text{Residui } 1 - 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{7} + \frac{1}{9} - \frac{1}{5} \&c: = 0.$$

284. Antequam horumce sine numero Factorum speculationi ferias
edicam nonnulla de Cyclometricis Propositionibus a Vallisio ad Oughtre-
dum missis labente anno MDCLV., nec non triennio prius Schootenio,
Christianoque Hugenio frustra tunc solutionem molientibus partim commu-
nicatis, differenda manent. Curvam, cuius Ordinatum-applicatae aequali-

bus

bus super Abscissarum Axem intervallis dispositae valores haberent $\frac{1}{1}$,

$$\frac{1.3}{1.2}, \frac{1.3.5}{1.2.4}, \frac{1.3.5.7}{1.2.4.6}, \frac{1.3.5.7.9}{1.2.4.6.8}, \frac{1.3.5.7.9.11}{1.2.4.6.8.10}, \&c: \text{ in Infinitum,}$$

adseruit Vallisius talem esse, qualem pro Geometrica, tametsi non satis commodae expressionis, habuisset Cartesius. Hoc autem Vallisii effatum castigationem meretur, eoquod revera Aequatio illius Curvae nonnisi *Transcendentaliter* obtineatur, atque universalissime expressa pendeat ab Ae-

qualitate $\int (1 - x^2)^z dx = y^{-1}$, in qua Integratio a Summatorio

Signo indicata unicum constantem valorem $x = 1$ respiciat. Qui Lineam istam Exponentialem, facilis constructionis expertem, quadam imagine sibi cupiat repraesentare, Locum ad Superficiem Aequatione praedicta significatum mentis acie persequatur: namque definito quolibet valore Exponentis z , qui munere fungatur Abscissae, et simul dato valore quovis numerico x , ope saltem Quadraturarum elicietur tertiae Coordinatae y valor; adeo, ut si Superficies exhibita Plano traiciatur ad distantiam τ & $x = 1$ Axibus Coordinatarum z , et y parallelo, communis Superficie, ac Plani Sectio Curvam illam designet a Vallisio quaesitam, et mendoso praecipitis nimium iudicii vaticinio Geometricis Lineis adscriptam. Confimilis error recurrit in gemina etiam Curva ibidem notata, cuius Ordinatum-appli-

catae aequaliter ad invicem distantes sint $\frac{1}{1}, \frac{1}{1} \cdot \frac{2.3}{1^2}, \frac{1}{1} \cdot \frac{2.3}{1^2} \cdot \frac{4.5}{2^2}, \frac{1}{1} \cdot$

$\frac{2.3}{1^2} \cdot \frac{4.5}{2^2} \cdot \frac{6.7}{3^2}, \frac{1}{1} \cdot \frac{2.3}{1^2} \cdot \frac{4.5}{2^2} \cdot \frac{6.7}{3^2} \cdot \frac{8.9}{4^2}, \&c:$, atque $\frac{1}{1}, \frac{1}{1} + \frac{1.3}{1.2}, \frac{1}{1}$

$+ \frac{1.3}{1.2} + \frac{1.3.5}{1.2.4}, \frac{1}{1} + \frac{1.3}{1.2} + \frac{1.3.5}{1.2.4} + \frac{1.3.5.7}{1.2.4.6}, \frac{1}{1} + \frac{1.3}{1.2} + \frac{1.3.5}{1.2.4}$

$+ \frac{1.3.5.7}{1.2.4.6} + \frac{1.3.5.7.9}{1.2.4.6.8}, \&c: \text{ sine Limite. Erit siquidem oecumenica}$

priorum omnium Aequatio $\int (x - x^2)^z dx = y^{-1}$; sed aliarum Ae-

qualitas haud secus exhiberi poterit praeterquam inconcinna, et subsultans,

quia nonnisi saltu derivetur ab Aequatione $\int dz \int (1 - x^2)^z dx = y^{-1}$,

prouti,

prouti, solita hypothefi facta variabilis $x = 1$, luculenter Analyftis conftabit fi maturius confulant Numericas Series, et Schemata ad calcem pofita *Arithmeticae Infinitorum* improbo labore exfoliandae.

285. Summo igitur Vallifii in Mathematicis rebus acumini, nec parum fortunae debemus *Interpolationis* felicitatem, qua Circuli Quadraturam innumeris Factoribus expreffit, quoniam Problematis refolutio omnibus numeris absoluta in tam fublimi pofita loco eminebat, viresque Analyfeos eo temporis tanto fuperabat intervallo, ut eximius ille Auctor ab interiectis Seriebus Numerorum Figuratorum, quos Lineae imitari poffunt Geometricae Aequationibus praeditae $y = 1$, $y = x$, $y = \frac{x^2+x}{2}$,

$$y = \frac{x^3+3x^2+2x}{6}, y = \frac{x^4+6x^3+11x^2+6x}{24} \&c.; \text{ delufus (*) Aequationes}$$

quoque Curvarum interponendarum Ordinis, vel Potestatis intermediae finitorum Analyfeos methodis assignabiles fibi quafi polliceretur. Unica profecto parata via erat, nec tum accessibilis, nondum existente Infinitesimalium Algorithmi, Aequationum ope Summatorio-Exponentialium fuperius latarum ad terminos univerfos earum Serierum rigida Lege Analytica fimul iungendos, cui Legi nusquam faciunt fatis indirectae Interpolationum methodi, quae quamproxime tantum folvunt Problemata, eoquod descriptione reguntur Curvarum Parabolici Generis. Ut vero propitia, quae Vallifio arrifit fortuna, meliori in fede elucefcant, obfervandum eft

terminum primae Seriei inter $\frac{1}{1}$, atque $\frac{1.2}{1.3}$ ponendum a Summatione τ

$(1-x^2)^{\frac{1}{2}} dx$, quam postmodum iniit Neutonus, pendere, nimirum effe

$$1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152} \&c.; \text{ aut } 1 - \frac{1}{2.3} - \frac{1}{2.4.5} - \frac{1.3}{2.4.6.7}$$

$-\frac{1.3.5}{2.4.6.8.9} - \&c.$ in Infinitum; hicque valor intercalandus nova adhuc

exigebat subsidia ad ipsum in Seriem Factoribus conflata innumeris re-

Dddd solvendum

(*) Confulat Lector numerum 386.

solvendum. Oportebat namque Seriem illam tanto ingenii conatu adinventam destituere, et secum mente volutare Aream Quadrantis petitam cum Arcus - Octantis Longitudine coincidentem, ideoque $1 - \frac{1}{6} -$

$$\frac{1}{40} - \frac{1}{112} - \frac{5}{1152} - \&c: = \frac{P}{4} =$$

$$1 + \frac{1}{6} + \frac{3}{40} + \frac{5}{112} + \frac{35}{1152} + \&c: \quad \text{ac praeterea ope Regressus Se-$$

$$\text{rierum nancisci } 1 = \frac{P}{2} \left(1 - \frac{1}{6} \left(\frac{P}{2} \right)^2 + \frac{1}{120} \left(\frac{P}{2} \right)^4 - \frac{1}{5040} \left(\frac{P}{2} \right)^6 + \right. \\ \left. \frac{1}{362880} \left(\frac{P}{2} \right)^8 - \&c: \right), \text{ et denuo naturam Circuli profundius speculari ad}$$

$$\text{acquirendam evolutionem Aequalitatum } 1 = \frac{P}{2} \left(\left(1 - \frac{\left(\frac{P}{2} \right)^2}{P^2} \right) \left(1 - \frac{\left(\frac{P}{2} \right)^2}{4P^2} \right) \right.$$

$$\left. \left(1 - \frac{\left(\frac{P}{2} \right)^2}{9P^2} \right) \left(1 - \frac{\left(\frac{P}{2} \right)^2}{16P^2} \right) \left(1 - \frac{\left(\frac{P}{2} \right)^2}{25P^2} \right) \&c: \right) = \frac{P}{2} \left(\frac{3}{4} \cdot \frac{15}{16} \cdot \frac{35}{36} \cdot \frac{63}{64} \cdot \frac{99}{100} \cdot \&c: \right)$$

$$= \frac{P}{2} \left(\frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot \&c:}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot \&c:} \right); \text{ nempe } \frac{P}{2} =$$

$$\frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot \&c:}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot \&c:}, \text{ vel demum } \frac{P}{4} =$$

$$\frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot \&c:}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot \&c:} = 1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} -$$

$$\frac{5}{1152} - \&c:, \text{ scilicet aequalem Rationi Areae Circuli ad circumscriptum}$$

Quadratum, aut quaesito Numero interpolando. Qui huius ratiocinationis circuitum, atque aliquantulam perplexitatem etiam concessa Recentiorum Analyfi rite perpenderit, mecum mirari non desinet Anglum illum Geometram intimo prope analogiae sensu excitatum eo pervenisse, quo tunc absolutissimae demonstrationis rigore vel accedere nefas. Methodi directae ad hoc conficiendum inscitiam firmiter comprobant labiles nimium

super

super Tabulam interpolandarum Serierum arguendi modi, quibus utitur saepenumero Vallisus, nonnulla demonstrationum optandarum vacua hinc inde interspersa, et, quod rei summa est, de reliquis interpolandis Seriebus ad Circuli Tetragonismum aequae suo iudicio ducentibus nec par sermo, nec labor. Terminum equidem inter $\frac{1}{1}$, ac $\frac{1}{1} \cdot \frac{1^2}{2 \cdot 3}$ secunda in

Serie ponendum non ipse, sed vidit Neutonus esse $\frac{2}{3} - \frac{1}{5} + \frac{1}{28} -$

$\frac{1}{72} - \frac{5}{704} - \&c$: adiumento Integrationis notissimae Formulae Fluxio-

nalis $(x - x^2)^{\frac{1}{2}} dx$. Verumtamen qui valoris illius mallet transformatione potiri in Factores innumeros, primum animadvertere deberet Semi-Circuli Aream, posita Diametro 1, exhiberi a dimidio Numero, quo Circumferentiae exprimitur Octans, dum Radius Unitatem peraequet. Sicuti

vero Formula Sinuum-Verforum praebet $Q = 2^{\frac{1}{2}} \left(1 + \frac{1}{12} + \frac{3}{160} + \frac{5}{896} \right.$

$\left. + \frac{35}{18432} + \&c \right)$ contemplanda erit Aequatio $\frac{2}{3} - \frac{1}{5} + \frac{1}{4 \cdot 7} - \frac{3}{4 \cdot 6 \cdot 9}$

$- \frac{3 \cdot 5}{4 \cdot 6 \cdot 8 \cdot 11} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8 \cdot 10 \cdot 13} - \&c =$

$1 + \frac{1^2}{2^2 \cdot 3} + \frac{1^2 \cdot 3^2}{2^3 \cdot 3 \cdot 4 \cdot 5} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^4 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{2^5 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \&c$

$\frac{1}{2 \cdot 2^2} \cdot \text{Heic tan}$

tum sistendo nulli adhuc proveniunt Factores, quibus adipiscendis, ut in Serierum priore, conversam necesse est Calculo subigere Versi-Sinus Seriem superius descriptam, quae ita se habebit, $2 =$

$\frac{P^2}{2} \left(1 - \frac{1}{4} + \frac{1}{4} - \frac{1}{16} + \frac{1}{16} - \frac{1}{36} + \frac{1}{36} - \frac{1}{64} + \frac{1}{64} - \&c \right)$

$=$

$\frac{P^2}{2}$

$$\frac{P^2}{2} \left(\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{15}{16} \cdot \frac{15}{16} \cdot \frac{35}{36} \cdot \frac{35}{36} \cdot \frac{63}{64} \cdot \frac{63}{64} \cdot \&c: \right), \text{ vel } \frac{p^2}{4} = \frac{4}{3} \cdot \frac{4}{3} :$$

$$\frac{16}{15} \cdot \frac{16}{15} \cdot \frac{36}{35} \cdot \frac{36}{35} \cdot \frac{64}{63} \cdot \frac{64}{63} \cdot \&c: , \text{ aut } \frac{P}{2} = \frac{4}{3} \cdot \frac{16}{15} \cdot \frac{36}{35} \cdot \frac{64}{63} \cdot \&c: , \text{ scilicet}$$

$$Q = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot \&c:}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot \&c:} , \text{ ac denique } \frac{Q}{4} =$$

$$\frac{1 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot \&c:}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot \&c:} = \frac{2}{3} - \frac{1}{5} - \frac{1}{28} - \frac{1}{72}$$

$$- \frac{5}{704} - \&c: = \text{termino interpolando Rationem Geometricam determi-}$$

nanti Areae Semi-Circuli ad Quadratum Diametri (*).

286. Ea insuper, quae Vallisus aut non persensit, aut fortunata potius, quam tuta inductione tantum ostendit, frustra quaeruntur maiori argumentorum robore firmata in Ismaelis Bullialdi Commentario satis amplo *ad Arithmeticae Infinitorum*, coeteroquin laudabili, Lutetiae Parisiorum edito, dum annus laberetur MDCLXXXII., iam Neutono, ac Leibnitio Infinitesimalis Algorithmi novum organum impertitis. Nihilominus fatendum erit, quod non solummodo Vallisus, eiusque fidus Scholastes Circulum versantes subtilius saepenumero se gesserint quam illi Geometriae aetati par esset, verum etiam multa sagacitatis miracula complecti Leotaudi *Cyclometricam* anno MDCLXIII. vulgatam, et a Leibnitio avide perlectam, *Cyclometricum* Snellii, Opus Hugenii *de Circuli Magnitudine* &c.; ita, ut Auctores ipsi vere lyncaei Calculo Fluxionum, Fluentiumque praeceuntes non-

nulla

$$(*) \text{ Serierum Infinitarum } 1 - \frac{1 \cdot 1}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8} -$$

$$\&c: \text{ atque } 1 - \frac{1}{2 \cdot 3} - \frac{1}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3}{2 \cdot 4 \cdot 6 \cdot 7} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} - \&c: \text{ Productum peraequari}$$

huic simplicissimae Fractioni $\frac{1}{2}$ videram adhuc adolescens, et ex istis facillime manat.

nulla definiverint, in quibus enucleandis huiusce quoque potentioris instrumenti subsidio non parum foret adlaborandum. Nonne ipsemet Gregorius a Sancto Vincentio *Quadraturam Circuli, et Hyperbolae* anno MDCXLVII. fallaciter adferens rara tamen, et ingeniosissima prodidit de Curvis Lineis, de Hyperbolica Logometria ac de Seriebus inventa, quae Ainscomii Sarassaeque mentes perperam in Quadraturae illius propugnatione operam, et oleum perdentium ad Auctoris promeritam laudem tuendam meliori iure exercere potuissent!



C A P U T VII.

MEDITATIONES IN UNITATIS RADICES IMAGINARIAS
 UBI DE NONNULLARUM AEQUATIONUM RESOLUTIONE
 AC DE THEOREMATIBUS QUOQUE COTESIANO ET MOIVREANO.

287.  Alter Capitis V. post Circuli Theoricen illustratam uberrimus equidem fructus latet in Formula a communis Binomii hausta fonte $C^{x\sqrt{-1}}$
 $= \text{Cofin. C. } x \rightarrow (\text{Sin. C. } x) \sqrt{-1}$, subindeque ex Potestatum nativis
 regulis $C^{z \cdot x \sqrt{-1}} = (\text{Cofin. C. } x \rightarrow (\text{Sin. C. } x) \sqrt{-1})^z = \text{Cofin. C. } zx \rightarrow (\text{Sin. C. } zx) \sqrt{-1}$, quum non solum idonea sit Logarithmis, et Exponentialibus analytice perficiendis, verum etiam nodos quasi Gordianos difficillimarum in Algebra Cartesiana quaestionum admodum simpliciter solvat. Mei interest argumenti quae ad Logometricam doctrinam absolvendam pertineant diligentius colligere; nam de aliis usibus agens in limine tantum subsistam. Quae autem compendio maiore dicenda erunt perraro, uti opinor, claritatem offendent.

288. Proponatur Aequalitas Ordinis indefiniti $x^m - 1 = 0$, quae Aequatio Binomia, aut aliis, praeter primum, ac ultimum, terminis non affecta, dum oecumenice resolvi possit, Radices Unitatis cuiusvis Gradus, etiam irrationalis, vel transcendentalis, et debito numero determinabit. Esto

itaque universaliter resolvenda $x^m - 1 = 0$, seu potius $1^{\frac{1}{m}} = x$; et cuiuslibet patebit talem concinnari debere expressionem valoris τx , ut eius Potestas Ordinis m Unitatem adaequet. Hoc autem efficietur facillime praestensorum adiumento statim atque x ponatur $= \text{Cofin. C. } v \rightarrow (\text{Sin. C. } v) \sqrt{-1}$:

quandoquidem ista ex hypothese profluat $x^m = \text{Cofin. C. } mv \rightarrow (\text{Sin. C. } mv) \sqrt{-1} = 1$, nimirum Arcus $mv = \pm n\pi$, veluti toties notavimus, ubi π Circularem Peripheriam, et n quemlibet designat Numerum integrum, quinimo
 etiam

etiam ipsam Nullitatem. Quo facto erit $v = \pm \frac{n\Pi}{m}$, ideoque $x = i^{\frac{n}{m}} = \text{Cofin. C. } \pm \frac{n\Pi}{m} + \left(\text{Sin. C. } \pm \frac{n\Pi}{m} \right) \sqrt{-1}$, scilicet aequalis Functioni Multiformi, et quandoque Infinitiformi, id, quod aliquam illustrationem meretur. Dum im-

primis Index m fit Numerus integer impar positivus, Radices $i^{\frac{1}{m}}$ evadent Cofin. C. 0 + (Sin. C. 0) $\sqrt{-1}$, five 1; Cofin. C. $\frac{\Pi}{m} + \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1}$; Cofin. C. $\frac{2\Pi}{m} + \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1}$; Cofin. C. $\frac{3\Pi}{m} + \left(\text{Sin. C. } \frac{3\Pi}{m} \right) \sqrt{-1}$; Cofin. C. $\frac{4\Pi}{m} + \left(\text{Sin. C. } \frac{4\Pi}{m} \right) \sqrt{-1}$; Cofin. C. $\frac{5\Pi}{m} + \left(\text{Sin. C. } \frac{5\Pi}{m} \right) \sqrt{-1}$ &c., donec inclu-

sive Coefficiens Circumferentiae Π Unitate deficiat ab Indice m , quoniam postea recurrerent inani repetitionum innumerarum periodo eadem iam inventae Radices. Obtinebuntur hac methodo Radices numero m Aequationis $x^m - 1 = 0$, et singulae ad unguem congruentes cum Algebrae regulis, quarum Radicum unica Realis $= 1$, et ceterae omnes Imaginariae. Difficultate etiam caret observatio, quod firma manente hypothese Numeri integri positivi, sed addita conditione m Numeri paris, inter Radices Imaginarias, forma tenus, latebit una Realis, videlicet ea sic expres-

sa Cofin. C. $\frac{\frac{1}{2}m\Pi}{m} + \left(\text{Sin. C. } \frac{\frac{1}{2}m\Pi}{m} \right) \sqrt{-1} = \text{Cofin. C. } \frac{\Pi}{2} +$

$\left(\text{Sin. C. } \frac{\Pi}{2} \right) \sqrt{-1} = -1$; adeo, ut in Aequationibus paris Gradus uni-

versis $x^m - 1 = 0$ habeantur geminae Reales Radices ± 1 , ac reliquae omnes numero $m - 2$ Imaginariae, et ex adverso, dum Ordo Aequationis sit impar, unica tantummodo Radix Realem praebeat valorem $+1$, residuaeque numero $m - 1$ compositum-Imaginarium.

289. Ne vero duplex Signum $\pm$ praecedens Arcus $\frac{n\Pi}{m}$ confusionem pa-

riat,

riat, neve Radicum copia nos premat, quae prima fronte, si utrumque Signum attendatur, nimis excrescit usque ad ipsarum Numerum $2m$ Cartesianae Analyseos canonibus adversantem, meliori quam potero ordine valores disponam $1^{\frac{x}{m}}$, scribendi modo varios, substantialiter autem aequipolentes. Selecta hypothese Exponentis m parisi erunt, quemadmodum et in altera dispari

$$\begin{aligned} \text{Cofin. C. } \frac{0}{m} + \left(\text{Sin. C. } \frac{0}{m} \right) \sqrt{-1} &= \text{Cofin. C. } \frac{-0}{m} + \left(\text{Sin. C. } \frac{-0}{m} \right) \sqrt{-1} = 1 \\ \text{Cofin. C. } \frac{\Pi}{m} + \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1} &= \text{Cofin. C. } -\left(\frac{m-1}{m} \right) \Pi + \left(\text{Sin. C. } -\left(\frac{m-1}{m} \right) \Pi \right) \sqrt{-1} \\ \text{Cofin. C. } \frac{2\Pi}{m} + \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1} &= \text{Cofin. C. } -\left(\frac{m-2}{m} \right) \Pi + \left(\text{Sin. C. } -\left(\frac{m-2}{m} \right) \Pi \right) \sqrt{-1} \\ \text{Cofin. C. } \frac{3\Pi}{m} + \left(\text{Sin. C. } \frac{3\Pi}{m} \right) \sqrt{-1} &= \text{Cofin. C. } -\left(\frac{m-3}{m} \right) \Pi + \left(\text{Sin. C. } -\left(\frac{m-3}{m} \right) \Pi \right) \sqrt{-1} \\ &\text{---} \\ \text{Cofin. C. } \left(\frac{m-3}{m} \right) \Pi + \left(\text{Sin. C. } \left(\frac{m-3}{m} \right) \Pi \right) \sqrt{-1} &= \text{Cofin. C. } -\frac{3\Pi}{m} + \left(\text{Sin. C. } -\frac{3\Pi}{m} \right) \sqrt{-1} \\ \text{Cofin. C. } \left(\frac{m-2}{m} \right) \Pi + \left(\text{Sin. C. } \left(\frac{m-2}{m} \right) \Pi \right) \sqrt{-1} &= \text{Cofin. C. } -\frac{2\Pi}{m} + \left(\text{Sin. C. } -\frac{2\Pi}{m} \right) \sqrt{-1} \\ \text{Cofin. C. } \left(\frac{m-1}{m} \right) \Pi + \left(\text{Sin. C. } \left(\frac{m-1}{m} \right) \Pi \right) \sqrt{-1} &= \text{Cofin. C. } -\frac{\Pi}{m} + \left(\text{Sin. C. } -\frac{\Pi}{m} \right) \sqrt{-1}; \end{aligned}$$

adeo, ut vel Arcus sumantur positivi, vel negativi, Radicum vere diversarum Numerus Indicem m nullatenus praetergrediatur. Apparens haec replicatio veri Radicum Numeri in Quadraticis etiam Aequationibus resolvendis, ne altiores memorem, locum habet. Negativos itaque Arcus —

$\frac{n\Pi}{m}$ excludere absque Radicum quaerendarum diminutione licebit; nec dicere audeas congemmatum esse nihilominus a Formula $\text{Cofin. C. } \pm \frac{n\Pi}{m} +$

(Sin. C.

$(\text{Sin. C.} \pm \frac{n\Pi}{m})\sqrt{-1}$ non valorum, sed Radicum numerum, quomodo-
 cumque illi se habeant tametsi bis repetiti. Namque nimis probaret
 hic arguendi modus, eoquod tam post $\text{Cofin. C.} (\frac{m-1}{m})\Pi \rightarrow (\text{Sin. C.}$
 $(\frac{m-1}{m})\Pi)\sqrt{-1}$, quam post $\text{Cofin. C.} - \frac{\Pi}{m} \rightarrow (\text{Sin. C.} - \frac{\Pi}{m})\sqrt{-1}$

illimitatae denuo recurrant eorundem valorum repetitionis periodi, et Al-
 gebrae notionibus primis repugnet perfecta congeminatio Radicum universa-
 rum in Aequatione, quae in duas admodum identicas resolvi non possit,

cuius quidem speciei non est $x^m - 1 = 0$ superius proposita. Hisce

praestens consequitur esse $x^{\frac{1}{m}}$ Functionem mformem; ac, quoties m fue-

rit par, in superiori Radicum omnium Prospectu locari debere $\text{Cofin. C.} \frac{\Pi}{2}$

$\rightarrow (\text{Sin. C.} \frac{\Pi}{2})\sqrt{-1} = \text{Cofin. C.} - \frac{\Pi}{2} \rightarrow (\text{Sin. C.} - \frac{\Pi}{2})\sqrt{-1} = -1,$

cui vacuam sedem consulto reliquimus.

290. Duo vellem, priusquam ultra progrediar, in hac Unitatis Radi-
 cum Theorice demonstrare. Primum in gravissima affectione repositum est,
 quae Radices illas universaliter comitatur, et qua fit, ut, praeter Cofin.

$\text{C.} \frac{0}{m} \rightarrow (\text{Sin. C.} \frac{0}{m})\sqrt{-1}$, vel $\text{Cofin. C.} 0 \rightarrow (\text{Sin. C.} 0)\sqrt{-1}$, aut 1, at-

que, dum adfit casu Indicis paris, $\text{Cofin. C.} \frac{\Pi}{2} \rightarrow (\text{Sin. C.} \frac{\Pi}{2})\sqrt{-1}$ five

-1 , dimidius reliquarum Radicum Numerus simpliciori forma donari pos-
 sit iuxta compendiarium Aequationum Tabulam heic recensitam, et ab Ele-
 mentis Trigonometriae derivatam.

$$\text{Cofin. C. } \left(\frac{m-1}{m}\right) \Pi \rightarrow \left(\text{Sin. C. } \left(\frac{m-1}{m}\right) \Pi\right) \sqrt{-1} = \text{Cofin. C. } \frac{\Pi}{m} - \left(\text{Sin. C. } \frac{\Pi}{m}\right) \sqrt{-1}$$

$$\text{Cofin. C. } \left(\frac{m-2}{m}\right) \Pi \rightarrow \left(\text{Sin. C. } \left(\frac{m-2}{m}\right) \Pi\right) \sqrt{-1} = \text{Cofin. C. } \frac{2\Pi}{m} - \left(\text{Sin. C. } \frac{2\Pi}{m}\right) \sqrt{-1}$$

$$\text{Cofin. C. } \left(\frac{m-3}{m}\right) \Pi \rightarrow \left(\text{Sin. C. } \left(\frac{m-3}{m}\right) \Pi\right) \sqrt{-1} = \text{Cofin. C. } \frac{3\Pi}{m} - \left(\text{Sin. C. } \frac{3\Pi}{m}\right) \sqrt{-1}$$

— — — — —

Subinde fuit contractior praeatas Radices scribendi modus illarum numero m facto testo, nempe

$\rightarrow 1$

$\rightarrow 1$

$$\text{Cofin. C. } \frac{\Pi}{m} \pm \left(\text{Sin. C. } \frac{\Pi}{m}\right) \sqrt{-1}, \text{ aut vicissim, } \text{Cofin. C. } \left(\frac{m-1}{m}\right) \Pi \pm \left(\text{Sin. C. } \left(\frac{m-1}{m}\right) \Pi\right) \sqrt{-1}$$

$$\text{Cofin. C. } \frac{2\Pi}{m} \pm \left(\text{Sin. C. } \frac{2\Pi}{m}\right) \sqrt{-1} \quad \text{et} \quad \text{Cofin. C. } \left(\frac{m-2}{m}\right) \Pi \pm \left(\text{Sin. C. } \left(\frac{m-2}{m}\right) \Pi\right) \sqrt{-1}$$

$$\text{Cofin. C. } \frac{3\Pi}{m} \pm \left(\text{Sin. C. } \frac{3\Pi}{m}\right) \sqrt{-1} \text{ eadem ratione } \text{Cofin. C. } \left(\frac{m-3}{m}\right) \Pi \pm \left(\text{Sin. C. } \left(\frac{m-3}{m}\right) \Pi\right) \sqrt{-1}$$

— — — — —

quibus omnibus, dum m fit par, addi debet -1 , altera nimirum Radix Realis expressa per

$$\text{Cofin. C. } \frac{\frac{1}{2} m \Pi}{m} \pm \left(\text{Sin. C. } \frac{\frac{1}{2} m \Pi}{m}\right) \sqrt{-1}, \text{ vel}$$

$$\text{Cofin. C. } \left(\frac{m - \frac{1}{2} m}{m}\right) \Pi \pm \left(\text{Sin. C. } \left(\frac{m - \frac{1}{2} m}{m}\right) \Pi\right) \sqrt{-1},$$

ubi sistit supputatio Radicum, quae, si m foret impar, eo loci cederet, quo perventum esset ad includendam Radicem

$$\text{Cofin. C. } \left(\frac{m-1}{2}\right) \frac{\Pi}{m} \pm \left(\text{Sin. C. } \left(\frac{m-1}{2}\right) \frac{\Pi}{m}\right) \sqrt{-1}, \text{ aut}$$

$$\text{Cofin. C. } \left(m - \left(\frac{m-1}{2}\right)\right) \frac{\Pi}{m} \pm \left(\text{Sin. C. } \left(m - \left(\frac{m-1}{2}\right)\right) \frac{\Pi}{m}\right) \sqrt{-1},$$

quemadmodum luculentissime patet.

291. Aliud

291. Aliud periucundum symptoma Radices istas exornans decernit vi numeri 159. eosdem esse valores, utut varie conscriptos,

$$\text{Cofin. C. } \frac{\Pi}{m} - \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1}, \text{ vel } \frac{1}{\text{Cofin. C. } \frac{\Pi}{m} + \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1}}$$

$$\text{Cofin. C. } \frac{2\Pi}{m} - \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1}, \text{ vel } \frac{1}{\text{Cofin. C. } \frac{2\Pi}{m} + \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1}}$$

$$\text{Cofin. C. } \frac{3\Pi}{m} - \left(\text{Sin. C. } \frac{3\Pi}{m} \right) \sqrt{-1}, \text{ vel } \frac{1}{\text{Cofin. C. } \frac{3\Pi}{m} + \left(\text{Sin. C. } \frac{3\Pi}{m} \right) \sqrt{-1}}$$

— — — — —

quod idem est ac vicissim

$$\text{Cofin. C. } \frac{\Pi}{m} + \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1}, \text{ aut } \frac{1}{\text{Cofin. C. } \frac{\Pi}{m} - \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1}}$$

$$\text{Cofin. C. } \frac{2\Pi}{m} + \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1}, \text{ aut } \frac{1}{\text{Cofin. C. } \frac{2\Pi}{m} - \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1}}$$

$$\text{Cofin. C. } \frac{3\Pi}{m} + \left(\text{Sin. C. } \frac{3\Pi}{m} \right) \sqrt{-1}, \text{ aut } \frac{1}{\text{Cofin. C. } \frac{3\Pi}{m} - \left(\text{Sin. C. } \frac{3\Pi}{m} \right) \sqrt{-1}}$$

— — — — —

nec negotium ullum facesset haec ipsa consimiliter praedicare de altera quoque Radicum Prospectus Linea primae aequipollente, ac e regione disposita. Diversi igitur Radicum scribendarum modi erunt

$$\begin{array}{lcl}
 \text{Cofin. C. } \frac{0}{m} \pm \left(\text{Sin. C. } \frac{0}{m} \right) \sqrt{-1} = +1 & | & \left(\text{Cofin. C. } \frac{0}{m} \mp \left(\text{Sin. C. } \frac{0}{m} \right) \sqrt{-1} \right)^{-1} = +1 \\
 \text{Cofin. C. } \frac{\Pi}{m} \pm \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1} & | & \left(\text{Cofin. C. } \frac{\Pi}{m} \mp \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1} \right)^{-1} \\
 \text{Cofin. C. } \frac{2\Pi}{m} \pm \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1} & | & \left(\text{Cofin. C. } \frac{2\Pi}{m} \mp \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1} \right)^{-1} \\
 \text{Cofin. C. } \frac{3\Pi}{m} \pm \left(\text{Sin. C. } \frac{3\Pi}{m} \right) \sqrt{-1} & | & \left(\text{Cofin. C. } \frac{3\Pi}{m} \mp \left(\text{Sin. C. } \frac{3\Pi}{m} \right) \sqrt{-1} \right)^{-1} \\
 \text{---} & | & \text{---} \\
 \text{Cofin. C. } \frac{\frac{1}{2} m \Pi}{m} \pm \left(\text{Sin. C. } \frac{\frac{1}{2} m \Pi}{m} \right) \sqrt{-1} = -1 & | & \left(\text{Cofin. C. } \frac{\frac{1}{2} m \Pi}{m} \mp \left(\text{Sin. C. } \frac{\frac{1}{2} m \Pi}{m} \right) \sqrt{-1} \right)^{-1} = -1
 \end{array}$$

$$\begin{array}{lcl}
 \text{Cofin. C. } \frac{0}{m} \pm \left(\text{Sin. C. } \frac{0}{m} \right) \sqrt{-1} = +1 & | & \left(\text{Cofin. C. } \frac{0}{m} \mp \left(\text{Sin. C. } \frac{0}{m} \right) \sqrt{-1} \right)^{-1} = +1 \\
 \text{Cofin. C. } \frac{\Pi}{m} \pm \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1} & | & \left(\text{Cofin. C. } \frac{\Pi}{m} \mp \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1} \right)^{-1} \\
 \text{Cofin. C. } \frac{2\Pi}{m} \pm \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1} & | & \left(\text{Cofin. C. } \frac{2\Pi}{m} \mp \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1} \right)^{-1} \\
 \text{Cofin. C. } \frac{3\Pi}{m} \pm \left(\text{Sin. C. } \frac{3\Pi}{m} \right) \sqrt{-1} & | & \left(\text{Cofin. C. } \frac{3\Pi}{m} \mp \left(\text{Sin. C. } \frac{3\Pi}{m} \right) \sqrt{-1} \right)^{-1} \\
 \text{---} & | & \text{---} \\
 \text{Cofin. C. } \left(\frac{m+1}{2} \right) \frac{\Pi}{m} \pm \left(\text{Sin. C. } \left(\frac{m+1}{2} \right) \frac{\Pi}{m} \right) \sqrt{-1} & | & \left(\text{Cofin. C. } \left(\frac{m+1}{2} \right) \frac{\Pi}{m} \mp \left(\text{Sin. C. } \left(\frac{m+1}{2} \right) \frac{\Pi}{m} \right) \sqrt{-1} \right)^{-1}
 \end{array}$$

m Pari

$$\begin{array}{l}
 \text{Cofin. C. } \left(\frac{m-m}{m}\right) \Pi \pm \left(\text{Sin. C. } \left(\frac{m-m}{m}\right) \Pi\right) \sqrt{-1} = +1 \quad \left| \quad \left(\text{Cofin. C. } \left(\frac{m-m}{m}\right) \Pi \mp \left(\text{Sin. C. } \left(\frac{m-m}{m}\right) \Pi\right) \sqrt{-1}\right)^{-1} = +1 \right. \\
 \text{Cofin. C. } \left(\frac{m-\frac{1}{2}m}{m}\right) \Pi \pm \left(\text{Sin. C. } \left(\frac{m-\frac{1}{2}m}{m}\right) \Pi\right) \sqrt{-1} = -1 \quad \left| \quad \left(\text{Cofin. C. } \left(\frac{m-\frac{1}{2}m}{m}\right) \Pi \mp \left(\text{Sin. C. } \left(\frac{m-\frac{1}{2}m}{m}\right) \Pi\right) \sqrt{-1}\right)^{-1} = -1 \right. \\
 \text{Cofin. C. } \left(\frac{m-3}{m}\right) \Pi \pm \left(\text{Sin. C. } \left(\frac{m-3}{m}\right) \Pi\right) \sqrt{-1} \quad \left| \quad \left(\text{Cofin. C. } \left(\frac{m-3}{m}\right) \Pi \mp \left(\text{Sin. C. } \left(\frac{m-3}{m}\right) \Pi\right) \sqrt{-1}\right)^{-1} \right. \\
 \text{Cofin. C. } \left(\frac{m-2}{m}\right) \Pi \pm \left(\text{Sin. C. } \left(\frac{m-2}{m}\right) \Pi\right) \sqrt{-1} \quad \left| \quad \left(\text{Cofin. C. } \left(\frac{m-2}{m}\right) \Pi \mp \left(\text{Sin. C. } \left(\frac{m-2}{m}\right) \Pi\right) \sqrt{-1}\right)^{-1} \right. \\
 \text{Cofin. C. } \left(\frac{m-1}{m}\right) \Pi \pm \left(\text{Sin. C. } \left(\frac{m-1}{m}\right) \Pi\right) \sqrt{-1} \quad \left| \quad \left(\text{Cofin. C. } \left(\frac{m-1}{m}\right) \Pi \mp \left(\text{Sin. C. } \left(\frac{m-1}{m}\right) \Pi\right) \sqrt{-1}\right)^{-1} \right.
 \end{array}$$

m Impari

$$\begin{array}{l}
 \text{Cofin. C. } (m-m) \frac{\Pi}{m} \pm \left(\text{Sin. C. } (m-m) \frac{\Pi}{m}\right) \sqrt{-1} = +1 \quad \left| \quad \left(\text{Cofin. C. } (m-m) \frac{\Pi}{m} \mp \left(\text{Sin. C. } (m-m) \frac{\Pi}{m}\right) \sqrt{-1}\right)^{-1} = +1 \right. \\
 \text{Cofin. C. } \left(m - \left(\frac{m-1}{2}\right)\right) \frac{\Pi}{m} \pm \left(\text{Sin. C. } \left(m - \left(\frac{m-1}{2}\right)\right) \frac{\Pi}{m}\right) \sqrt{-1} \quad \left| \quad \left(\text{Cofin. C. } \left(m - \left(\frac{m-1}{2}\right)\right) \frac{\Pi}{m} \mp \left(\text{Sin. C. } \left(m - \left(\frac{m-1}{2}\right)\right) \frac{\Pi}{m}\right) \sqrt{-1}\right)^{-1} \right. \\
 \text{Cofin. C. } (m-3) \frac{\Pi}{m} \pm \left(\text{Sin. C. } (m-3) \frac{\Pi}{m}\right) \sqrt{-1} \quad \left| \quad \left(\text{Cofin. C. } (m-3) \frac{\Pi}{m} \mp \left(\text{Sin. C. } (m-3) \frac{\Pi}{m}\right) \sqrt{-1}\right)^{-1} \right. \\
 \text{Cofin. C. } (m-2) \frac{\Pi}{m} \pm \left(\text{Sin. C. } (m-2) \frac{\Pi}{m}\right) \sqrt{-1} \quad \left| \quad \left(\text{Cofin. C. } (m-2) \frac{\Pi}{m} \mp \left(\text{Sin. C. } (m-2) \frac{\Pi}{m}\right) \sqrt{-1}\right)^{-1} \right. \\
 \text{Cofin. C. } (m-1) \frac{\Pi}{m} \pm \left(\text{Sin. C. } (m-1) \frac{\Pi}{m}\right) \sqrt{-1} \quad \left| \quad \left(\text{Cofin. C. } (m-1) \frac{\Pi}{m} \mp \left(\text{Sin. C. } (m-1) \frac{\Pi}{m}\right) \sqrt{-1}\right)^{-1} \right.
 \end{array}$$

291. Transitu facto ad hypotesin Exponentis m integri, sed negativi, petantur

Radices $1^{\frac{1}{-m}}$ vel $1^{-\frac{1}{m}}$, five Radices Aequationis $x^{-m} - 1 = 0$.

In aperto est novam hanc Aequationem forma tantum differre ab ea superius enucleata, propterea quod $\frac{1}{x^m} - 1 = 0$ ope multiplicationis in x^m .

$1 = 0$ non gravate vertatur, et ideo iisdem gaudeat admissim Radicibus praenotatis. Hoc etiam concluditur a Radicum Formulis numero 289. fusius expositis; siquidem ibi permutato, uti exigit praefens Aequatio, Indice $+m$ in $-m$ prima Radicum inventarum Linea abit in alteri aequipollentem, seu, quoad singulos valores perfecte identicam, ac vicissim.

Comprobat insuper veritatem identitatis adsertae Aequalitas notissima $1^{\frac{1}{-m}} = \frac{1}{1^{\frac{1}{m}}}$, ob quam fit, ut reciproca cuiusvis Radicis inventae, dum erat

m positivus, Radicem praebeat respondentem hypothese Exponentis m negativi: illas autem reciprocas, si numeri 290. et 291. simul iuncti consulantur,

universas Radices, quae referuntur ad $1^{\frac{1}{m}}$, inverso ordine reproducere demonstravi. Accedit palmaria quoque ad id confirmandum investigatarum Radicum genesis, praesidio adquisita sat familiaris exaltationis Binomii Trigonometrici $(\text{Cosin. C. } v + (\text{Sin. C. } v) \sqrt{-1})^{-m} = \text{Cosin. C. } -mv + (\text{Sin. C. } -mv) \sqrt{-1} = 1$, a qua statim profluit $v = \pm \frac{n\pi}{m}$

non secus ac in prioribus fuit recensitum. Supervacaneum est igitur in hisce minutiori oratione immorari Radicibus.

293. Ultimus veluti Limes Unitatis Radicum, de quibus sermo instituitur extat in casu $1^{\frac{\pm 1}{\infty}}$, vel, ex iam dictis, tantummodo $1^{\frac{1}{\infty}}$; hic-

que

que nobis occurrit dum Index m Rationalis, atque Integer in Infinitum excrescat, videlicet eo perveniat, quo Numeri non amplius assignabiles, Integri, Fractiones, Rationales, Irrationales, Transcendentes omnigeni confunduntur.

Substituto in superioribus Formulis ∞^{to} vice Indicis indeterminati m faci-

le obtinemus $1^{\frac{1}{\infty}} = \text{Cofin. C.} \pm \frac{n\pi}{\infty} + \left(\text{Sin. C.} \pm \frac{n\pi}{\infty} \right) \sqrt{-1} = 1$ Arith-

metice, si n supponatur etiam Numerus Infinitus, vel ipsemet Coefficientens n

confecta hypothese adversa in Nihilum abeat. Et re quidem vera $1^{\frac{1}{\infty}} =$

$1^0 = 1$ iuxta magis obvia Analyseos praecepta, quae ab Exponentialium

Formulis nullo exantlato labore convalidantur. Nec perturbata extemplo

videatur ab unico hoc valore Arithmetico $+1$ Continuitatis Analyticae Lex,

perinde ac si nullo gradu facto, sed, quod absit!, saltu quodam Radices illae

numero m utut semper semperque excrescente in Aequationibus indefinitis

$x^m - 1 = 0$ unicam demum addicant Aequationi Seriem omnium possibilium

claudenti $x^\infty - 1 = 0$; idque nefas absurdumque futurum esse non inficias

ibo. Unitas enim valoris multiplicitati Radicum non refragatur, ita, ut

farta testis foret equidem Continuitas, quae ab $1^{\frac{1}{m}}$ Functione m -formi iubet

transitum fieri ad $1^{\frac{1}{\infty}}$ Functionem ∞ -formem, tametsi unus quoque ma-
neret valor $+1$, ad sententiam subsequentis evolutionis.

$$\text{Cofin. C.} \pm \frac{0.\Pi}{\infty} + \left(\text{Sin. C.} \pm \frac{0.\Pi}{\infty} \right) \sqrt{-1} = 1$$

$$\text{Cofin. C.} \pm \frac{1.\Pi}{\infty} + \left(\text{Sin. C.} \pm \frac{1.\Pi}{\infty} \right) \sqrt{-1} = 1$$

$$\text{Cofin. C.} \pm \frac{2.\Pi}{\infty} + \left(\text{Sin. C.} \pm \frac{2.\Pi}{\infty} \right) \sqrt{-1} = 1$$

Radicum

Numerus ∞ .

$$\text{Cofin. C.} \pm \frac{3.\Pi}{\infty} + \left(\text{Sin. C.} \pm \frac{3.\Pi}{\infty} \right) \sqrt{-1} = 1$$

$$\text{Cofin. C.} \pm \frac{\lambda.\Pi}{\infty} + \left(\text{Sin. C.} \pm \frac{\lambda.\Pi}{\infty} \right) \sqrt{-1} = 1$$

$$\text{Cofin. C.} \pm \frac{(\lambda+1)\Pi}{\infty} + \left(\text{Sin. C.} \pm \frac{(\lambda+1)\Pi}{\infty} \right) \sqrt{-1} = 1$$

$$\text{Cofin. C.} \pm \frac{(\infty-1)\Pi}{\infty} + \left(\text{Sin. C.} \pm \frac{(\infty-1)\Pi}{\infty} \right) \sqrt{-1} = 1$$

Arithmeti-
ce Loquen-
do, seu Fi-
nitis tan-
tum Nu-
meris sup-
putatis.

Verum enimvero Infinitas numero nec singulas invicem aequales Radices $1^{\frac{1}{\infty}}$, sed Infinitesima primi Ordinis Magnitudine discrepantes, quod propter Indicem ∞ accidere iure potest, re melius perpenſa evincit Canon Hud-denii Aequationi admotus $x^{\infty} - 1 = 0$. Profecto $\tau \delta \infty . x^{\infty-1}$, vel ſi mavis, facultate ab Infinito conceſſa, $\infty . x^{\infty}$ non eſt, iuxta lata praecepta, diviſor Binomii $x^{\infty} - 1$, nec Polynomium exiſtit, quod ſit exactus commu-nis Diviſor tam $x^{\infty} - 1$, quam $\infty . x^{\infty}$, ut omnibus patet. Conſectarium hinc elicietur, quod nimirum identicae non ſint, quinimo toto coelo di-ſtent, Aequationes Ordinis Infiniti $x^{\infty} - 1$, ſeu $1 - x^{\infty} = 0$, atque $(x - 1)^{\infty}$, ſive $(1 - x)^{\infty} = 0$. Nec mirum, quia Aequatio poſtrema

$$\text{nil aliud eſt quam } 1 - \frac{\infty}{1} . x^1 + \frac{\infty^2 . x^2}{1.2} - \frac{\infty^3 . x^3}{1.2.3} + \frac{\infty^4 . x^4}{1.2.3.4} -$$

$$\frac{\infty^5 . x^5}{1.2.3.4.5}$$

$$\frac{\infty^5 \cdot x^5}{1.2.3.4.5} + \dots \frac{\infty^{2n} \cdot x^{2n}}{1.2.3 \dots 2n} - \frac{\infty^{2n+1} \cdot x^{2n+1}}{1.2.3 \dots 2n.2n+1} \dots$$

+ — $1 \cdot x^\infty$, quum Coefficiens ultimi termini omnium peraequare debeat Radicum Productum, et ideo satis adparet summopere diversa Aequalitas ab $1 - x^\infty = 0$, quae primum dumtaxat, ac postremum illius continet terminum, nullamque teneat rationem ceterorum — $\frac{\infty}{1} x$

$$+ \frac{\infty^2 \cdot x^2}{1.2} - \frac{\infty^3 \cdot x^3}{1.2.3} + \frac{\infty^4 \cdot x^4}{1.2.3.4} - \frac{\infty^5 \cdot x^5}{1.2.3.4.5} + \dots$$

$$\frac{\infty^{2n} \cdot x^{2n}}{1.2.3 \dots 2n} - \frac{\infty^{2n+1} \cdot x^{2n+1}}{1.2.3 \dots 2n+1} \dots + \&c., \text{ quorum Summa}$$

non evanescit, nec evanescere potest nisi altera efformetur a superius traditis remotissima Aequatio

$$1 - \frac{\infty \cdot x}{1.2} + \frac{\infty^2 x^2}{1.2.3} - \frac{\infty^3 x^3}{1.2.3.4} + \frac{\infty^4 x^4}{1.2.3.4.5} \dots$$

$$\frac{\infty^{2n-1} \cdot x^{2n-1}}{1.2.3 \dots 2n} + \frac{\infty^{2n} \cdot x^{2n}}{1.2.3 \dots 2n+1} \dots - \dots \&c. = 0, \text{ nullatenus}$$

praesenti animadversioni opportuna. Communis autem Algebra docet inaniter quaeri in Binomiali Aequatione quacumque Radices aequales, quum ex intima sui natura illas respuat. Nec ab Huddenio datum experimentum ullius vitii arguendum, quia dum eo uti velimus, Radicum aequalium praesumptarum illico detegat absurditatem.

294. Alia tamen verior methodus adest Radices $1^{\frac{1}{\infty}}$ condendi, usui futura in re perficienda Logometrica, ac Trigonometrica sublimiore. Est

$$\text{enim } 1^{\frac{1}{\infty}} = \text{Cofin. C.} \pm \frac{n\Pi}{\infty} + \left(\text{Sin. C.} \pm \frac{n\Pi}{\infty} \right) \sqrt{-1} = 1 - \frac{n^2 \Pi^2}{1.2. \infty^2} \pm$$

$$\frac{n\Pi}{\infty} \cdot \sqrt{-1} = 1 \pm \frac{n\Pi}{\infty} \cdot \sqrt{-1}, \text{ etiam in Magnitudinum Limite evanescens, sive nascentium. Eccine novus Radicum ordo.}$$

$$1 \pm \frac{0.\Pi}{\infty} \cdot \sqrt{-1} = 1$$

Rigidae Analytico $1 \pm \frac{1.\Pi}{\infty} \cdot \sqrt{-1} = 1$ Arithmeticae, aut Pseudo-Analyticae
sensu Radices.

$$1 \pm \frac{2.\Pi}{\infty} \cdot \sqrt{-1} = 1 \text{ Et sic deinceps in Infinitum.}$$

$$1 \pm \frac{3.\Pi}{\infty} \cdot \sqrt{-1} = 1$$

$$1 \pm \frac{n.\Pi}{\infty} \cdot \sqrt{-1} = 1$$

Aequatio igitur toties in praecedentibus usurpata $\infty \left(\left(1^{\frac{1}{\infty}} \cdot \frac{1}{1+z} \right)^{\frac{1}{\infty}} - 1 \right)$
 $= L(1+z)$ plenissimam resolutionem admittet ab Unitatis Radicibus
derivatam

$$\text{Num.} = 1 + \frac{L^1}{1} + \frac{L^2}{1.2} + \frac{L^3}{1.2.3} + \frac{L^4}{1.2.3.4} + \frac{L^5}{1.2.3.4.5} +$$

$$\text{Log. Hyp.} = \frac{(N-1)^1}{1} - \frac{(N-1)^2}{2} + \frac{(N-1)^3}{3} - \frac{(N-1)^4}{4} + \frac{(N-1)^5}{5} -$$

$$\log. N = M. \text{Log. Hyp.} = M \left(\frac{(N-1)^1}{1} - \frac{(N-1)^2}{2} + \frac{(N-1)^3}{3} - \frac{(N-1)^4}{4} + \frac{(N-1)^5}{5} - \right)$$

Tametsi hanc Logarithmi cuiusque Infinitiformitatem in Capite V. deduxerimus a canonibus ipsis Binomii Newtoniani absque ulla Indicis, aut Exponentis Infinite magni necessitate, variaque conspersa fuerint huius fundamentalis affectionis Logometricae semina etiam in Capitibus II., ac III.; nihilominus novam comprobationem vixdum nactus sedem Theoremati magis evoluto digniorem hic loci assignare decrevi.

295. Utillima haec quoque sunt ad Problematis Alembertiani plenior
rem

derivatam. Inito namque Calculo fluet $\infty \left(\left(1 \pm \frac{n\pi}{\infty} \cdot \sqrt{-1} \right) (1+z)^{\frac{1}{\infty}} - 1 \right)$

$$\text{nimirum } \left(\infty \pm n\pi \sqrt{-1} \right) \left(1 + \frac{z^1}{\infty} - \frac{z^2}{2.\infty} + \frac{z^3}{3.\infty} - \frac{z^4}{4.\infty} + \frac{z^5}{5.\infty} - \frac{z^6}{6.\infty} + \frac{z^7}{7.\infty} - \&c: \right) - \infty, \text{ vel } + \frac{z^1}{1} - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \&c: \pm n\pi \sqrt{-1} = L(1+z), \text{ qui}$$

Logarithmus Hyperbolicus erit idcirco Functio reapse Infinitiformis; eademque dicas de alio quovis Logarithmorum Systemate, quum Modulus, vel Coefficiens tantummodo diversitatem affingat levissimam. Quod num°. 184. iam polliciti fuimus nunc adimpletum constabit, fidamque tot methodorum Algebrae copiam quisquis non iniuria mirabitur. Interea, ne Capituli IX., a quo dubitationis vitiive omnis notam ei fortasse inustam auferre fatagam, optata desit lux, ad instar Lemmatis geminas Series conscribo omnium Magnitudinum pertractatarum veluti cardines.

Num. =

$$\frac{L^6}{1.2.3.4.5.6} + \frac{L^7}{1.2.3.4.5.6.7} + \&c: \text{ Functio Uniformis,}$$

$$\frac{(N-1)^6}{6} + \frac{(N-1)^7}{7} - \&c: \pm n\pi \cdot \sqrt{-1} \text{ Functio Infinitiformis,}$$

$$- \frac{(N-1)^6}{6} + \frac{(N-1)^7}{7} - \&c:) \pm Mn\pi \cdot \sqrt{-1} \text{ Functio pariter Infinitiformis.}$$

Tametsi

rem illustrationem, cuius praesertim in num°. 230. prima Theorices tulimus elementa. Iisdem porro retentis denominationibus, et Siglis, quae inibi occurrunt, erat illic Arcus $x = \frac{\infty}{\sqrt{-1}} \left((v\sqrt{-1} + \sqrt{1-v^2})^{\frac{1}{\infty}} \cdot 1^{\frac{1}{\infty}} - 1 \right).$

Suffecto autem pro $(v\sqrt{-1} + \sqrt{1-v^2})^{\frac{1}{\infty}}$, vel potius pro (Cofin. C. x + (Sin.

$(\text{Sin. C. } x) \sqrt{-1}^{\frac{1}{\infty}}$ eius valore $\text{Cofin. C. } \frac{x}{\infty} \rightarrow (\text{Sin. C. } \frac{x}{\infty}) \sqrt{-1}$ scilicet $1 + \frac{x\sqrt{-1}}{\infty}$, habebitur $x = \frac{\infty}{\sqrt{-1}} \left(\left(1 + \frac{x\sqrt{-1}}{\infty} \right) \left(1 \pm \frac{n\pi\sqrt{-1}}{\infty} \right) - 1 \right)$

$= x \pm n\pi$, iuxta Trigonometricae Facultatis omnimodam extensionem. Pari gradu incedentes, dum Tangentium z rite interpretanda sit Formula,

obtinemus, praemissa Aequatione $C^{2x\sqrt{-1}} = \frac{1 + z\sqrt{-1}}{1 - z\sqrt{-1}}$, ideoque

$\left(\frac{1 + z\sqrt{-1}}{1 - z\sqrt{-1}} \right)^{\frac{1}{\infty}} = C^{\frac{2x\sqrt{-1}}{\infty}} = \text{Cofin. C. } \frac{2x}{\infty} \rightarrow (\text{Sin. C. } \frac{2x}{\infty}) \sqrt{-1}$
 $= 1 + \frac{2x\sqrt{-1}}{\infty}$, transformationem Aequalitatis praecitato num°. appositae,

nempe $x = \frac{\infty}{2\sqrt{-1}} \left(\left(1 \pm \frac{n\pi\sqrt{-1}}{\infty} \right) \left(\frac{1 + z\sqrt{-1}}{1 - z\sqrt{-1}} \right)^{\frac{1}{\infty}} - 1 \right)$ in alteram aequipollen-

tem $\frac{\infty}{2\sqrt{-1}} \left(\left(1 \pm \frac{n\pi\sqrt{-1}}{\infty} \right) \left(1 + \frac{2x\sqrt{-1}}{\infty} \right) - 1 \right) = x$, vel multiplica-

tione peracta in $x = x \pm \frac{n\pi}{2} = x \pm nP$, prouti Elementa Circuli do-

cent. Ad hoc igitur, ut Circulari Trigonometriae accessio fiat absolutissima, quae Dalembertio erat in votis, subsequentes Functiones separatim evolvere huic palaestrae non inutile censeo.

$$\text{Sin. C.} = \frac{A^1}{1} - \frac{A^3}{1.2.3} + \frac{A^5}{1.2.3.4.5} - \frac{A^7}{1.2.3.4.5.6.7} + \frac{A^9}{1.2.3.4.5.6.7.8.9} - \frac{A^{11}}{1.2.3.4.5.6.7.8.9.10.11} + \&c.; \text{ Functio Uniformis;}$$

$\text{Arc. C.} = x \pm n\pi$ Functio Infinitiformis; quod Symbolum x minimum designat Arcum Sin. C;

$$\text{Tang. C.} = \frac{A^1}{1} + \frac{A^3}{1^2.3} + \frac{2A^5}{1^2.3.5} + \frac{17A^7}{1^2.3^2.5.7} + \frac{62A^9}{1^2.3^2.5.7.9} + \frac{1382A^{11}}{1^2.3^2.5^2.7.9.11} + \&c; \text{ Functio Uniformis;}$$

$\text{Arc. C.} = x \pm nP$ Functio Infinitiformis; ubi pariter x minimum denotat Arcum Tang. C.

Poterunt etiam, si libeat, Circulares Arcus praedicti minori compendio, sed clarius fortasse, significari.

Arc. C.

$$\text{Arc. C.} = S^1 + \frac{1.S^3}{2.3} + \frac{1.3.S^5}{2.4.5} + \frac{1.3.5.S^7}{2.4.6.7} + \frac{1.3.5.7.S^9}{2.4.6.8.9} + \frac{1.3.5.7.9.S^{11}}{2.4.6.8.10.11} + \&c: \pm n\pi \text{ Functio } \infty \text{formis};$$

$$\text{Arc. C.} = \frac{T^1}{1} - \frac{T^3}{3} + \frac{T^5}{5} - \frac{T^7}{7} + \frac{T^9}{9} - \frac{T^{11}}{11} + \frac{T^{13}}{13} - \frac{T^{15}}{15} + \&c: \pm nP \text{ Functio prouti superius } \infty \text{formis},$$

eademque manarent dum ceterae Formulae Trigonometricae hac methodo duce ad examen revocarentur.

296. Veruntamen aliquis obiiciet Sinui-recto cuilibet non tantum referri innumeros Arcus $x \pm n\pi$, quin etiam $\lambda P - x$, vel universalius $\pm \lambda P - x$ dum λ Numerum imparem peraequet, quos valores superius traditae Formulae silentio premunt. Illius autem resolutionis veritas plenissima in eo fita est, quod Arcus ex natura Aequationis propositae suppeditare debeat Sinui, et Cosinui dato correspondentes, idque aliter certe non poterat nisi Functione prolata $x \pm n\pi$, cum alteri $\pm \lambda P - x$ idem Sinus, at non Cosinus idem additus sit. Qui nauci habito Cosinus Signo in eadem Functione $x \pm n\pi$ alteram quoque Infinitiformem $\pm \lambda P - x$ perlegere velit facile fiet voti compos rite sub initium feligendo minimum Arcuum, de quo sit quaestio, sinui-recto dato, sed Cosinui magnitudine, non qualitate, determinato competens, illeque, quum geminum habeat valorem, nempe x , et $P - x = v$, provenient tum $x \pm n\pi$, tum $v \pm n\pi = P - x \pm 2nP$, aut $(\pm 2n + 1)P - x$, vel denique $\pm \lambda P - x$; adeo ut omnia pendeant a prima minimi Arcus in casu quolibet assignato substitutione. Similia similibus huius ratiocinationis ope adplicentur. Sic in Hyperbolicæ Trigonometriæ Sinibus, Tangentibusve, aliisque Lineis Circulo analogis essent toto iure Analytico deducenda ab Aequationibus $x = \infty$

$$\left(\left(1^{\frac{1}{\infty}} \cdot (\text{Cosin. H. } x + \text{Sin. H. } x)^{\frac{1}{\infty}} \right) - 1 \right) = \infty \left(\left(1^{\frac{1}{\infty}} (\text{Cosin. H. } \frac{x}{\infty} + \text{Sin. H. } \frac{x}{\infty}) \right) - 1 \right) = \infty \left(\left(1 \pm \frac{n\pi\sqrt{-1}}{\infty} \right) \left(1 + \frac{x}{\infty} \right) - 1 \right) = x \pm n\pi\sqrt{-1},$$

$$\text{H. } \frac{x}{\infty} \left(1 \pm \frac{n\pi\sqrt{-1}}{\infty} \right) - 1 = \infty \left(\left(1 \pm \frac{n\pi\sqrt{-1}}{\infty} \right) \left(1 + \frac{x}{\infty} \right) - 1 \right) = x \pm n\pi\sqrt{-1},$$

$$\text{necnon } 2x = \infty \left((\text{Cosin. H. } 2x + \text{Sin. H. } 2x)^{\frac{1}{\infty}} \cdot 1^{\frac{1}{\infty}} - 1 \right) = \infty \left(\left(1 + \frac{2x}{\infty} \right) \left(1 \pm \frac{n\pi\sqrt{-1}}{\infty} \right) - 1 \right), \text{ ideoque } x = x \pm nP\sqrt{-1}, \text{ quae simul tribuunt,}$$

$$\left(1 \pm \frac{n\pi\sqrt{-1}}{\infty} \right) - 1, \text{ ideoque } x = x \pm nP\sqrt{-1}, \text{ quae simul tribuunt,}$$

ponendo $B = 2 \text{ Sect. H}$, aut Logarithmo Arcubus Circuli analogo

I i i i

Sin. H. =

$$\text{Sin. H.} = \frac{B^1}{1} + \frac{B^3}{1.2.3} + \frac{B^5}{1.2.3.4.5} + \frac{B^7}{1.2.3.4.5.6.7} + \frac{B^9}{1.2.3.4.5.6.7.8.9} + \frac{B^{11}}{1.2.3.4.5.6.7.8.9.10.11} + \&c: \text{ Functio Uniformis,}$$

$$B = S^1 - \frac{1.S^3}{2.3} + \frac{1.3.S^5}{2.4.5} - \frac{1.3.5.S^7}{2.4.6.7} + \frac{1.3.5.7.S^9}{2.4.6.8.9} - \frac{1.3.5.7.9.S^{11}}{2.4.6.8.10.11} + \&c: \pm n\prod \sqrt{-1} \text{ Functio } \infty \text{formis,}$$

$$\text{Tan. H.} = \frac{B^1}{1} - \frac{B^3}{1^2.3} + \frac{2B^5}{1^2.3.5} - \frac{17B^7}{1^2.3^2.5.7} + \frac{62B^9}{1^2.3^2.5.7.9} - \frac{1382B^{11}}{1^2.3^2.5^2.7.9.11} + \&c: \text{ Functio Uniformis,}$$

$$B = \frac{T^1}{1} + \frac{T^3}{3} + \frac{T^5}{5} + \frac{T^7}{7} + \frac{T^9}{9} + \frac{T^{11}}{11} + \frac{T^{13}}{13} + \frac{T^{15}}{15} + \&c: \pm nP \sqrt{-1} \text{ Functio similiter Infinitiformis,}$$

Eodem praeterea pacto methodum renovantes, qua supra usi sumus, si vice initialis Logarithmi analogi B alter identico Sinu-recto, sed Cofinu negativo affectus subrogetur $P \sqrt{-1} - B$ obtinebimus

$$\text{Log. Anal.} = \frac{S^1}{1} - \frac{1.S^3}{2.3} + \frac{1.3.S^5}{2.4.5} - \frac{1.3.5.S^7}{2.4.6.7} + \&c: \pm \lambda P \sqrt{-1} - B \text{ Functionem etiam Infinitiformem.}$$

Nec dubium manebit circa substitutionem praefatam $P \sqrt{-1} - B$, ex eo quod picto Hyperbolae Aequilaterae Schemate, et ratiocinio ad ipsum translato pateant omnia evidenter, novaque affulgeat lux ab Aequatione

$$\begin{aligned} \frac{P \sqrt{-1} - B}{C} &= \frac{P \sqrt{-1}}{C} - \frac{B}{C} = -1 (\text{Cofin. Hyp.} - B + \text{Sin. Hyp.} - B) \\ &= -1 (\text{Cofin. Hyp. B} - \text{Sin. Hyp. B}) = -\text{Cofin. Hyp. B} + \text{Sin. Hyp. B.}, \end{aligned}$$

nam iste valor postremus satis ostendit $P \sqrt{-1} - B$ esse analogum Logarithmum, cui idem Sinus respectu ad B, Cofinus autem contingat, uti nuperrime visum in Circulo, eiusdem magnitudinis negativae.

297. Accedat nunc resolvenda Binomialis Aequatio $x^{\frac{r}{v}} - 1 = 0$,

scilicet quaerantur Radices $1^{\frac{v}{r}}$ Indice aut Exponente quolibet fracto distinctae. Nititur equidem iisdem principiis investigatio exoptata: namque,

facta hypothesi $x = \text{Cofin. C. } z + (\text{Sin. C. } z) \sqrt{-1}$, emerget $x^{\frac{r}{v}} = \text{Cofin.}$

$$C. \frac{rz}{v} + (\text{Sin. C. } \frac{rz}{v}) \sqrt{-1} = 1 \text{ dummodo sit } \frac{rz}{v} = \pm n\pi, \text{ nempe } z$$

$= \pm \frac{v\sqrt{-1}}{r}$, adeo, ut valorem hunc ipsum, substituto vice τ in praesenti

Indice $\frac{r}{v}$, a posteriori etiam eruere licuisset. Igitur petita Radices $\frac{v}{r}$

erunt $\text{Cofin. C.} \pm \frac{v\sqrt{-1}}{r} + \left(\text{Sin. C.} \pm \frac{v\sqrt{-1}}{r} \right) \sqrt{-1}$, seu

$$\text{Cofin. C.} \pm 0 \cdot \frac{v\sqrt{-1}}{r} + \left(\text{Sin. C.} \pm 0 \cdot \frac{v\sqrt{-1}}{r} \right) \sqrt{-1} = 1 \quad \frac{1}{\text{Cofin. C.} \pm 0 \cdot \frac{v\sqrt{-1}}{r} - \left(\text{Sin. C.} \pm 0 \cdot \frac{v\sqrt{-1}}{r} \right) \sqrt{-1}} = 1$$

$$\text{Cofin. C.} \pm 1 \cdot \frac{v\sqrt{-1}}{r} + \left(\text{Sin. C.} \pm 1 \cdot \frac{v\sqrt{-1}}{r} \right) \sqrt{-1} \quad \frac{1}{\text{Cofin. C.} \pm 1 \cdot \frac{v\sqrt{-1}}{r} - \left(\text{Sin. C.} \pm 1 \cdot \frac{v\sqrt{-1}}{r} \right) \sqrt{-1}}$$

$$\text{Cofin. C.} \pm 2 \cdot \frac{v\sqrt{-1}}{r} + \left(\text{Sin. C.} \pm 2 \cdot \frac{v\sqrt{-1}}{r} \right) \sqrt{-1} \quad \text{aut} \quad \frac{1}{\text{Cofin. C.} \pm 2 \cdot \frac{v\sqrt{-1}}{r} - \left(\text{Sin. C.} \pm 2 \cdot \frac{v\sqrt{-1}}{r} \right) \sqrt{-1}}$$

$$\text{Cofin. C.} \pm 3 \cdot \frac{v\sqrt{-1}}{r} + \left(\text{Sin. C.} \pm 3 \cdot \frac{v\sqrt{-1}}{r} \right) \sqrt{-1} \quad \frac{1}{\text{Cofin. C.} \pm 3 \cdot \frac{v\sqrt{-1}}{r} - \left(\text{Sin. C.} \pm 3 \cdot \frac{v\sqrt{-1}}{r} \right) \sqrt{-1}}$$

$$\text{Cofin. C.} \pm (r-1) \frac{v\sqrt{-1}}{r} + \left(\text{Sin. C.} \pm (r-1) \frac{v\sqrt{-1}}{r} \right) \sqrt{-1} \quad \frac{1}{\text{Cofin. C.} \pm (r-1) \frac{v\sqrt{-1}}{r} - \left(\text{Sin. C.} \pm (r-1) \frac{v\sqrt{-1}}{r} \right) \sqrt{-1}}$$

Abrumpitur necessario Series inveniendarum Radicum in postrema adnota-
ta, propterea quod neminem fugiat fore subsequentem ordine numerico
 $\text{Cofin. C.} \pm v\sqrt{-1} + \left(\text{Sin. C.} \pm v\sqrt{-1} \right) \sqrt{-1} = 1$, ac denuo $\text{Cofin. C.} \pm (r+1)$

$$\frac{v\sqrt{-1}}{r} + \left(\text{Sin. C.} \pm (r+1) \frac{v\sqrt{-1}}{r} \right) \sqrt{-1} = \text{Cofin. C.} \pm 1 \frac{v\sqrt{-1}}{r} + \left(\text{Sin. C.} \right.$$

$$\left. \pm 1 \cdot \frac{v\sqrt{-1}}{r} \right) \sqrt{-1}, \text{ nimirum alteram eorumdem valorum Periodum incipere,}$$

eaque

eaque vixdum completa alias sine Limite aequipollentes, quaestioque penitus inutiles laboriri. Interea toti erimus in stabiliendo Theoremate „ tot

esse diversas Radices $1^{\frac{v}{r}}$, quot Indicis aut Exponentis Denominator r Unitates contineat, statim ac Fractio $\frac{v}{r}$ sit minimis terminis comparata „ Observandum denique est, quod, supposito Radicis Indice, idest r , Num-

ro pari Radix adsit intermedia Cofin. C. $\pm \frac{\frac{1}{2} r \cdot v \Pi}{r} \rightarrow$

$$\left(\text{Sin. C} \pm \frac{\frac{1}{2} r \cdot v \Pi}{r} \right) \sqrt{-1} = \text{Cofin. C.} \pm vP \rightarrow (\text{Sin. C.} \pm vP) \sqrt{-1} =$$

-1 , quia v impar Numerus erit, et idcirco geminae tunc Reales Radices evadent ± 1 , atque numero $r - 2$ Imaginariae, cum ex adverso r

impare manente habeantur $\rightarrow 1$ unica Radix Realis $1^{\frac{v}{r}}$, ac reliquae omnes numero $r - 1$ Imaginariae. Universis itaque casibus hactenus evolutis aut

Indicis m , aut $\frac{v}{r}$, positivi seu negativi, repertae semper fuere Imaginariae Unitatis Radices numero pares, ac ideo consonae universalibus Analyseos Cartesiana praeeptis.

298. Quae sint iam dictis communia omitto libenter; sed nefas existimo silentio praeterire, quod nusquam abruptitur Radicum Numerus tum

cum Index Unitati appositus fuerit Irrationalis. Exemplum praebeat $1^{\frac{1}{r} \sqrt{p}}$

ab Aequatione Interfendente $x^{\frac{r}{\sqrt{p}}} - 1 = 0$ deductus. Sciunt omnes $\sqrt[p]{p}$ nullo modo Rationalibus Numeris exprimi posse, quamvis liceat Rationalium

tionalium ope infinite crescentium ad Surdos Numeros sine Limite accedere (*). Quum ergo vi methodi eiusdem hucusque usitatae, et in Capite I., quod praecipuum illius pretium est, Potestatum Indicibus Irrationabilibus vel etiam Transcendentibus addictae, Formula universalis Cofin. C.

$$\frac{\pm n\Pi}{r\sqrt{p}} + \left(\text{Sin. C. } \frac{\pm n\Pi}{r\sqrt{p}}\right)\sqrt{-1} \text{ quaesitas Radices complectatur, hae in}$$

subsequentes singillatim abibunt;

$$\text{Cofin. C. } \frac{\pm 0.\Pi}{r\sqrt{p}} + \left(\text{Sin. C. } \frac{\pm 0.\Pi}{r\sqrt{p}}\right)\sqrt{-1} = 1 \qquad \frac{\pm 0.\Pi}{r\sqrt{p}} - \left(\text{Sin. C. } \frac{\pm 0.\Pi}{r\sqrt{p}}\right)\sqrt{-1} = 1$$

$$\text{Cofin. C. } \frac{\pm 1.\Pi}{r\sqrt{p}} + \left(\text{Sin. C. } \frac{\pm 1.\Pi}{r\sqrt{p}}\right)\sqrt{-1} \qquad \frac{\pm 1.\Pi}{r\sqrt{p}} - \left(\text{Sin. C. } \frac{\pm 1.\Pi}{r\sqrt{p}}\right)\sqrt{-1}$$

$$\text{Cofin. C. } \frac{\pm 2.\Pi}{r\sqrt{p}} + \left(\text{Sin. C. } \frac{\pm 2.\Pi}{r\sqrt{p}}\right)\sqrt{-1} \quad \text{vel} \quad \frac{\pm 2.\Pi}{r\sqrt{p}} - \left(\text{Sin. C. } \frac{\pm 2.\Pi}{r\sqrt{p}}\right)\sqrt{-1}$$

$$\text{Cofin. C. } \frac{\pm 3.\Pi}{r\sqrt{p}} + \left(\text{Sin. C. } \frac{\pm 3.\Pi}{r\sqrt{p}}\right)\sqrt{-1} \qquad \frac{\pm 3.\Pi}{r\sqrt{p}} - \left(\text{Sin. C. } \frac{\pm 3.\Pi}{r\sqrt{p}}\right)\sqrt{-1}$$

$$\text{Cofin. C. } \frac{\pm \beta.\Pi}{r\sqrt{p}} + \left(\text{Sin. C. } \frac{\pm \beta.\Pi}{r\sqrt{p}}\right)\sqrt{-1} \qquad \frac{\pm \beta.\Pi}{r\sqrt{p}} - \left(\text{Sin. C. } \frac{\pm \beta.\Pi}{r\sqrt{p}}\right)\sqrt{-1}$$

Kkkk

nusquam

(*) Huic fundamento tota innititur *Incommensurabilium* tractatio, cuius in num°. 12. aliqua recensui, subtiliterque Pascalius meminit inter *Opuscula Mathematica* in Epistola ad Carcavium de Problematum circa Trochoidem resolutione.

nusquam absumpta, ut palam est, unica Radicum ipsarum periodo, ob incom-
mensurabilitatem inter Factorem Integrum β , atque Irrationalem $\sqrt[r]{p}$ sem-

per semperque manentem. Ex adverso Radices $\sqrt[r]{p}$ ab Aequatione geni-

tas pariter Interfcedente $x^{\frac{1}{\sqrt[r]{p}}} - 1 = 0$ Formula oecumenica designabit

Cofin. C. $\pm \sqrt[r]{p} \cdot n\Pi \rightarrow (\text{Sin. C.} \pm \sqrt[r]{p} \cdot n\Pi) \sqrt{-1}$, quae tribuet ali-
quantulum evoluta

$$\text{Cofin. C.} \pm \sqrt[r]{p} \cdot 0\Pi \rightarrow (\text{Sin. C.} \pm \sqrt[r]{p} \cdot 0\Pi) \sqrt{-1} = 1$$

$$\frac{1}{\text{Cofin. C.} \pm \sqrt[r]{p} \cdot 0\Pi - (\text{Sin. C.} \pm \sqrt[r]{p} \cdot 0\Pi) \sqrt{-1}} = 1$$

$$\text{Cofin. C.} \pm \sqrt[r]{p} \cdot 1\Pi \rightarrow (\text{Sin. C.} \pm \sqrt[r]{p} \cdot 1\Pi) \sqrt{-1}$$

$$\frac{1}{\text{Cofin. C.} \pm \sqrt[r]{p} \cdot 1\Pi - (\text{Sin. C.} \pm \sqrt[r]{p} \cdot 1\Pi) \sqrt{-1}}$$

$$\text{Cofin. C.} \pm \sqrt[r]{p} \cdot 2\Pi \rightarrow (\text{Sin. C.} \pm \sqrt[r]{p} \cdot 2\Pi) \sqrt{-1}$$

$$\frac{1}{\text{Cofin. C.} \pm \sqrt[r]{p} \cdot 2\Pi - (\text{Sin. C.} \pm \sqrt[r]{p} \cdot 2\Pi) \sqrt{-1}}$$

$$\text{Cofin. C.} \pm \sqrt[r]{p} \cdot 3\Pi \rightarrow (\text{Sin. C.} \pm \sqrt[r]{p} \cdot 3\Pi) \sqrt{-1}$$

$$\frac{1}{\text{Cofin. C.} \pm \sqrt[r]{p} \cdot 3\Pi - (\text{Sin. C.} \pm \sqrt[r]{p} \cdot 3\Pi) \sqrt{-1}}$$

$$\text{Cofin. C.} \pm \sqrt[r]{p} \cdot \beta\Pi \rightarrow (\text{Sin. C.} \pm \sqrt[r]{p} \cdot \beta\Pi) \sqrt{-1}$$

$$\frac{1}{\text{Cofin. C.} \pm \sqrt[r]{p} \cdot \beta\Pi - (\text{Sin. C.} \pm \sqrt[r]{p} \cdot \beta\Pi) \sqrt{-1}} ;$$

et singulae istae Radices propter asymmetriam Numeri $\sqrt[r]{p}$ infinite diver-
sae nullum Limitem noscunt. Nec putandum innumeras inter Radices,

quae spectant vel ad $\sqrt[r]{p}$, vel ad $\sqrt[r]{p}$, inesse unquam 1 ,
quia $\pm$

quia $\pm \frac{n\Pi}{\sqrt[r]{p}}$, seu $\pm n\Pi (\sqrt[r]{p})$ peraequare nunquam poterit aut P, aut

eius quodlibet Multiplum a Coefficiente impari definitum. Ut memineris autem sum auctor, negativa Signa, quae Arcus afficiunt, non duplicare Radicum Numerum, nec quaere si maneant etiam in casu Exponentium Irrationalium Radices Imaginariae Numero pares, cum hae parium impariumque distinctiones de Numeris tantum finitis evidenter aptaque significatione praedicari possint, omniaque consimilia symptomata in Infinito confusa veluti vasto in gurgite absorbeantur. Hoc solummodo adfirmo fuisse tutissima de-

monstratione absolutum Numeros $1 \sqrt[r]{p}$, et $1 \sqrt[r]{p}$, analytice loquendo, Infinitiformibus iniuria eripi Functionibus.

299. Libet ad initium prope reverti, et universalius rem istam momenti perquam maximi futuram promovere. Subiiciatur Index imperfectae

Potentiae Unitatis expressus per $\frac{\Delta}{\Theta}$ ad Numeros omnes non modo integros, fractos, rationales, irrationales, positivos, vel negativos, quin etiam Transcendentes cuiusque speciei comprehendendos, et idcirco resolvendam se

praebeat $1 - x^{\frac{\Theta}{\Delta}} = 0$ Exponentialis Aequatio. Constat hanc resolutionem

pendere ab Analytica Formula $x = 1^{\frac{\Delta}{\Theta}} = \text{Cofin. C. } z \pm (\text{Sin. C. } z) \sqrt{-1}$,

ex qua fit $1 = (\text{Cofin. C. } z \pm (\text{Sin. C. } z) \sqrt{-1})^{\frac{\Theta}{\Delta}} = \text{Cofin. C. } \frac{\Theta z}{\Delta} \pm$

$(\text{Sin. C. } \frac{\Theta z}{\Delta}) \sqrt{-1}$, cui conditioni respondet unica Aequalitas $\frac{\Theta z}{\Delta} = \pm n\Pi$,

vel $z = \pm \frac{\Delta}{\Theta} \cdot n\Pi$. Inde dimanat universaliter $1^{\frac{\Delta}{\Theta}} = \text{Cofin. C. } \pm \frac{\Delta}{\Theta} \cdot n\Pi$

$\pm (\text{Sin.}$

$\pm (\text{Sin. } C. \pm \frac{\Delta}{\Theta} \cdot n\Pi) \sqrt{-1}$ valore Infinitiformi praedita dum $\Delta \Theta$ aut ambo fuerint Irrationales, et eo magis Numeri Transcendentes, quos inter valores innumeros unicus Realis ± 1 tantummodo existet. A singulari superfedeo Radicum evolutione, quum eadem recurrant mutatis mutandis superius tradita, lucemque universa non parum mutuentur ab expressionibus in Capite V. contentis, et praesertim num^o. 237., nonnullisque sequentibus. Illa equidem Formula hoc servat generatim symptoma, quod aequipolleant substantia, sin minus forma, pro Radicibus eruendis scribendi modi Cofin. C. $\pm \frac{\Delta}{\Theta} \cdot n\Pi \pm (\text{Sin. } C. \pm \frac{\Delta}{\Theta} \cdot n\Pi) \sqrt{-1}$, et

$$\frac{1}{\text{Cofin. } C. \pm \frac{\Delta}{\Theta} \cdot n\Pi \mp (\text{Sin. } C. \pm \frac{\Delta}{\Theta} \cdot n\Pi) \sqrt{-1}}, \text{ prouti nativa Circuli in-}$$

doles ipsa suadet. Quid vero sentiendum sit de Formulis huiusmodi $z^{x\sqrt{-1}}$,

vel simplicius $1^{x\sqrt{-1}}$, aut $z^{\frac{1}{x\sqrt{-1}}}$, $1^{\frac{1}{x\sqrt{-1}}}$, scilicet $z^{-\left(\frac{1}{x}\right)\sqrt{-1}}$,

$1^{-\left(\frac{1}{x}\right)\sqrt{-1}}$, seu $z^{\Phi\sqrt{-1}}$, $1^{\Phi\sqrt{-1}}$, uti supra, indicante x vel Φ quem-

libet Numerum, etiam Irrationalem, ac Transcendentem, paucis aperiam. Ostenfa etenim fuit, ac saepenumero in hac Exercitatione diversis Loga-

richmorum Basibus aptata Aequalitas $B^z = C^{(LB)z}$; quamobrem $z^{x\sqrt{-1}}$
 $= C^{(Lz)x\sqrt{-1}} = C^{x\sqrt{-1}((Lz) \pm n\Pi \sqrt{-1})} = C^{(Lz)x\sqrt{-1} \mp xn\Pi}$; unde, quod

nostri nunc magis interest, $1^{x\sqrt{-1}} = C^{(L1)x\sqrt{-1}} = C^{(\pm n\Pi \sqrt{-1})x\sqrt{-1}}$

$= C^{\mp x \cdot n\Pi}$. Valores ergo innumeri Functionis Infinitiformis $1^{x\sqrt{-1}}$ hunc prospectum acquirant minutatim enucleati,

$$C^{\pm x \cdot n\Pi}$$

$$\begin{aligned}
 C^{\mp x \cdot 0 \Pi} & \text{ vel } C^0 \text{ aut } +1 \\
 C^{\mp x \cdot 1 \Pi} & \text{ vel Cofin. H. } x\Pi \mp \text{Sin. H. } x\Pi \\
 C^{\mp x \cdot 2 \Pi} & \text{ vel Cofin. H. } 2x\Pi \mp \text{Sin. H. } 2x\Pi \\
 C^{\mp x \cdot 3 \Pi} & \text{ vel Cofin. H. } 3x\Pi \mp \text{Sin. H. } 3x\Pi \\
 C^{\mp x \cdot 4 \Pi} & \text{ vel Cofin. H. } 4x\Pi \mp \text{Sin. H. } 4x\Pi \\
 \text{uno verbo } C^{\mp x \cdot n \Pi} & \text{ vel Cofin. H. } nx\Pi \mp \text{Sin. H. } nx\Pi, \text{ qui valores iidem si-}
 \end{aligned}$$

ne Limite, seu Radices Unitatis Indice Imaginario $x\sqrt{-1}$ distinctae, erunt omnes Reales. Translatio ista a Circularibus ad Hyperbolae Trigonometricas Lineas, qua plenius elucescit Circuli et Hyperbolae ipsius Harmonia, nusquam satis commendanda ab Analystis sublimia colentibus, pendet solummodo a variatione Exponentis x , primum Realis, ac postmodum Imaginari, prouti sequens Tabella comparationis evidentissime ob oculos ponit.

Valores $\tau \tilde{x}^x$, Functionis Potentialis.	Valores $\tau \tilde{x}^{x\sqrt{-1}}$, Functionis Potentialis.
Cofin. C. $\rightarrow 0.x\Pi \pm (\text{Sin. C.} \rightarrow 0.x\Pi)\sqrt{-1} = 1$	Cofin. H. $\rightarrow 0.x\Pi \mp (\text{Sin. H.} \rightarrow 0.x\Pi) = 1$
Cofin. C. $\rightarrow 1x\Pi \pm (\text{Sin. C.} \rightarrow 1x\Pi)\sqrt{-1}$	Cofin. H. $\rightarrow 1x\Pi \mp (\text{Sin. H.} \rightarrow 1x\Pi)$
Cofin. C. $\rightarrow 2x\Pi \pm (\text{Sin. C.} \rightarrow 2x\Pi)\sqrt{-1}$	Cofin. H. $\rightarrow 2x\Pi \mp (\text{Sin. H.} \rightarrow 2x\Pi)$
Cofin. C. $\rightarrow 3x\Pi \pm (\text{Sin. C.} \rightarrow 3x\Pi)\sqrt{-1}$	Cofin. H. $\rightarrow 3x\Pi \mp (\text{Sin. H.} \rightarrow 3x\Pi)$
— — — — —	— — — — —
Cofin. C. $\rightarrow nx\Pi \pm (\text{Sin. C.} \rightarrow nx\Pi)\sqrt{-1}$	Cofin. H. $\rightarrow nx\Pi \mp (\text{Sin. H.} \rightarrow nx\Pi)$

Quaelibetunque effingatur unius variabilis pluriumve variabilium Functio $\tau \tilde{x}$ latere potest dumtaxat valores inter forma Imaginarios alter valor Realis — 1.

300. Comprobant declaratam evolutionem ea, quae in Capite V. fufius innuimus: namque ibidem ostenditur effe Cofin. C. $\frac{v}{\sqrt{-1}}$, fcilicet Cofin. C.

$$-v\sqrt{-1} = \text{Cofin. H. } v, \text{ et } \left(\text{Sin. C. } \frac{v}{\sqrt{-1}} \right) \sqrt{-1}, \text{ aut } (\text{Sin. C.} - v\sqrt{-1})$$

$\sqrt{-1} = \text{Sin. H. } v$, nimirum ad maiorem univerfalitatem, $\text{Cofin. C. } \pm v\sqrt{-1} = \text{Cofin. H. } v$, necnon $(\text{Sin. C. } \pm v\sqrt{-1})\sqrt{-1} = \mp \text{Sin. H. } v$. Quibus pofitis, cum fit $1^x = \text{Cofin. C. } x(n\pi) \pm (\text{Sin. C. } x(n\pi))\sqrt{-1}$, habebitur

etiam ad inftar Coronidis $1^{x\sqrt{-1}} = \text{Cofin. C. } x(n\pi)\sqrt{-1} \pm (\text{Sin. C. } x(n\pi)\sqrt{-1})\sqrt{-1}$, aut ex praemiffis $= \text{Cofin. H. } x(n\pi) \mp \text{Sin. H. } x(n\pi)$, mirifico certe cum fuperius allatis confenfu. Sponte illico profluit perinfinis affectio nihil caeteris Exponentialium proprietatibus, fane eximiis,

concedens, quae ftatuit Functionem Infinitiformem $1^{x\sqrt{-1}}$ univerfos tribuere, tametfi innumeros, valores Reales, nempe Aequationem Exponentialem

$$1^{x\sqrt{-1}} - z = 0 \text{ Radicibus gaudere, numero Infinitis, Realibus. Dum } x$$

foret Unitas valores ii fine fine $1^{\sqrt{-1}}$ nil aliud forent, praeter unitatem, nifi Numeri, quibus pro Hyperbolicis Logarithmis respondent perfecta Mul-

tipla pofitiva vel negativa Peripheriae Circularis, videlicet $C^{\pm n\pi}$ ob oecumenicam Formulam fuperius productam. Hoc ideo Theorema praefert confequentes terminos in Geometrica Progreffione difpofitos.

$$1^{\sqrt{-1}} = \left. \begin{array}{l} C^0, C^\pi, C^{2\pi}, C^{3\pi}, C^{4\pi}, C^{5\pi}, C^{6\pi}, C^{7\pi}, C^{8\pi}, C^{9\pi}, C^{10\pi}, C^{11\pi} \&c: \\ C^{-0}, C^{-\pi}, C^{-2\pi}, C^{-3\pi}, C^{-4\pi}, C^{-5\pi}, C^{-6\pi}, C^{-7\pi}, C^{-8\pi}, C^{-9\pi}, C^{-10\pi}, C^{-11\pi} \&c: \end{array} \right\} \text{in Infinitum}$$

Liceret etiam methodum eandem transferre ad investigationem valorum, 1^x , ne aliquis fortaffe opinetur obfcuram, fufpectamque fuiſſe Radicum

$1^{x\sqrt{-1}}$ iam proditarum rationem, quae fimul coniungit omnimodas Unitatis Radices, Potestates cuiuſcumque Indicis Realis, quinimo et Imaginariis,

necnon

necnon Logarithmos Unitatis innumeros ex alio fonte superius, nec una

tantum vice, deductos. Est enim $1^x = C^{(L_1)x} = C^{x(\pm n\pi\sqrt{-1})} =$

$C^{\pm x \cdot n\pi\sqrt{-1}} = \text{Cofin. } C.x(n\pi) \pm (\text{Sin. } C.x(n\pi))\sqrt{-1}$, veluti in num^o. praecedente. Differentia autem extat cautissime adnotanda inter 1^x ,

atque $1^{x\sqrt{-1}}$, propterea quod in postrema Formula nunquam non habeantur diversi ac Numero Infiniti valores, sed ex adverso in priore eorum Numerus sit definitus, Seriesque inventae definant in Infinitum tum cum Exponens x in Rationalium classem cooptetur.

301. Qui haec diligentius colligere, illisque maiorem, si fortassis aliquando non fuerit explicita, universalitatem donare curaverit, facillime Potestatum cuiuslibet generis Algorithmum, sine hoc praesidio angustissimos intra cancellos oclusum, perficere poterit, prouti in anteaactis Capitibus erat in votis. Ordinem sequar parum a dissertis haecenus, ac supra pollicitis recedentem.

$$(a^2 \pm x^2)^{\frac{1}{2}} = (\text{Cofin. } C.0 + (\text{Sin. } C.0)\sqrt{-1})(a^2 \pm x^2)^{\frac{1}{2}} = \pm 1 (a^2 \pm x^2)^{\frac{1}{2}}, \text{ Functio Biformis;}$$

$$(\text{Cofin. } C.\frac{\pi}{2} + (\text{Sin. } C.\frac{\pi}{2})\sqrt{-1})$$

$$(1 \pm x^2)^{-\frac{1}{2}} = (\text{Cofin. } C.0 + (\text{Sin. } C.0)\sqrt{-1})(1 \pm x^2)^{-\frac{1}{2}} = \pm 1 (1 \pm x^2)^{-\frac{1}{2}}, \text{ Functio pariter Biformis.}$$

$$(\text{Cofin. } C.-\frac{\pi}{2} + (\text{Sin. } C.-\frac{\pi}{2})\sqrt{-1})$$

Haec simul cum postsequentibus innituntur Theoriae passim in 1. Capite aedificatae, quod nempe Functionum Potentialium Polyformitas compendio maximo per Multiplicationem Unitatis ad consimilem exaltandae Potestatem

$$\frac{1}{1^2}, 1^{\frac{1}{2}} \text{ \&c: obtineatur (*).}$$

$$(a^3 \pm$$

(*) Rursus perlege potissimum numeros 21. 22. 23. 24.

$$(a^3 \pm z^3)^{\frac{1}{3}} = 1^{\frac{1}{3}} (a^3 \pm z^3)^{\frac{1}{3}} = \text{Cofin. C. } 0 \rightarrow (\text{Sin. C. } 0) \sqrt{-1} (a^3 \pm z^3)^{\frac{1}{3}} =$$

$$\text{Cofin. C. } \frac{\Pi}{3} \rightarrow (\text{Sin. C. } \frac{\Pi}{3}) \sqrt{-1}$$

, Functio Triformis,

$$\text{Cofin. C. } \frac{2\Pi}{3} \rightarrow (\text{Sin. C. } \frac{2\Pi}{3}) \sqrt{-1}$$

$$\left. \begin{aligned} & - \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \sqrt{-1} \\ & - \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot \sqrt{-1} \end{aligned} \right\} \times \frac{1}{a^3 \pm z^3}$$

$$(1 \pm z)^{\frac{n}{m}} = 1^{\frac{n}{m}} (1 \pm z)^{\frac{n}{m}} = \left(\text{Cofin. C. } \frac{n}{m} \beta \Pi \rightarrow (\text{Sin. C. } \frac{n}{m} \beta \Pi) \sqrt{-1} \right) \left(1 \pm \text{Cofin. C. } \frac{\beta \Pi}{n} \rightarrow (\text{Sin. C. } \frac{\beta \Pi}{n}) \sqrt{-1} \right) \times m^{\frac{1}{n}} z^{\frac{1}{n}}$$

$$(1 \pm z)^T = 1^T (1 \pm z)^T = \left(\text{Cofin. C. } T \beta \Pi \rightarrow (\text{Sin. C. } T \beta \Pi) \sqrt{-1} \right) \left(1 \pm T' \cdot z^1 \rightarrow T' \cdot T' - 1 \cdot z^2 \right.$$

$$T'' \quad T'' \cdot T'' - 1$$

$$T''' \quad T''' \cdot T''' - 1$$

$$T^{iv} \quad T^{iv} \cdot T^{iv} - 1$$

$$- \quad - \quad -$$

$$\frac{T^p}{1} \quad \frac{T^p \cdot T^p - 1}{1 \cdot 2}$$

$$(1 \pm z)^\Pi = 1^\Pi (1 \pm z)^\Pi = \left(\text{Cofin. C. } \Pi (\beta \Pi) \rightarrow (\text{Sin. C. } \Pi (\beta \Pi)) \sqrt{-1} \right) \left(1 \pm \frac{\Pi}{1} \cdot z^1 \rightarrow \frac{\Pi \cdot \Pi - 1}{1 \cdot 2} \cdot z^2 \right.$$

$$(1 \pm z)^H = 1^H (1 \pm z)^H = \left(\text{Cofin. C. } H (\beta \Pi) \rightarrow (\text{Sin. C. } H (\beta \Pi)) \sqrt{-1} \right) \left(1 \pm \frac{H}{1} \cdot z^1 \rightarrow \frac{H \cdot H - 1}{1 \cdot 2} \cdot z^2 \right.$$

$$(1 \pm z)^{S \hat{x} dx} = 1^{S \hat{x} dx} (1 \pm z)^{S \hat{x} dx} = \left(\text{Cofin. C. } S \hat{x} dx (\beta \Pi) \rightarrow (\text{Sin. C. } S \hat{x} dx (\beta \Pi)) \sqrt{-1} \right) \left(1 \pm \frac{(S' \hat{x} dx)}{1} \cdot z^1 \right.$$

$$\frac{(S'' \hat{x} dx)}{1}$$

$$\frac{(S''' \hat{x} dx)}{1}$$

$$\frac{(S^{iv} \hat{x} dx)}{1}$$

$$\frac{(S^r \hat{x} dx)}{1}$$

$$(a \pm z)^{\frac{m}{n}} = 1^{\frac{m}{n}} (a \pm z)^{\frac{m}{n}} = \left(\text{Cofin. C. } \frac{m}{n} \cdot \beta \Pi + (\text{Sin. C. } \frac{m}{n} \cdot \beta \Pi) \sqrt{-1} \right) (a \pm z)^{\frac{m}{n}}, \text{ Functio } \overline{n} \text{ formis};$$

in qua monendum Coefficientem β positivo-negativum quemlibet Numerum integrum cum Nihilo repraesentare. Praeterea

$$(1 \pm z)^{\frac{n}{\sqrt{m}}}$$

$$+ \left(\text{Cofin. C. } \frac{\beta \Pi}{n} + (\text{Sin. C. } \frac{\beta \Pi}{n}) \sqrt{-1} \right) m^{\frac{1}{n}} \times \left(\text{Cofin. C. } \frac{\beta \Pi}{n} + (\text{Sin. C. } \frac{\beta \Pi}{n}) \sqrt{-1} \right) m^{\frac{1}{n}-1} \cdot z^2 \&c: \Big), \overline{n. \infty} \text{ formis Functio};$$

$$\pm T' \cdot T' - 1 \cdot T' - 2 \cdot z^3 + \&c: \Big), \text{ Functio saltem } \overline{p} \text{ formis si omnes } T', T'', T''' \&c: \text{ fuerint Numeri integri};$$

$$T'' \cdot T'' - 1 \cdot T'' - 2$$

$$T''' \cdot T''' - 1 \cdot T''' - 2$$

$$T^{IV} \cdot T^{IV} - 1 \cdot T^{IV} - 2$$

$$- - - - -$$

$$\frac{T^p \cdot T^p - 1 \cdot T^p - 2}{1 \cdot 2 \cdot 3}$$

$$\pm \frac{\Pi \cdot \Pi - 1 \cdot \Pi - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \frac{\Pi \cdot \Pi - 1 \cdot \Pi - 2 \cdot \Pi - 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot z^4 \pm \&c: \Big), \text{ Functio Infinitiformis};$$

$$\pm \frac{H \cdot H - 1 \cdot H - 2}{1 \cdot 2 \cdot 3} \cdot z^3 + \&c: \Big), \text{ Functio pariter } \infty \text{ formis, quum } H \text{ fit Sigla Areae Hyperbolicae};$$

$$+ \frac{(S' \hat{x} dx) \cdot (S' \hat{x} dx) - 1 \cdot z^2}{(S'' \hat{x} dx) \cdot (S'' \hat{x} dx) - 1} \pm \frac{(S' \hat{x} dx) \cdot (S' \hat{x} dx) - 1 \cdot (S' \hat{x} dx) - 2 \cdot z^3}{(S'' \hat{x} dx) \cdot (S'' \hat{x} dx) - 1 \cdot (S'' \hat{x} dx) - 2} + \&c: \Big), \text{ Functio, } \overline{r. \infty} \text{ formis, integratione confecta pro } x = q \text{ Numero indefinito,}$$

$$(S'' \hat{x} dx) \cdot (S'' \hat{x} dx) - 1$$

$$(S'' \hat{x} dx) \cdot (S'' \hat{x} dx) - 1 \cdot (S'' \hat{x} dx) - 2$$

$$(S''' \hat{x} dx) \cdot (S''' \hat{x} dx) - 1$$

$$(S''' \hat{x} dx) \cdot (S''' \hat{x} dx) - 1 \cdot (S''' \hat{x} dx) - 2$$

$$(S^{IV} \hat{x} dx) \cdot (S^{IV} \hat{x} dx) - 1$$

$$(S^{IV} \hat{x} dx) \cdot (S^{IV} \hat{x} dx) - 1 \cdot (S^{IV} \hat{x} dx) - 2$$

$$- - - - -$$

$$\frac{(S^r \hat{x} dx) \cdot (S^r \hat{x} dx) - 1}{1 \cdot 2}$$

$$\frac{(S^r \hat{x} dx) \cdot (S^r \hat{x} dx) - 1 \cdot (S^r \hat{x} dx) - 2}{1 \cdot 2 \cdot 3}$$

M m m m

in qua

in qua Formula $\tau \delta S \dot{x} dx$ Integrationem respuat, habeatque numero quolibet r diversos, et persaepe Infinitos valores, uti contingit Quadraturis omnium Curvarum

$$\frac{A. \sin. b}{(1 \pm z)} = 1 \quad \frac{A. \sin. b}{(1 \pm z)} = \left(\cosin. C. (A. \sin. b)(\beta \Pi) \mp (\sin. C(A. \sin. b)(\beta \Pi)) \sqrt{-1} \right)$$

ubi adnotandum Coefficientem $\infty \mp 1$ Symbolum solummodo esse inaffignabilis extremi Numeri imparis.

varum in se redeuntium, aut Infinitis numero Ramis gaudentium. Pari progressu

$$\frac{A. \sin. b}{(1 \pm z)}$$

$$(1 \pm (A. \sin. b) \cdot z^1 + (A. \sin. b) \cdot (A. \sin. b)^{-1} \cdot z^2 \pm (A. \sin. b) \cdot (A. \sin. b)^{-1} \cdot (A. \sin. b)^{-2} \cdot z^3 + \&c.,$$

Functio

$$(\Pi \rightarrow A) \quad (\Pi \rightarrow A) \cdot (\Pi \rightarrow A)^{-1} \quad (\Pi \rightarrow A) \cdot (\Pi \rightarrow A)^{-1} \cdot (\Pi \rightarrow A)^{-2} \quad \infty \cdot \infty \text{formis;}$$

$$(2 \Pi \rightarrow A) \quad (2 \Pi \rightarrow A) \cdot (2 \Pi \rightarrow A)^{-1} \quad (2 \Pi \rightarrow A) \cdot (2 \Pi \rightarrow A)^{-1} \cdot (2 \Pi \rightarrow A)^{-2}$$

$$(3 \Pi \rightarrow A) \quad (3 \Pi \rightarrow A) \cdot (3 \Pi \rightarrow A)^{-1} \quad (3 \Pi \rightarrow A) \cdot (3 \Pi \rightarrow A)^{-1} \cdot (3 \Pi \rightarrow A)^{-2}$$

$$-----$$

$$(n \Pi \rightarrow A) \quad (n \Pi \rightarrow A) \cdot (n \Pi \rightarrow A)^{-1} \quad (n \Pi \rightarrow A) \cdot (n \Pi \rightarrow A)^{-1} \cdot (n \Pi \rightarrow A)^{-2}$$

$$-----$$

$$(\infty \Pi \rightarrow A) \quad (\infty \Pi \rightarrow A) \cdot (\infty \Pi \rightarrow A)^{-1} \quad (\infty \Pi \rightarrow A) \cdot (\infty \Pi \rightarrow A)^{-1} \cdot (\infty \Pi \rightarrow A)^{-2}$$

$$(P \rightarrow A) \quad (P \rightarrow A) \cdot (P \rightarrow A)^{-1} \quad (P \rightarrow A) \cdot (P \rightarrow A)^{-1} \cdot (P \rightarrow A)^{-2}$$

$$(3 P \rightarrow A) \quad (3 P \rightarrow A) \cdot (3 P \rightarrow A)^{-1} \quad (3 P \rightarrow A) \cdot (3 P \rightarrow A)^{-1} \cdot (3 P \rightarrow A)^{-2}$$

$$(5 P \rightarrow A) \quad (5 P \rightarrow A) \cdot (5 P \rightarrow A)^{-1} \quad (5 P \rightarrow A) \cdot (5 P \rightarrow A)^{-1} \cdot (5 P \rightarrow A)^{-2}$$

$$-----$$

$$(\overline{n+1} \cdot P \rightarrow A) \quad (\overline{n+1} \cdot P \rightarrow A) \cdot (\overline{n+1} \cdot P \rightarrow A)^{-1} \quad (\overline{n+1} \cdot P \rightarrow A) \cdot (\overline{n+1} \cdot P \rightarrow A)^{-1} \cdot (\overline{n+1} \cdot P \rightarrow A)^{-2}$$

$$-----$$

$$(\overline{\infty+1} \cdot P \rightarrow A) \quad (\overline{\infty+1} \cdot P \rightarrow A) \cdot (\overline{\infty+1} \cdot P \rightarrow A)^{-1} \quad (\overline{\infty+1} \cdot P \rightarrow A) \cdot (\overline{\infty+1} \cdot P \rightarrow A)^{-1} \cdot (\overline{\infty+1} \cdot P \rightarrow A)^{-2}$$

$$\frac{1}{1} \quad \frac{1}{1} \quad \frac{2}{2} \quad \frac{1}{1} \quad \frac{2}{2} \quad \frac{3}{3}$$

ubi adno-

302. Quae adhuc forte remaneant consimiliter absoluntur exempla, lucemque affundent non levem tot aliis brevitatis causa praetermissis. Sic

$$\frac{L(d)}{(1 \pm z)}$$

$+(L(d)).$ $(L(d) - 1 \cdot z^2 \pm \&c:)$, posito $L(d)$ Signo unius Logarithmi Realis,

$$(L(d) \pm \Pi\sqrt{-1}).(L(d) \pm \Pi\sqrt{-1}) - 1$$

$$(L(d) \pm 2\Pi\sqrt{-1}).(L(d) \pm 2\Pi\sqrt{-1}) - 1$$

$$(L(d) \pm 3\Pi\sqrt{-1}).(L(d) \pm 3\Pi\sqrt{-1}) - 1$$

, Functio infinities Infinitiformis;

$$(L(d) \pm 4\Pi\sqrt{-1}).(L(d) \pm 4\Pi\sqrt{-1}) - 1$$

$$\frac{(L(d) \pm n\Pi\sqrt{-1}).(L(d) \pm n\Pi\sqrt{-1}) - 1}{1 \quad 2}$$

$$(1 \pm z)^x$$

$+\frac{(Lv)^3 x^3}{1.2.3} + \frac{(Lv)^4 x^4}{1.2.3.4} + \frac{(Lv)^5 x^5}{1.2.3.4.5} + \&c:)$, ubi Lv unicum Realem designat, Functio ∞ formis, dummodo non sit x Rationalis;

vel $(\text{Cofin.H.}x(n\Pi) \mp \text{Sin.H.}x(n\Pi))(\text{Cofin.C.L}(z^x) + (\text{Sin.C.L}(z^x))\sqrt{-1})$, Functio ∞ formis.

Exponentes

sicuti Algebra praecipit unicum Realemque tribuunt valorem, quamvis infinities repetitum, Periodis universis in unico termino definiendo consentientibus. Censenda nequidem erit adeo sterilis, parcique usus Theoria superius exposita, ut Unitatis Radicibus variis determinandis solummodo idonea sit: vix etenim mentionem desiderat plenissima illius Theorices amplificatio ad Radicum cuiuslibet Numeri evolutionem. Hoc efficit universaliter subnexa Calculi Linea.

$$z^{\frac{n}{m}} =$$

$$\left(\text{Cofin.C.} \frac{-n}{m} \cdot \Pi + (\text{Sin.C.} \frac{-n}{m} \cdot \Pi) \sqrt{-1}\right)^m \sqrt[n]{z^n}, \left(\text{Cofin.C.} \frac{2n}{m} \cdot \Pi + (\text{Sin.C.} \frac{2n}{m} \cdot \Pi) \sqrt{-1}\right)^m \sqrt[n]{z^n}, \&c: \&c:$$

N n n n

ad numerum

ad numerum m se componentibus universis $\sqrt[m]{z}$ Potestatis imperfectae Ra-

dicibus. Exinde oritur $z^{\frac{n}{m}}$ Functio $\bar{m}$ formis, et in casu m paris capax valoris

gemini Realis $\pm \sqrt[m]{z^n}$, quae adhibita expressio denotat illum Radicalem Nu-

merum $\sqrt[m]{z^n}$ in communi Irrationalium Calculo contemplatum. Eodem pa-

cto, quum $(-z)^{\frac{n}{m}} = (-1)^{\frac{n}{m}} \cdot z^{\frac{n}{m}}$, reperientur Radices omnes Numeri

negativi ad imperfectam Potestatem exaltati $(-z)^{\frac{n}{m}}$ vixdum absolutissima

detur resolutio Aequationis $x^{\frac{m}{n}} + 1 = 0$, quam nunc versandam suscipio.

303. Esto primum $n = 1$, ideoque abeat Aequatio in Binomiale x^m

$+ 1 = 0$, aut $x = (-1)^{\frac{1}{m}}$ analogam illi num°. 288. perpensae, ac sequentibus. Renovata hypothesi $\sqrt[m]{x} = \text{Cofin. C. } v + (\text{Sin. C. } v) \sqrt{-1}$, vertetur Aequalitas in alteram $\text{Cofin. C. } mv + (\text{Sin. C. } mv) \sqrt{-1} = -1$, ni-

mirum vi Trigonometricorum canonum in $mv = \pm \lambda P$, seu $v = \pm \frac{\lambda P}{m}$

$= \pm \frac{\lambda \Pi}{2m}$, ac denique $(-1)^{\frac{1}{m}} = \text{Cofin. C. } \pm \frac{\lambda P}{m} + (\text{Sin. C. } \pm$

$\frac{\lambda P}{m}) \sqrt{-1}$, necnon generalius $(-z)^{\frac{1}{m}} = (\text{Cofin. C. } \pm \frac{\lambda P}{m} + (\text{Sin. C.}$

$\pm \frac{\lambda P}{m}) \sqrt{-1})^m \sqrt{z}$, prouti superius iam monitum. Evidens fit, quod, exce-

ptis casibus Indicis m imparis aeque ac λ , nullus haberi queat Realis va-

lor $\sqrt[m]{-1}$, et si forte hoc contingat, sit semper unicus -1 , respondens extremo

extremo Periodi termino Cofin. C. $\pm \frac{mP}{m} \rightarrow (\text{Sin. C.} \pm \frac{mP}{m})\sqrt{-1}$. Priusquam totam complectamur oecumenicae Formulae vim, eius veritatem experiamur utraque in Radice $(-1)^{\frac{1}{2}}$, $(-1)^{\frac{1}{4}}$ num°. 146. obiter enunciata. Subrogentur ergo loco $\frac{1}{m}$ discriminatim $\frac{1}{2}$, et $\frac{1}{4}$; eritque

$$\sqrt[2]{-1} = \text{Cofin. C.} \pm \frac{1.P}{2} \rightarrow (\text{Sin. C.} \pm \frac{1.P}{2})\sqrt{-1} = 0 \pm \sqrt{-1} = \pm \sqrt{-1}, \text{ Functio Biformis}$$

$$\text{Cofin. C.} \pm \frac{3.P}{2} \rightarrow (\text{Sin. C.} \pm \frac{3.P}{2})\sqrt{-1} = \text{Cofin. C.} \pm \frac{1.P}{2} \rightarrow (\text{Sin. C.} \pm \frac{1.P}{2})\sqrt{-1} = 0 \pm \sqrt{-1} = \pm \sqrt{-1};$$

ita ut post primam expressionem caeterae evadant inutiles, Periodo statim ipsomet recurrense sine ullo Limite in Infinitum. Insuper

$$\sqrt[4]{-1} = \text{Cofin. C.} \pm \frac{1.P}{4} \rightarrow (\text{Sin. C.} \pm \frac{1.P}{4})\sqrt{-1} = \sqrt[2]{\frac{1}{2}} \pm \sqrt[2]{\frac{1}{2}} \cdot \sqrt{-1}$$

$$\text{Cofin. C.} \pm \frac{3.P}{4} \rightarrow (\text{Sin. C.} \pm \frac{3.P}{4})\sqrt{-1} = -\sqrt[2]{\frac{1}{2}} \pm \sqrt[2]{\frac{1}{2}} \cdot \sqrt{-1}, \text{ Functio Tetraformis,}$$

$$\text{Cofin. C.} \pm \frac{5.P}{4} \rightarrow (\text{Sin. C.} \pm \frac{5.P}{4})\sqrt{-1} = -\sqrt[2]{\frac{1}{2}} \mp \sqrt[2]{\frac{1}{2}} \cdot \sqrt{-1} \dots \dots \text{ inutiliter sane quia hic nova incipit earundem Radicum Periodus.}$$

&c:

&c:

Addatur specimen aliquod Ordinis imparis in

$$\sqrt[5]{-1} = \text{Cofin. C.} \pm \frac{1.P}{5} \rightarrow (\text{Sin. C.} \pm \frac{1.P}{5})\sqrt{-1} = \sqrt[3]{\frac{3}{8}} \pm \frac{1}{8}\sqrt{5} \pm \frac{1}{2}\sqrt{\frac{5}{2}} - \frac{1}{2}\sqrt{5}$$

$$\text{Cofin. C.} \pm \frac{3.P}{5} \rightarrow (\text{Sin. C.} \pm \frac{3.P}{5})\sqrt{-1} = \frac{1}{4} - \frac{1}{4}\sqrt{5} \pm \sqrt{\frac{5}{8}} + \frac{1}{8}\sqrt{5}, \text{ Functio Pentaformis,}$$

$$\text{Cofin. C.} \pm \frac{5.P}{5} \rightarrow (\text{Sin. C.} \pm \frac{5.P}{5})\sqrt{-1} = -1 \dots \dots \dots \text{ subindeque renovatur Periodus;}$$

quibus antea confectis cuiuslibet gradus Radices $(-1)^{\frac{1}{m}}$ solito disponam systemate.

304.

Pro Indice m Impari

$$(-1)^{\frac{1}{m}} =$$

$$\text{Cofin. C. } \frac{P}{m} \pm \left(\text{Sin. C. } \frac{P}{m} \right) \sqrt{-1} \quad \left(\text{Cofin. C. } \frac{P}{m} \mp \left(\text{Sin. C. } \frac{P}{m} \right) \sqrt{-1} \right)^{-1}$$

$$\text{Cofin. C. } \frac{3P}{m} \pm \left(\text{Sin. C. } \frac{3P}{m} \right) \sqrt{-1} \quad \left(\text{Cofin. C. } \frac{3P}{m} \mp \left(\text{Sin. C. } \frac{3P}{m} \right) \sqrt{-1} \right)^{-1}$$

vel

$$\text{Cofin. C. } \frac{5P}{m} \pm \left(\text{Sin. C. } \frac{5P}{m} \right) \sqrt{-1} \quad \left(\text{Cofin. C. } \frac{5P}{m} \mp \left(\text{Sin. C. } \frac{5P}{m} \right) \sqrt{-1} \right)^{-1}$$

$$\text{Cofin. C. } \frac{7P}{m} \pm \left(\text{Sin. C. } \frac{7P}{m} \right) \sqrt{-1} \quad \left(\text{Cofin. C. } \frac{7P}{m} \mp \left(\text{Sin. C. } \frac{7P}{m} \right) \sqrt{-1} \right)^{-1}$$

$$\text{Cofin. C. } \frac{mP}{m} \pm \left(\text{Sin. C. } \frac{mP}{m} \right) \sqrt{-1} = -1 \quad \left(\text{Cofin. C. } \frac{mP}{m} \mp \left(\text{Sin. C. } \frac{mP}{m} \right) \sqrt{-1} \right)^{-1} = -1$$

aut

$$\text{Cofin. C. } \frac{(2m-1)P}{m} \mp \left(\text{Sin. C. } \frac{(2m-1)P}{m} \right) \sqrt{-1} \quad \left(\text{Cofin. C. } \frac{(2m-1)P}{m} \pm \left(\text{Sin. C. } \frac{(2m-1)P}{m} \right) \sqrt{-1} \right)^{-1}$$

$$\text{Cofin. C. } \frac{(2m-3)P}{m} \mp \left(\text{Sin. C. } \frac{(2m-3)P}{m} \right) \sqrt{-1} \quad \left(\text{Cofin. C. } \frac{(2m-3)P}{m} \pm \left(\text{Sin. C. } \frac{(2m-3)P}{m} \right) \sqrt{-1} \right)^{-1}$$

vel

$$\text{Cofin. C. } \frac{(2m-5)P}{m} \mp \left(\text{Sin. C. } \frac{(2m-5)P}{m} \right) \sqrt{-1} \quad \left(\text{Cofin. C. } \frac{(2m-5)P}{m} \pm \left(\text{Sin. C. } \frac{(2m-5)P}{m} \right) \sqrt{-1} \right)^{-1}$$

$$\text{Cofin. C. } \frac{(2m-7)P}{m} \mp \left(\text{Sin. C. } \frac{(2m-7)P}{m} \right) \sqrt{-1} \quad \left(\text{Cofin. C. } \frac{(2m-7)P}{m} \pm \left(\text{Sin. C. } \frac{(2m-7)P}{m} \right) \sqrt{-1} \right)^{-1}$$

$$\text{Cofin. C. } \frac{(2m-m)P}{m} \mp \left(\text{Sin. C. } \frac{(2m-m)P}{m} \right) \sqrt{-1} = -1 \quad \left(\text{Cofin. C. } \frac{(2m-m)P}{m} \pm \left(\text{Sin. C. } \frac{(2m-m)P}{m} \right) \sqrt{-1} \right)^{-1} = -1$$

Pro Indice

Pro Indice m Pariter - Pari

$$(-1)^{\frac{1}{m}} =$$

$$\text{Cofin. C. } \frac{P}{m} \pm \left(\text{Sin. C. } \frac{P}{m} \right) \sqrt{-1}$$

$$\left(\text{Cofin. C. } \frac{P}{m} \mp \left(\text{Sin. C. } \frac{P}{m} \right) \sqrt{-1} \right)^{-1}$$

$$\text{Cofin. C. } \frac{3P}{m} \pm \left(\text{Sin. C. } \frac{3P}{m} \right) \sqrt{-1}$$

$$\left(\text{Cofin. C. } \frac{3P}{m} \mp \left(\text{Sin. C. } \frac{3P}{m} \right) \sqrt{-1} \right)^{-1}$$

vel

$$\text{Cofin. C. } \frac{5P}{m} \pm \left(\text{Sin. C. } \frac{5P}{m} \right) \sqrt{-1}$$

$$\left(\text{Cofin. C. } \frac{5P}{m} \mp \left(\text{Sin. C. } \frac{5P}{m} \right) \sqrt{-1} \right)^{-1}$$

$$\text{Cofin. C. } \frac{7P}{m} \pm \left(\text{Sin. C. } \frac{7P}{m} \right) \sqrt{-1}$$

$$\left(\text{Cofin. C. } \frac{7P}{m} \mp \left(\text{Sin. C. } \frac{7P}{m} \right) \sqrt{-1} \right)^{-1}$$

$$\text{Cofin. C. } \frac{(m-1)P}{m} \pm \left(\text{Sin. C. } \frac{(m-1)P}{m} \right) \sqrt{-1}$$

$$\left(\text{Cofin. C. } \frac{(m-1)P}{m} \mp \left(\text{Sin. C. } \frac{(m-1)P}{m} \right) \sqrt{-1} \right)^{-1}$$

aut

$$\text{Cofin. C. } \frac{(2m-1)P}{m} \mp \left(\text{Sin. C. } \frac{(2m-1)P}{m} \right) \sqrt{-1}$$

$$\left(\text{Cofin. C. } \frac{(2m-1)P}{m} \pm \left(\text{Sin. C. } \frac{(2m-1)P}{m} \right) \sqrt{-1} \right)^{-1}$$

$$\text{Cofin. C. } \frac{(2m-3)P}{m} \mp \left(\text{Sin. C. } \frac{(2m-3)P}{m} \right) \sqrt{-1}$$

$$\left(\text{Cofin. C. } \frac{(2m-3)P}{m} \pm \left(\text{Sin. C. } \frac{(2m-3)P}{m} \right) \sqrt{-1} \right)^{-1}$$

vel

$$\text{Cofin. C. } \frac{(2m-5)P}{m} \mp \left(\text{Sin. C. } \frac{(2m-5)P}{m} \right) \sqrt{-1}$$

$$\left(\text{Cofin. C. } \frac{(2m-5)P}{m} \pm \left(\text{Sin. C. } \frac{(2m-5)P}{m} \right) \sqrt{-1} \right)^{-1}$$

$$\text{Cofin. C. } \frac{(2m-7)P}{m} \mp \left(\text{Sin. C. } \frac{(2m-7)P}{m} \right) \sqrt{-1}$$

$$\left(\text{Cofin. C. } \frac{(2m-7)P}{m} \pm \left(\text{Sin. C. } \frac{(2m-7)P}{m} \right) \sqrt{-1} \right)^{-1}$$

$$\text{Cofin. C. } \frac{(2m-\overline{m-1})P}{m} \mp \left(\text{Sin. C. } \frac{(2m-\overline{m-1})P}{m} \right) \sqrt{-1}$$

$$\left(\text{Cofin. C. } \frac{(2m-\overline{m-1})P}{m} \pm \left(\text{Sin. C. } \frac{(2m-\overline{m-1})P}{m} \right) \sqrt{-1} \right)^{-1}$$

O o o o

Pro Indice m

$$\text{Cofin. C. } \frac{P}{m} \pm \left(\text{Sin. C. } \frac{P}{m} \right) \sqrt{-1}$$

$$\text{Cofin. C. } \frac{(2m-1)P}{m} \mp \left(\text{Sin. C. } \frac{(2m-1)P}{m} \right) \sqrt{-1}$$

$$\text{Cofin. C. } \frac{3P}{m} \pm \left(\text{Sin. C. } \frac{3P}{m} \right) \sqrt{-1}$$

aut

$$\text{Cofin. C. } \frac{(2m-3)P}{m} \mp \left(\text{Sin. C. } \frac{(2m-3)P}{m} \right) \sqrt{-1}$$

vel

$$\text{Cofin. C. } \frac{5P}{m} \pm \left(\text{Sin. C. } \frac{5P}{m} \right) \sqrt{-1}$$

$$\text{Cofin. C. } \frac{(2m-5)P}{m} \mp \left(\text{Sin. C. } \frac{(2m-5)P}{m} \right) \sqrt{-1}$$

$$\text{Cofin. C. } \frac{\frac{1}{2}mP}{m} \pm \left(\text{Sin. C. } \frac{\frac{1}{2}mP}{m} \right) \sqrt{-1} = \pm \sqrt{-1}$$

$$\text{Cofin. C. } \frac{(2m-\frac{1}{2}m)P}{m} \mp \left(\text{Sin. C. } \frac{(2m-\frac{1}{2}m)P}{m} \right) \sqrt{-1} = \mp \sqrt{-1}$$

$$\text{Cofin. C. } \frac{(m-1)P}{m} \pm \left(\text{Sin. C. } \frac{(m-1)P}{m} \right) \sqrt{-1}$$

$$\text{Cofin. C. } \frac{(2m-m-1)P}{m} \mp \left(\text{Sin. C. } \frac{(2m-m-1)P}{m} \right) \sqrt{-1}$$

305. Radices itaque $(-1)^{\frac{1}{m}}$ emergent semper cumulatim sumptae numero m , nec ultra, nec minus, et earum Imaginariae erunt numero pares, nimirum $m-1$ dum m Impar fuerit, atque m in hypothesi adversa. De negativis Signis, et negativo Indice $-m$ eadem superius dicta repeti nul-

lus nescit. Ratione consimili, si obveniat Aequatio resolvenda $x^{\frac{m}{r}} + 1 = 0$,

quod idem sonat ac reperire Radices omnimodas $(-1)^{\frac{r}{m}}$, facillime quaestio dirimetur ope subiuncti Diagrammatis, a cuius indefinita in vacuis sedibus continuatione nihil est quod prohibeat.

Cofin. C. $\pm$

Impariter - Pari

$$\begin{array}{lcl}
 \left(\text{Cofin. C. } \frac{P}{m} \mp \left(\text{Sin. C. } \frac{P}{m} \right) \sqrt{-1} \right)^{-1} & | & \left(\text{Cofin. C. } \frac{(2m-1)P}{m} \pm \left(\text{Sin. C. } \frac{(2m-1)P}{m} \right) \sqrt{-1} \right)^{-1} \\
 \left(\text{Cofin. C. } \frac{3P}{m} \mp \left(\text{Sin. C. } \frac{3P}{m} \right) \sqrt{-1} \right)^{-1} & \text{aut} & \left(\text{Cofin. C. } \frac{(2m-3)P}{m} \pm \left(\text{Sin. C. } \frac{(2m-3)P}{m} \right) \sqrt{-1} \right)^{-1} \\
 \left(\text{Cofin. C. } \frac{5P}{m} \mp \left(\text{Sin. C. } \frac{5P}{m} \right) \sqrt{-1} \right)^{-1} & | & \left(\text{Cofin. C. } \frac{(2m-5)P}{m} \pm \left(\text{Sin. C. } \frac{(2m-5)P}{m} \right) \sqrt{-1} \right)^{-1} \\
 \text{---} & | & \text{---} \\
 \left(\text{Cofin. C. } \frac{\frac{1}{2}mP}{m} \mp \left(\text{Sin. C. } \frac{\frac{1}{2}mP}{m} \right) \sqrt{-1} \right)^{-1} = \pm \sqrt{-1} & | & \left(\text{Cofin. C. } \frac{(2m-\frac{1}{2}m)P}{m} \pm \left(\text{Sin. C. } \frac{(2m-\frac{1}{2}m)P}{m} \right) \sqrt{-1} \right)^{-1} = \mp \sqrt{-1} \\
 \text{---} & | & \text{---} \\
 \left(\text{Cofin. C. } \frac{(m-1)P}{m} \mp \left(\text{Sin. C. } \frac{(m-1)P}{m} \right) \sqrt{-1} \right)^{-1} & | & \left(\text{Cofin. C. } \frac{(2m-\overline{m-1})P}{m} \pm \left(\text{Sin. C. } \frac{(2m-\overline{m-1})P}{m} \right) \sqrt{-1} \right)^{-1}
 \end{array}$$

305. Radices

$$\text{Cofin. C. } \pm \frac{r.P}{m} + \left(\text{Sin. C. } \pm \frac{r.P}{m} \right) \sqrt{-1}$$

$$\text{Cofin. C. } \pm \frac{r.3P}{m} + \left(\text{Sin. C. } \pm \frac{r.3P}{m} \right) \sqrt{-1}$$

five

$$\text{Cofin. C. } \pm \frac{r.5P}{m} + \left(\text{Sin. C. } \pm \frac{r.5P}{m} \right) \sqrt{-1}$$

$$\text{Cofin. C. } \pm \frac{r.7P}{m} + \left(\text{Sin. C. } \pm \frac{r.7P}{m} \right) \sqrt{-1}$$

$$\text{Cofin. C. } \pm \frac{r(2m-1)P}{m} + \left(\text{Sin. C. } \pm \frac{r(2m-1)P}{m} \right) \sqrt{-1}$$

$$\text{Cofin. C. } \pm \frac{r.P}{m} - \left(\text{Sin. C. } \pm \frac{r.P}{m} \right) \sqrt{-1}$$

$$\text{Cofin. C. } \pm \frac{r.3P}{m} - \left(\text{Sin. C. } \pm \frac{r.3P}{m} \right) \sqrt{-1}$$

$$\text{Cofin. C. } \pm \frac{r.5P}{m} - \left(\text{Sin. C. } \pm \frac{r.5P}{m} \right) \sqrt{-1}$$

$$\text{Cofin. C. } \pm \frac{r.7P}{m} - \left(\text{Sin. C. } \pm \frac{r.7P}{m} \right) \sqrt{-1}$$

$$\text{Cofin. C. } \pm \frac{r(2m-1)P}{m} - \left(\text{Sin. C. } \pm \frac{r(2m-1)P}{m} \right) \sqrt{-1}$$

Hae

Hae Formulae generales luculenter ostendunt Numerum. diversarum Radicum

$(-1)^{\frac{r}{m}}$ nusquam praetergredi τ dm indicis Denominatorem minimis terminis expressi, namque subsequens Radix Cofin. C. $\pm \frac{r(2m+1)P}{m} \rightarrow (\text{Sin. C.} \pm \frac{r(2m+1)P}{m}) \sqrt{-1} = \text{Cofin. C.} \pm \frac{r.P}{m} \rightarrow (\text{Sin. C.} \pm \frac{r.P}{m}) \sqrt{-1}$, a qua *Epocha* incipiens renovaretur iam absumpta periodus. Gemina, quae Arcus universos praecedunt Signa, memoratum non turbant Radicum Numerum, quia firmis manentibus, uti Trigonometria docet, Rectis-Sinubus, ac Cofinibus idem est, exempli causa, Cofin. C. $-\frac{r.3P}{m} \rightarrow (\text{Sin. C.} - \frac{r.3P}{m}) \sqrt{-1}$ atque Cofin. C. $+\frac{r(2m-3)P}{m} \rightarrow (\text{Sin. C.} + \frac{r(2m-3)P}{m}) \sqrt{-1}$, et sic de singulis, clarius idipsum oculis demonstrante pictura Schematis opportuni.

306. Indicis autem $\frac{r}{m}$ ad maiorem intelligentiam curandam triplex fieri potest hypothesis, nempe τ r Imparis et m Paris; e contra r Paris atque m Imparis; ac demum utriusque Disparis, exclusa quarta suppositione utriusque

$$(-v)^x = (-1)^x \cdot v^x = (\text{Cofin. C.} \cdot x\lambda P \pm (\text{Sin. C.} \cdot x\lambda P) \sqrt{-1}) \left(1 + \frac{(Lv)^1 x^1}{1} + \frac{(Lv)^2 x^2}{1.2} \right)$$

nisi x contra suppositum Rationalis evadat. Nec amplius dubitari poterit

de Imaginarietate expressionis $(-v)^{\frac{m}{n}}$, quam in num°. 26. ac sequentibus adesse fancivimus cum primum n fuerit Par, eoquod

$$(-v)^{\frac{m}{n}} = (-1)^{\frac{m}{n}} \cdot v^{\frac{m}{n}} = (\text{Cofin. C.} \pm \frac{m}{n} \cdot \lambda P \rightarrow (\text{Sin. C.} \pm \frac{m}{n} \cdot \lambda P) \sqrt{-1}) v^{\frac{m}{n}}; \text{ quapropter}$$

$$(-v)^{\frac{1}{2}}$$

utriusque Paris ad hanc rem non faciente. Dum prima sistatur hypo-
thesis in Formulis recensitis nulla novitas occurrit quo ad Reales Radices,
verumtamen respectu habito ad Imaginarias, quando m fuerit Impariter-Par,

$$\text{adnotanda manet Formula } \text{Cofin. C.} \pm \frac{r \cdot \frac{1}{2} mP}{m} \rightarrow \left(\text{Sin. C.} \pm \frac{r \cdot \frac{1}{2} mP}{m} \right) \sqrt{-1}$$

$$= \pm \sqrt{-1}. \text{ Ad haec in altera hypothesi habebitur } \text{Cofin. C.} \pm \frac{r \cdot mP}{m} \rightarrow$$

$$\left(\text{Sin. C.} \pm \frac{r \cdot mP}{m} \right) \sqrt{-1} = 1, \text{ reliquis Radicibus in numero pari } m - 1$$

existentibus Imaginariis. Accessio quoque hypothesis tertiae novas aperit

$$\text{formas: fit etenim } \text{Cofin. C.} \pm \frac{r \cdot mP}{m} \rightarrow \left(\text{Sin. C.} \pm \frac{r \cdot mP}{m} \right) \sqrt{-1} = -1. \text{ Hi-}$$

scce peractis ponatur x quivis Numerus asymmetricus, five etiam Transcen-

dens loco Exponentis in Potestate $(-1)^x$, eruntque sine numero Radices

ab oecumenica Formula comprehensae $\text{Cofin. C.} \pm x \cdot \lambda P \rightarrow (\text{Sin. C.} \pm x \cdot \lambda P) \sqrt{-1}$,

quae idcirco resolutionem tribuit facillimam Aequationis Exponentialis non

affectae $z^{\frac{1}{x}} \rightarrow 1 = 0$, et variis adplicata casibus lacunas complet consulto
antea relictas. Ita

$$(-v)^x$$

$$+ \frac{(Lv)^3 x^3}{1 \cdot 2 \cdot 3} + \frac{(Lv)^4 x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{(Lv)^5 x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \&c:), \text{ Functio Infinitiformis,}$$

nisi x

$$(-v)^{\frac{1}{2}} = v^{\frac{1}{2}} \left(\text{Cofin. C.} \pm \frac{P}{2} \rightarrow (\text{Sin. C.} \pm \frac{P}{2}) \sqrt{-1} \right) = v^{\frac{1}{2}} \cdot \pm \sqrt{-1}, \text{ Biformis Imaginaria Functio;}$$

$$\left(\text{Cofin. C.} \pm \frac{3P}{2} \rightarrow (\text{Sin. C.} \pm \frac{3P}{2}) \sqrt{-1} \right)$$

P p p p

$$(-z)^{\frac{1}{2r}}$$

$$(-z)^{\frac{1}{2r}} = (-1)^{\frac{1}{2r}} \cdot z^{\frac{1}{2r}} = z^{\frac{1}{2r}} \left(\text{Cofin. } C. \pm \frac{\lambda P}{2r} + \left(\text{Sin. } C. \pm \frac{\lambda P}{2r} \right) \sqrt{-1} \right) =$$

$$(-z)^{\frac{2m-1}{2n}} = (-1)^{\frac{2m-1}{2n}} \cdot z^{\frac{2m-1}{2n}} = \left(z^{\frac{2m-1}{2n}} \right) \left(\text{Cofin. } C. \pm \frac{(2m-1)\lambda P}{2n} + \right.$$

$$\text{necnon } (-z)^{x\sqrt{-1}} = C^{x(L(-z))(\sqrt{-1})} = C^{(x\sqrt{-1})(Lz \pm \lambda P\sqrt{-1})} =$$

Functio nimirum Realis Infinitinomia. Ab ista profluit cum ea numⁱ 300. consentanea satis Aequatio

$$(-1)^{\sqrt{-1}} = \begin{matrix} P & 3P & 5P & 7P & 9P & 11P & 13P & 15P & 17P & 19P & 21P & 23P \\ C, & C, & C, & C, & C, & C, & C, & C, & C, & C, & C, & C \end{matrix} \&c; \\ \begin{matrix} -P & -3P & -5P & -7P & -9P & -11P & -13P & -15P & -17P & -19P & -21P & -23P \\ C, & C, & C, & C, & C, & C, & C, & C, & C, & C, & C, & C \end{matrix} \&c;$$

quorum Exponentialium Indices aliorum intervallis admodum bifectis inferuntur

$$\begin{matrix} 0 & 2P & 4P & 6P & 8P & 10P & 12P & 14P & 16P & 18P & 20P & 22P \\ C, & C, & C, & C, & C, & C, & C, & C, & C, & C, & C, & C \end{matrix} \&c; \\ \begin{matrix} -0 & -2P & -4P & -6P & -8P & -10P & -12P & -14P & -16P & -18P & -20P & -22P \\ C, & C, & C, & C, & C, & C, & C, & C, & C, & C, & C, & C \end{matrix} \&c; .$$

Itaque $(+1)^{\sqrt{-1}} = C^{\pm 2nP}$, atque $(-1)^{\sqrt{-1}} = C^{\pm (2n-1)P}$, scilicet Numero Infinitiformi, cuius Logarithmi positivo-negativi a Nullitate initium sumentes per intervalla Peripheriae Circulari Aequalia progrediantur.

Supervacaneum monitum duco, quod ad Aequationem $(-1)^{x\sqrt{-1}} - z = 0$, prouti in num^o. praecitato, infinities variae, sed omnes Reales

pertineant Radices. In aperto quoque est Radices Infinities diversas $(-1)^{\frac{1}{\infty}}$ vel Aequationis $x^{\infty} + 1 = 0$, complecti formulam oecumenicam Cofin. C.

$$\pm \frac{\lambda P}{\infty} + \left(\text{Sin. } C. \pm \frac{\lambda P}{\infty} \right) \sqrt{-1}, \text{ five } 1 - \frac{\lambda^2 P^2}{2 \cdot \infty^2} \pm \frac{\lambda P}{\infty} \sqrt{-1}, \text{ nimirum}$$

$$1 \pm \frac{\lambda P}{\infty} \sqrt{-1}, \text{ scilicet quae in singulas evolvitur}$$

$$z^{\frac{1}{2r}} \left(\text{Cofin. C.} \pm \frac{\lambda P}{2} + (\text{Sin. C.} \pm \frac{\lambda P}{2}) \sqrt{-1} \right)^{\frac{1}{r}} = (\pm \sqrt{-1})^{\frac{1}{r}} \cdot z^{\frac{1}{2r}}, \text{ Functio Imaginaria } \frac{1}{2r} \text{ formis;}$$

$$\left(\text{Sin. C.} \pm \frac{(2m-1)\lambda P}{2n} \right) \sqrt{-1}, \text{ five } (\pm \sqrt{-1})^{\frac{2m-1}{n}} \cdot z^{\frac{2m-1}{2n}}, \text{ nempe Functio } \frac{2m-1}{2n} \text{ formis Imaginaria;}$$

$$C^{Lz(x\sqrt{-1})} = x\lambda P; \text{ quamobrem } (-1)^{x\sqrt{-1}} = C^{\mp x\lambda P} = \text{Cofin. H. } x\lambda P \mp \text{Sin. H. } x\lambda P,$$

Functio nimi-

$$1 \pm \frac{1 \cdot P}{\infty} \cdot \sqrt{-1} = 1$$

$$1 \pm \frac{3 \cdot P}{\infty} \cdot \sqrt{-1} = 1$$

$$\text{Veras Radices Analyticas } 1 \pm \frac{5 \cdot P}{\infty} \cdot \sqrt{-1} = 1 \quad \text{Pseudo-Radices Arithmeticas}$$

$$1 \pm \frac{7 \cdot P}{\infty} \cdot \sqrt{-1} = 1$$

$$1 \pm \frac{\lambda \cdot P}{\infty} \cdot \sqrt{-1} = 1$$

sed omnes Imaginarias, nisi velimus $\tau \theta \infty$ prouti Limitem Numerorum Integrorum Imparium excogitare, qua in hypothesi $\text{Cofin. C.} \pm \frac{\infty P}{\infty} + (\text{Sin. C.} \pm \frac{\infty P}{\infty}) \sqrt{-1} = -1$ aliis Radicibus in calce accessisset.

307. Sunt qui arbitrentur non pauci *Casum* Aequationum tertii Gradus *Irreducibilem* in tanta Analyseos luce aequae obscurum uti antiquitus delitescere, nec a primo Algebrae aevo ad haec tempora meliorem subiisse fortunam. Universalis ipse Matheseos Historiographus ait in II. Volumine difficul-

difficultatem illam Cubicarum Aequationum celeberrimam eodem adhuc robore directis methodis rebellem obistere, qualem Vieta eius solutionem tantum modo Linearem nactus anno MDXCH.; pleniusque Girardus anno MDCXXIX. ope Trisectionis Circularis Arcus reliquere. Factum exinde est, quod obli- quas peramaverint semitas Analystae, et ad Series demum Infinitas confu- gerint, quos omnes antecelluit Nicolius in *Parisiensibus Commentariis* anni MDCCXXXVIII. quaerens pro *Casu Irreductibili* expressionem triplicis Ra- dicis analyticam eliminatis Imaginariis. Huic tamen eliminationi nec Series sufficere, quia Summationis expertes, nec ingeniosa alia quaedam molimina prodesse visa sunt, inter quae perinsigne fuit illud Harrioti, non satis no-

tum, substitutionis $z \rightarrow \frac{b^2}{z}$ in Aequatione $x^3 + 3b^2x - 2c^3 = 0$ loco

incognitae x , testante Ioanne Vallisio. Quin, etiam doctissimi *Encyclopediae* Gallicae Editores in Articulo *Casus Irreductibilis* adeo desperatum huius disquisitionis exitum adfirmarunt, ut aera numerata ab inventis Bombellii ab anno scilicet MDLXXIX., Geometrae nec hilum profecerint, et labor omnis impensus post duorum pene saeculorum intervallum irritus fuerit. Nisi autem mea me fallat opinio in sententiam eo iamdudum maximam pos- sibilem perfectionem adiecisse Analystas Cardanicae expressioni praedicti *Ca- sus Irreductibilis*, Imaginarietatem omnigenam feliciter oblitterasse, Series- que huic rei idoneas in Summam coegisse, quantum Algebrae viribus da- tum erat, canonicam, defectusque, si qui sint adhuc manentes, non ab Analyseos vitio, sed a Radicum inveniendarum peculiari charactere ortum ducere. Hae quippe per raro usitatum Algebraicum quinimo Trigonome- trico stylo tantummodo generaliter exprimibilem, aut Formula finita spuriam Imaginarietatem comprehendente, vel Serie Infinita Algebraicae Summationis saepius experte, utut non Transcendentem, nec indefinito Cir- culi vel Hyperbolae Tetragonismo illaqueatum, habent valorem. Eadem sane detinuit opinio incomparabilem Dalembercium, cum scriberet Disqui-

tionem ^{tam} 34. quinti Opusculorum Voluminis; at multocius ante Alge- braico more tractatam ab Eulero Trigonometriam res in comperto fuisset dummodo Binomii Neutoniani amplissimum usum impensius perspexissent Geo- metrae. Argumentum resolutionis Aequationum, quod est singularis momen- ti in Analyticis disciplinis, tametsi a tot tantisque Scriptoribus exornatum

perpolire

perpolire aliquantulum, et compendio continere superius iacta principia me non parum sollicitant. Sublimiora autem linquo, quae gratiam, seu novitatem non exuerint, de natura Aequationum, ipsarumque aliis methodis ducibus evolutione, iis, quos gloria maneat longe lateque propagandi Fontainii, Bezoutii, Uaringii, Euleri, LaGrangii, Vandermondi, &c: subtilissimas investigationes *Memorabilibus* insertas Academiarum Parisinae, Petropolitanae, ac Berolinensis, *Meditationibus nuperrimis Algebraicis* &c:, et immenso quasi intervallo distitas a primoribus lineamentis Canonum Huddenianorum, vel potius Merreyani MS. in Oxoniensi Bibliotheca adfervati.

308. In forma expressionis Trigonometricae, num. 233. ac 236. impensius animadversa, maioris simplicitatis gratia sit $\text{Cofin. } C.x = y$, et $\text{Cofin. } C.zx = A$, necnon $\text{Cofin. } H.zx = B$, atque $\text{Cofin. } H.x = v$, fluetque, supposito Coefficiente numerico $z=3$,

1^{um}. casum $y^3 - \frac{3y}{4} - \frac{A}{4} = 0$ Aequatio generalis Irreductibilis, quoad

2^{um}. casum $v^3 - \frac{3v}{4} - \frac{B}{4} = 0$ Aequatio analogica Reductibilis;

et hae duae Aequationes in hoc solummodo differunt, videlicet, quod $A < 1$, et e-contra $B > 1$, medio interiacente valore $A=1=B$, tum cum ambae Aequationes discrimine omni sublato in unam coalescunt. Nec mireris illos Cubicarum Aequationum Typos generalium munere fungi, cum revera, si Radius vel Semi-latus Transversum R sedem occupet implicitae Unitatis, Signumque apte introducatur positivo-negativum Cofinubus semper additum Circularibus et Hyperbolicis, ii formam acquirant

$$y^3 - \frac{3R^2y}{4} \mp \frac{R^2A}{4} = 0$$

$$v^3 - \frac{3R^2v}{4} \mp \frac{R^2B}{4} = 0,$$

quod idem de sequentibus intellectum velim absque nova singularis mentionis necessitate. Ut Limitem in primis, qui Aequationum eiusdem formae

Qqqq

intimum

intimum separat sensum, contemplemur, versioque unius in alteram facilius elucescat, proderit non leviter memoriae rursus commendare universales Ae-

$$\text{qualitates Cofin. C. } x = y = \frac{(\text{Cofin. C. } 3x + (\text{Sin. C. } 3x) \sqrt{-1})^{\frac{1}{3}}}{2} +$$

$$\frac{(\text{Cofin. C. } 3x - (\text{Sin. C. } 3x) \sqrt{-1})^{\frac{1}{3}}}{2}, \text{ atque eodem pacto Cofin. H. } x = v =$$

$$\frac{(\text{Cofin. H. } 3x + (\text{Sin. H. } 3x)^{\frac{1}{3}})^{\frac{1}{3}}}{2} + \frac{(\text{Cofin. H. } 3x - (\text{Sin. H. } 3x)^{\frac{1}{3}})^{\frac{1}{3}}}{2}, \text{ prouti constat ex}$$

iam notis Formulis Trigonometricis. Igitur alterutra Aequatio, vixdum

$$A = B = R = 1, \text{ tribuet } y = v = \frac{(\pm 1)^{\frac{1}{3}} + (\pm 1)^{\frac{1}{3}}}{2},$$

$$\frac{\pm 1 \pm \sqrt{-3}}{2}, \frac{\pm 1 \mp \sqrt{-3}}{2}, \frac{\pm 1 \mp \sqrt{-3}}{2}, \frac{\pm 1 \pm \sqrt{-3}}{2} = \pm 1, \mp \frac{1}{2}, \mp \frac{1}{2}, \text{ quas}$$

ipsas Rationales Radices longiori Calculi labore invenit Nicolius, quippe qui non ad ea tempora viderit suae Formulae evolutionem illis ad Multifsectionem Angulorum, et Logarithmorum pertinentibus adamussim congruere. Obiter moneo, quanta adseruerit iniuria Montuclas in praelaudato II°. celeberrimae *Historiae* Volumine (*) Radices Imaginarias ambiguas semper esse, idest positivas aut negativas existimari ad libitum simul posse: nisi etenim ipsarum Signum vel qualitas rite recteque fuisset superius distributa, prouti Aequationis *Data* patiebantur, gravissimi errores hac in quaestione lapsi veritatem Radicum maximopere perturbassent.

309. Universalius vero rem peragentibus obviam fit $y =$

$$\frac{\text{Cofin. C. } \frac{3x}{3} + (\text{Sin. C. } \frac{3x}{3}) \sqrt{-1}}{2} + \frac{\text{Cofin. C. } \frac{3x}{3} - (\text{Sin. C. } \frac{3x}{3}) \sqrt{-1}}{2} =$$

$$\text{Cofin. C. } x = \text{Cofin. } \left(\frac{\text{Arc. C. Cofin. } \pm A}{3} \right), \text{ et vi familiaris Harmonices Circuli}$$

(*) Parte IV^a. Lib. 2°. Art. 4°.

culi, atque Aequilaterae Hyperbolae $v = \text{Cofin.} \left(\frac{(\text{Arc. C. Cofin.} \pm B) \sqrt{-1}}{3} \right)$

$$= \text{Cofin. H. } x = \text{Cofin. C. } x \sqrt{-1} = \frac{\text{Cofin. H. } \frac{3x}{3} + \text{Sin. H. } \frac{3x}{3}}{2} +$$

$$\frac{\text{Cofin. H. } \frac{3x}{3} - \text{Sin. H. } \frac{3x}{3}}{2}. \text{ Igitur tam Radix Cofin. H. } x, \text{ five Cofin. C. } x \sqrt{-1},$$

quam Radix altera Cofin. C. x sunt Reales; verumtamen, quum ab expres-

sione unius, nempe $\frac{1}{2} (\text{Cofin. C. } 3x + (\text{Sin. C. } 3x) \sqrt{-1})^{\frac{1}{3}} + \frac{1}{2} (\text{Cofin. C.}$

$3x - (\text{Sin. C. } 3x) \sqrt{-1})^{\frac{1}{3}}$, aut $\frac{1}{2} (\pm A + \sqrt{1-A^2} \cdot \sqrt{-1})^{\frac{1}{3}} + \frac{1}{2} (\pm A$

$- \sqrt{1-A^2} \cdot \sqrt{-1})^{\frac{1}{3}}$ ad alteram $\frac{1}{2} (\text{Cofin. H. } 3x + \text{Sin. H. } 3x)^{\frac{1}{3}} +$

$\frac{1}{2} (\text{Cofin. H. } 3x - \text{Sin. H. } 3x)^{\frac{1}{3}}$, five $\frac{1}{2} (\text{Cofin. C. } 3x \sqrt{-1} - (\text{Sin. C. } 3x$

$\sqrt{-1}) \sqrt{-1})^{\frac{1}{3}} + \frac{1}{2} (\text{Cofin. C. } 3x \sqrt{-1} + (\text{Sin. C. } 3x \sqrt{-1}) \sqrt{-1})^{\frac{1}{3}}$, scili-

cet $\frac{1}{2} (\pm B + \sqrt{1-B^2} \cdot \sqrt{-1})^{\frac{1}{3}} + \frac{1}{2} (\pm B - \sqrt{1-B^2} \cdot \sqrt{-1})^{\frac{1}{3}}$ aditus

non pateat praeterquam adiumento Symboli Imaginarietatis, quod Reales valores, ubicunque adsint, nihil immutat, debeatque unica Algebraica Expressio communi continere formae ac societatis vinculo geminae Aequationis eodem typo praeditae Radices, necesse est alterutram affici Signo $\sqrt{-1}$. Quisquis incommodae Imaginarietati auferendae remedium quaereret in praecipua Analyseos peccaret principia, quae Lege praemissorum inviolabili discernunt unam tantum ex Formulis Radicum, de quibus sermo, ab apparentibus Imaginariis non perturbatum iri. Id commodi, lata sententia inappellabili, contigit casui $B > 1$, quo fit $\sqrt{1-B^2} \cdot \sqrt{-1} = \sqrt{B^2-1}$, quantitati nempe Reali, videlicet quando Cofinus, Sinusque Hyperbolici, vel trisectio Logarithmi Naturalis, aut duo Mediarum Proportionalium inventio, in

tio, in subsidium vocentur, negatumque consequenter fuit hypothefi adverfae $A < I$, fcilicet illi, cuius proprium eft $\tau\delta \sqrt{1-A^2} \cdot \sqrt{-1} = \sqrt{A^2-1}$, quantitati nimirum Imaginariae, et cui ope Signi Imaginarietatis Cofinus Sinusque Hyperbolici in Circulares verfi, ac trifectione Logarithmi in Anguli trifectionem mutata respondent. Tempus itaque oleamque perdunt, fruſtraque improbo labore cruciantur univerfi minus habentes Analyſtae, qui falva univerſalitate Cardanicam expreſſionem *Cafus Irreductibilis* praefumant reddere, ut aiunt, eliminatis Imaginariis abſque Infinitarum Serierum methodo, et Circularis Trigonometriae ſuppetiis tractabiliorem; haec enim mendosa ratiocinatio concluderet variationem etiam conſimilem in Formula Radicum ad *Cafum Reductibilem* pertinentium. Iſtud autem poſtremum aequè abſurdum videtur ac alia forma donare generalem Radicum Quadraticarum expoſitionem, Irrationalitatem vincere perſaepe moleſtam in Radicum Formulis univerſalibus dum indirectis methodis ſpeciali caſu invenirentur Rationales, Algebraicas vel Tranſcendentes ſexcentas Functiones finitis numero terminis inexprimibiles, niſi forte fortuna per Imaginariorum commixtionem, communi Finitorum Calculo ſubigere, nexus deſtruere, quos Natura poſuit aeternos Hyperbolen inter ac Circulum, Veritatem, ſi ſuperis placeat, caſtigare, denegata quamvis artificii potentia ab intimis rei proprietatibus, aliaque commenta moliri Algebrae labefactantia Leges, quae nuſquam ulli fas erit efficere, ut unicum, vel ſingulares ſolum eventus prae tota ac neceſſaria poſſibilitatis plenitudine reſpiciant.

310. Radices ergo Aequationis $y^3 - \frac{3R^2y}{4} \mp \frac{R^2A}{4}$ diverſimode, ſarta tecta earum magnitudine, ſic poterant exhiberi, nimirum R Cofin.

$$\left(\frac{\text{Arc. C. Cofin.} \pm \frac{A}{R}}{3} \right), R. \text{ Cofin.} \left(\frac{\Pi \mp \text{Arc. C. Cofin.} \pm \frac{A}{R}}{3} \right), R. \text{ Cofin.}$$

$$\left(\frac{2\Pi \mp \text{Arc. C. Cofin.} \pm \frac{A}{R}}{3} \right), \text{ vel } R. \text{ Cofin.} \left(\frac{\text{Arc. C. Cofin.} \pm \frac{A}{R}}{3} \right), R. \text{ Cofin.}$$

$$\left(\frac{2\Pi \mp \text{Arc. C. Cofin.} \pm \frac{A}{R}}{3} \right), R. \text{ Cofin.} \left(\frac{\Pi \mp \text{Arc. C. Cofin.} \pm \frac{A}{R}}{3} \right), \text{ aut,}$$

poſito

posito Radio = 1 simplicitatis gratia, $\frac{1}{2} \cdot (1)^{\frac{1}{3}} (\pm A + \sqrt{A^2 - 1})^{\frac{1}{3}} +$

$\frac{1}{2} \cdot (1)^{\frac{1}{3}} (\pm A - \sqrt{A^2 - 1})^{\frac{1}{3}}$, videlicet $1 \left(\frac{1}{2} (\pm A + \sqrt{A^2 - 1})^{\frac{1}{3}} + \right.$

$\left. \pm A - \sqrt{A^2 - 1} \right)^{\frac{1}{3}} \Bigg), \frac{-1 - \sqrt{-3}}{2} \left(\frac{1}{2} (\pm A + \sqrt{A^2 - 1})^{\frac{1}{3}} \right)$

$\frac{-1 + \sqrt{-3}}{2} \left(\frac{1}{2} (\pm A - \sqrt{A^2 - 1})^{\frac{1}{3}} \right)$, et $\frac{-1 + \sqrt{-3}}{2}$

$\left(\frac{1}{2} (\pm A + \sqrt{A^2 - 1})^{\frac{1}{3}} \right) \frac{-1 - \sqrt{-3}}{2} \left(\frac{1}{2} (\pm A - \sqrt{A^2 - 1})^{\frac{1}{3}} \right)$, omnes

triplici ista forma Reales. Hoc, cum nequaquam de prima et altera, de po-

strema brevissime sic demonstratur. Est namque $\pm A \pm \sqrt{A^2 - 1}^{\frac{1}{3}} = \text{Cofin.}$

C. x $\pm (\text{Sin. C. x}) \sqrt{-1}$, ac subinde dimanant

$$1 \left(\frac{1}{2} (\pm A + \sqrt{A^2 - 1})^{\frac{1}{3}} + \frac{1}{2} (\pm A - \sqrt{A^2 - 1})^{\frac{1}{3}} \right) = \frac{1}{2} \text{Cofin. C. x} +$$

$$\frac{1}{2} (\text{Sin. C. x}) \sqrt{-1} + \frac{1}{2} \text{Cofin. C. x} - \frac{1}{2} (\text{Sin. C. x}) \sqrt{-1} = \text{Cofin. C. x}$$

$$\frac{-1-\sqrt{-3}}{2} \cdot \frac{1}{2} (\text{Cofin. C. x} + (\text{Sin. C. x}) \sqrt{-1}) \frac{-1+\sqrt{-3}}{2} \cdot \frac{1}{2} (\text{Cofin. C. x} - (\text{Sin. C. x}) \sqrt{-1})$$

$$\frac{-1+\sqrt{-3}}{2} \cdot \frac{1}{2} (\text{Cofin. C. x} + (\text{Sin. C. x}) \sqrt{-1}) \frac{-1-\sqrt{-3}}{2} \cdot \frac{1}{2} (\text{Cofin. C. x} - (\text{Sin. C. x}) \sqrt{-1})$$

cuius methodi ope non solum omnis eliminatur Imaginarietas, verum etiam, quod pene omnes negligunt, terni diducuntur Realium Radicum valores

$$\text{Cofin. C. x}, \frac{-\text{Cofin. C. x} + \sqrt{3} \cdot \text{Sin. C. x}}{2}, \frac{-\text{Cofin. C. x} - \sqrt{3} \cdot \text{Sin. C. x}}{2}. \text{Ho-}$$

rumce Summa, prouti Aequatio exigit, Nullitatem peraequat; et Trianguli Aequilateri inscriptione mirum iucundumque est, quantam haec omnia in Circulo nitorem, ac facilitatem, Aequationum Cubicarum affectionibus uno oculi ictu intelligendis, et demonstrandis idoneam adipiscantur. Criterium etiam emergit, quo nosci tutissime possit, an ternae in Aequatione Cubica proposita Radices sint omnes Reales: ad hoc enim requiritur, ut

$$\left(\mp \frac{R^2 A}{4}\right)^2 < \left(\frac{R^3}{4}\right)^2 < \left(\frac{1}{3} \cdot \frac{3R^2}{4}\right)^3 \cdot 4 \text{ ideoque } \frac{\left(\mp \frac{R^2 A}{4}\right)^2}{4} < \frac{\left(\frac{3R^2}{4}\right)^3}{27}; \text{ quod}$$

recte congruit usitatae Analyistarum regulae praecipienti in generali Aequa-

$$\text{tione } y^3 - py \pm q \text{ Casui Irreductibili subiectae symptomatis } \frac{q^2}{4} < \frac{p^3}{27} \text{ ne-}$$

cessitatem. Nec minus firmanur facillime quae tanta Calculorum molestia in praenotatis *Commentariis* subiunxit Nicolius. Ac primum sit Aequatio tertii Ordinis adeo comparata, ut formam habeat

$$z^3 - pz + \left(\frac{p-d^2}{3\sqrt{3}}\right)(\sqrt{4p-d^2}) = 0, \text{ vel } z^3 - pz + \left(\frac{p-d^2}{3}\right)\left(\frac{\sqrt{4p-d^2}}{\sqrt{3}}\right) = 0;$$

et nullus dubito, quominus quisque videat, permutatis ad nutum valoribus p atque d , typum illum expositum *Casui Irreductibili* adscriptum fore. Profecto debet semper esse $p > \frac{d^2}{4}$, ne insurgat in postremo termino Imagina-

$$\text{rietas, et, sicuti in } q^2 = \frac{4p^3}{27} - \frac{+2pd^2 + 4pd^4 - 8p^2d^3 - p^3d^2}{27}, \text{ ob } p > \frac{d^2}{4},$$

negativus

$$= -\frac{1}{2} \text{Cofin. C. } x - \frac{\sqrt{3}}{2} \cdot \text{Sin. C. } x = -\frac{\sqrt{3}}{2} \cdot \sqrt{1 - (\text{Cofin. C. } x)^2} - \frac{1}{2} \cdot \text{Cofin. C. } x$$

$$= -\frac{1}{2} \text{Cofin. C. } x + \frac{\sqrt{3}}{2} \cdot \text{Sin. C. } x = +\frac{\sqrt{3}}{2} \cdot \sqrt{1 - (\text{Cofin. C. } x)^2} - \frac{1}{2} \cdot \text{Cofin. C. } x,$$

cuius me-

negativus semper manebit Fractionis alterius Numerator, fiet $q^2 = \frac{4p^2}{27}$

M, vel $q^2 < \frac{4p^2}{27}$, aut $\frac{q^2}{4} < \frac{p^2}{27}$ iuxta Legem superius proditi experimen-

ti. Calculus autem vel primo intuitu demonstrat praecedens Aequationis

Membrum nullefcere tum quum vice z ponatur $-\sqrt{\frac{4p-d^2}{3}}$; namque eli-

sis omnibus terminis obtinetur $-\left(\frac{4p-d^2}{3}\right)\sqrt{\frac{4p-d^2}{3}} + p\sqrt{\frac{4p-d^2}{3}} +$

$\left(\frac{p-d^2}{3}\right)\sqrt{\frac{4p-d^2}{3}} = 0$. Igitur ex tribus Realibus Radicibus una erit

$$\frac{(4p-d^2)^{\frac{1}{2}}}{\sqrt{3}}, \text{ geminaeque remanentes } \left(\frac{-1 \pm \sqrt{-3}}{2}\right)^{\frac{1}{2}} \left(-\sqrt{\frac{4p-d^2}{3}} + \frac{d}{\sqrt{3}} \cdot \sqrt{-1}\right) + \left(-\frac{1}{2} \mp \frac{\sqrt{-3}}{2}\right)^{\frac{1}{2}} \left(-\sqrt{\frac{4p-d^2}{3}} - \frac{d}{\sqrt{3}} \cdot \sqrt{-1}\right) =$$

$$+\frac{1}{2}\sqrt{\frac{4p-d^2}{3}} \mp \frac{1}{2}d, \text{ uti eruitur a Formula Cofin. C. } x = -\sqrt{\frac{4p-d^2}{3}},$$

quae, cum Radius R sit $\sqrt{\frac{4p}{3}}$ vi Terminorum homologorum comparationis

$$p = \frac{3R^2}{4}, \text{ tribuit } \pm (\text{Sin. C. } x) \sqrt{-1} = \pm \frac{d \cdot \sqrt{-1}}{\sqrt{3}}. \text{ Confimili facilitate ii,}$$

qui Algebraicum Algorithmum etiam non ultra Elementa tractare fueverint, luculenter perspicient formam alterius Cubicae Aequationis

$$z^3 - d^2 z + \left(\frac{4p-4d^2}{3\sqrt{3}}\right)\left(\sqrt{\frac{4p-d^2}{3}}\right), \text{ aut } z^3 - d^2 z + \left(\frac{4p-d^2}{3}\right)\left(\sqrt{\frac{4p-d^2}{3}}\right) = 0$$

eamdem Radicem admittere $-\sqrt{\frac{4p-d^2}{3}}$, quippe quum se invicem destruant

struant $-\left(\frac{4p-d^2}{3}\right)\left(\sqrt{\frac{4p-d^2}{3}}\right) + d^2\left(\sqrt{\frac{4p-d^2}{3}}\right) + \left(\frac{4p-4d^2}{3}\right)\left(\sqrt{\frac{4p-d^2}{3}}\right)$, et

idcirco, obtentis $R = \frac{2d}{\sqrt{3}}$, ac Cofin. C. $x = -\sqrt{\frac{4p-d^2}{3}}$, fluet Sin. C. $x =$

$\sqrt{\frac{5d^2-4p}{3}}$, nimirum reliquae manabunt Radices ope Aequationis

$$\frac{1}{2}\left(\frac{-1 \pm \sqrt{-3}}{2}\right)\left(-\sqrt{\frac{4p-d^2}{3}} + \sqrt{\frac{5d^2-4p}{3}} \cdot \sqrt{-1}\right) + \frac{1}{2}\left(\frac{-1 \mp \sqrt{-3}}{2}\right)\left(-\sqrt{\frac{4p-d^2}{3}} - \sqrt{\frac{5d^2-4p}{3}} \cdot \sqrt{-1}\right) =$$

$$\frac{1}{2}\sqrt{\frac{4p-d^2}{3}} \mp \frac{1}{2}\sqrt{\frac{5d^2-4p}{3}} \text{ in quarum investigatione totus pene consistit Ni-}$$

colii labor obliquo indirectoque artificio innixus Cardanicas ab Imaginarie-
tatis vitio Formulas liberandi, seu inversum solvendi Problema *Casus Irre-*
ductibilis.

311. Ostendit demum ipse Nicolius, non obscure idem multo antea
praedicente Leibnitio, (*) $\left(\frac{a}{b} + \sqrt{-1}\right)^n + \left(\frac{a}{b} - \sqrt{-1}\right)^n$ non modo

seriem praebere ex finito terminorum Numero existentem, neque ab Imagi-
nariis turbatam dum Index n fuerit Integer positivus, verum etiam tribue-
re Seriem Infinitam, at Summationis Algebraicae capacem, quando n fue-
rit integer negativus, specimenque dat in hypothesi $n = -3$, qua adferit esse

$$\left(\frac{a}{b} + \sqrt{-1}\right)^{-3} + \left(\frac{a}{b} - \sqrt{-1}\right)^{-3} = 2 \left(\frac{\left(\frac{a}{b}\right)^3 - 3\left(\frac{a}{b}\right)}{\left(\left(\frac{a}{b}\right)^2 + 1\right)^3} \right); \text{ idque quanta sim-}$$

plicitate a nostris deducatur principiis paulisper contemplari conveniet.
Porro est universaliter Cofin. C. $nx = \text{Cofin. C.} - nx$; qua de re Cofin.
C. $-3x = \text{Cofin. C. } 3x$. Atqui ex praecedentibus constat esse

$$\frac{1}{2} \left(\frac{a + b\sqrt{-1}}{(a^2 + b^2)^{\frac{1}{2}}} \right)^{-3} + \frac{1}{2} \left(\frac{a - b\sqrt{-1}}{(a^2 + b^2)^{\frac{1}{2}}} \right)^{-3} = \text{Cofin. C.} - 3x = \text{Cofin. C. } 3x$$

$$= \frac{1}{2}$$

(*) In eius Epistola anni MDCLXXVI. edita inter alias *Commercii Epistolici Col-*
linfsani pag. 63. 64. 95. Vide Num.^{um} 322.

$$= \frac{1}{2} \left(\frac{a + b\sqrt{-1}}{(a^2 + b^2)^{\frac{1}{2}}} \right)^3 + \frac{1}{2} \left(\frac{a - b\sqrt{-1}}{(a^2 + b^2)^{\frac{1}{2}}} \right)^3 = 4 (\text{Cofin. C. } x)^3 -$$

$$3 (\text{Cofin. C. } x) = \frac{4a^3}{(a^2 + b^2)^{\frac{3}{2}}} - \frac{3a}{(a^2 + b^2)^{\frac{1}{2}}} = \frac{a^3 - 3b^2 a}{(a^2 + b^2)^{\frac{3}{2}}}; \text{ unde consequi-}$$

$$\text{tur altera Aequatio } \left(\frac{a}{b} + \sqrt{-1} \right)^3 + \left(\frac{a}{b} - \sqrt{-1} \right)^3 = \left(\frac{a}{b} + \sqrt{-1} \right)^{-3}$$

$$+ \left(\frac{a}{b} - \sqrt{-1} \right)^{-3} = \left(\frac{a + b\sqrt{-1}}{(a^2 + b^2)^{\frac{1}{2}}} \right)^{-3} \times \left(\frac{(a^2 + b^2)^{\frac{1}{2}}}{b} \right)^{-3} +$$

$$\left(\frac{a - b\sqrt{-1}}{(a^2 + b^2)^{\frac{1}{2}}} \right)^{-3} \times \left(\frac{(a^2 + b^2)^{\frac{1}{2}}}{b} \right)^{-3} = 2 \left(\frac{\left(\frac{a^3}{b^3} - \frac{3b^2 a}{b^3} \right)}{\left(\frac{a^2}{b^2} + \frac{b^2}{b^2} \right)^3} \right). \text{ Dum enim}$$

$$\text{Summa reperta } \frac{a^3 - 3b^2 a}{(a^2 + b^2)^{\frac{3}{2}}} \text{ multiplicetur per Fractionem } \left(\frac{(a^2 + b^2)^{\frac{1}{2}}}{b} \right)^{-3}$$

$$\text{superius adlatam proveniet } \left(\frac{a^3 - 3b^2 a}{a^2 + b^2} \right) \left(\frac{(a^2 + b^2)^{\frac{1}{2}}}{b} \right)^{-3} = \frac{a^3 b^3 - 3b^5 a}{(a^2 + b^2)^3}$$

$$= \frac{\left(\frac{a}{b} \right)^3 - \frac{3a}{b}}{\left(\left(\frac{a}{b} \right)^2 + 1 \right)^3}, \text{ cuius duplum} = 2 \frac{\left(\frac{a}{b} \right)^3 - 3 \left(\frac{a}{b} \right)}{\left(\left(\frac{a}{b} \right)^2 + 1 \right)^3}, \text{ quemadmodum}$$

demonstrandum suscepi.

$$312. \text{ Sicuti vero } 2 \text{ Cofin. C. } - nx = \left(\frac{a + b\sqrt{-1}}{(a^2 + b^2)^{\frac{1}{2}}} \right)^{-n} +$$

$$\left(\frac{a - b\sqrt{-1}}{(a^2 + b^2)^{\frac{1}{2}}} \right)^{-n} = 2 \text{ Cofin. C. } + nx = \left(\frac{a + b\sqrt{-1}}{(a^2 + b^2)^{\frac{1}{2}}} \right)^n + \left(\frac{a - b\sqrt{-1}}{(a^2 + b^2)^{\frac{1}{2}}} \right)^n$$

$$= 2 \left(\left(\frac{a}{(a^2 + b^2)^{\frac{1}{2}}} \right)^n - \frac{n \cdot n - 1}{1 \cdot 2} \left(\frac{a}{(a^2 + b^2)^{\frac{1}{2}}} \right)^{n-2} \left(\frac{b}{(a^2 + b^2)^{\frac{1}{2}}} \right)^2 +$$

S s s s

$\frac{n \cdot n - 1}{1 \cdot 2}$

$$\frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} \left(\frac{a}{(a^2 + b^2)^{\frac{1}{2}}} \right)^{n-4} \left(\frac{b}{(a^2 + b^2)^{\frac{1}{2}}} \right)^4 -$$

$$\frac{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4 \cdot n-5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \left(\frac{a}{(a^2 + b^2)^{\frac{1}{2}}} \right)^{n-6} \left(\frac{b}{(a^2 + b^2)^{\frac{1}{2}}} \right)^6 + \&c;$$

atque ista Series abrumpitur, Summamque exhibet Algebraicam quoties n Numerum repraesentet Integrum positivum, idem respondebit Summae valor etiam casui Indicis n Integri negativi, tametsi in Infinitum Series progrediatur. Et ideo recte adseruit generali propositione Nicolius semper ha-

beri $\tau \left(\frac{a}{b} + \sqrt{-1} \right)^{-n} + \left(\frac{a}{b} - \sqrt{-1} \right)^{-n}$, quae paucis, veluti supra,

immutatis congruit cum $\left(\frac{a + b\sqrt{-1}}{(a^2 + b^2)^{\frac{1}{2}}} \right)^{-n} + \left(\frac{a - b\sqrt{-1}}{(a^2 + b^2)^{\frac{1}{2}}} \right)^{-n}$, Summam defi-

nito ter-

$$\text{Cofin. C. } zx = (\text{Cofin. C. } x)^z - \frac{z \cdot z-1}{1 \cdot 2} (\text{Cofin. C. } x)^{z-2} (\text{Sin. C. } x)^2 + \frac{z \cdot z-1 \cdot z-2 \cdot z-3}{1 \cdot 2 \cdot 3 \cdot 4}$$

dum e-converso, quamvis $\text{Cofin. C. } -zx = \text{Cofin. C. } zx$, tam diversa
Aequatio

$$\text{Cofin. C. } -zx = (\text{Cofin. C. } x)^{-z} - \frac{-z \cdot -z-1}{1 \cdot 2} (\text{Cofin. C. } x)^{-z-2} (\text{Sin. C. } x)^2 + \frac{-z \cdot -z-1 \cdot -z-2 \cdot -z-3}{1 \cdot 2 \cdot 3 \cdot 4}$$

ut vix earum aequipollentia, ob postremam Seriem numero terminorum
Infinitam, Algebrae regulis demonstrari unquam posse videatur? Anne, quan-
do Numerus z ex positivo vertitur in negativum, mendosa evadit omnisque
expers

$$\text{Sin. C. } zx = \frac{z}{1} (\text{Cofin. C. } x)^{z-1} (\text{Sin. C. } x)^1 - \frac{z \cdot z-1 \cdot z-2}{1 \cdot 2 \cdot 3} (\text{Cofin. C. } x)^{z-3} (\text{Sin. C. } x)^3$$

$$\text{Sin. C. } -zx = - \left(- \frac{z}{1} (\text{Cofin. C. } x)^{-z-1} (\text{Sin. C. } x)^1 - \frac{-z \cdot -z-1 \cdot -z-2}{1 \cdot 2 \cdot 3} (\text{Cofin. C. } x)^{-z-3} (\text{Sin. C. } x)^3 \right)$$

nito terminorum numero expositam. Interim qui velit haec experiri non

$$\text{obliviscatur oportet esse } \left(\frac{a + b\sqrt{-1}}{(a^2 + b^2)^{\frac{1}{2}}} \right)^{-n} + \left(\frac{a - b\sqrt{-1}}{(a^2 + b^2)^{\frac{1}{2}}} \right)^{-n} = \\ \left(\left(\frac{a}{b} + \sqrt{-1} \right)^{-n} + \left(\frac{a}{b} - \sqrt{-1} \right)^{-n} \right) \left(\frac{b}{(a^2 + b^2)^{\frac{1}{2}}} \right)^{-n}.$$

313. Sed, cum in argumento versemur utramque Trigonometriam sum-
mopere illustrante, proderit nonnihilum immorari in consideratione singula-
ris cuiusdam affectionis, qua fit, ut suspecta aliquibus videatur Analytistis
Formula canonica Multiplicationi aut Divisioni Angulorum, five Rationum
inserviens. Quonam pacto contingit, inquiunt illi, veritas Aequationis

Cofin. C.

$$(\text{Cofin. C. } x)^{z-4} (\text{Sin. C. } x)^4 = \frac{z \cdot z-1 \cdot z-2 \cdot z-3 \cdot z-4 \cdot z-5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} (\text{Cofin. C. } x)^{z-6} (\text{Sin. C. } x)^6 \&c.,$$

dum

Aequatio suboritur

Cofin. C.

$$(\text{Cofin. C. } x)^{-z-4} (\text{Sin. C. } x)^4 = \frac{-z \cdot -z-1 \cdot -z-2 \cdot -z-3 \cdot -z-4 \cdot -z-5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} (\text{Cofin. C. } x)^{-z-6} (\text{Sin. C. } x)^6 \&c.,$$

ut vix

expers usus praefata Formula Trigonometrica? Idem de Sinubus dicas:
namque

Sin. C.

$$+ \frac{z \cdot z-1 \cdot z-2 \cdot z-3 \cdot z-4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Cofin. C. } x)^{z-5} (\text{Sin. C. } x)^5 = \&c. =$$

$$(\text{Sin. C. } x)^3 + \frac{-z \cdot -z-1 \cdot -z-2 \cdot -z-3 \cdot -z-4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Cofin. C. } x)^{-z-5} (\text{Sin. C. } x)^5 = \&c.,$$

nec modus

nec modus apparet Finitam cum Infinita Serie ita conciliandi, ut permutatis tantummodo alterutrius Aequationis omnium terminorum Signis perfecta enascatur aequalitas. Ad huiusmodi solutionem, methodo duce prae omnibus demonstrativa, repetendum est id, quod in Capite II°. de Exponentialium genesi, atque evolutione agens fusius ostendi, dupliciter nempe exal-

tationem

$$(\text{Cofin. C. } x)^{\pm z} \left(1 + \frac{(\text{Sin. C. } x)}{\text{Cofin. C. } x} \sqrt{-1} \right)^{\pm z} = (\text{Cofin. C. } x)^{\pm z} - \frac{(\pm z \cdot \pm z - 1)}{1 \cdot 2} (\text{Cofin. C. } x)^{\pm z - 2} \\ + \frac{(\pm z)}{1} (\text{Cofin. C. } x)^{\pm z - 1} (\text{Sin. C. } x) \sqrt{-1} - \frac{(\pm z \cdot \pm z - 1 \cdot \pm z - 2)}{1 \cdot 2 \cdot 3} (\text{Cofin. C. } x)^{\pm z - 3} + \dots$$

atque

$$\{ (\text{Cofin. C. } x)^{\pm z} \left(1 + \frac{(\text{Sin. C. } x)}{\text{Cofin. C. } x} \sqrt{-1} \right)^{\pm z} = 1 + \frac{(L(\text{Cofin. C. } x + (\text{Sin. C. } x) \sqrt{-1}))^2 (\pm z)^2}{1 \cdot 2} + \dots \\ + \frac{(L(\text{Cofin. C. } x + (\text{Sin. C. } x) \sqrt{-1}))^1 (\pm z)^1}{1} + \dots$$

sive subrogato Logarithmo, eiusque Potestatibus respectivis

$$= 1 + \frac{(x\sqrt{-1})^2 (\pm z)^2}{1 \cdot 2} + \frac{(x\sqrt{-1})^4 (\pm z)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{(x\sqrt{-1})^6 (\pm z)^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{(x\sqrt{-1})^8 (\pm z)^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \dots \\ + \frac{(x\sqrt{-1})^1 (\pm z)^1}{1} + \frac{(x\sqrt{-1})^3 (\pm z)^3}{1 \cdot 2 \cdot 3} + \frac{(x\sqrt{-1})^5 (\pm z)^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{(x\sqrt{-1})^7 (\pm z)^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \dots$$

unde luculentissime patet τὸν Cofin. C. (±zx), ob Potestates ±z Gradus Paris, eiusdem esse valoris, et e-contra valorem τὸ Sin. C. (—zx) positivum haberi, sed Sin. C. (—zx) negativum, ideoque qualitate, non quantitate diversum, propter Potestates ±z Exponente Impari praeditas. Quod igitur Trigonometrica fide tantum constabat evolvendo, ad morem Newtoni, Binomium $(\text{Cofin. C. } x + (\text{Sin. C. } x) \sqrt{-1})^{\pm z}$, sensibili nunc prope iu-

dicio

tationem Binomii (Cofin. C. x) $(1 + \frac{(\text{Sin. C. x})}{\text{Cofin. C. x}} \sqrt{-1})$ ad Potestatem indefinitam, cui z Index aut Exponens respondeat, obtineri posse, salva nedum valoris identitate, verum etiam partium singularum aequipollentia, quarum diversus solum distributionis ordo variam formam producit. Eodem igitur redit scribendi modus

Cofin. C.

$$(\text{Sin. C. x})^2 + \frac{(\pm z \cdot \pm z - 1 \cdot \pm z - 2 \cdot \pm z - 3)}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cofin. C. x})^{\pm z - 4} (\text{Sin. C. x})^4 - \&c:$$

$$(\text{Cofin. C. x})^{\pm z - 3} (\text{Sin. C. x})^3 \sqrt{-1} + \frac{(\pm z \cdot \pm z - 1 \cdot \pm z - 2 \cdot \pm z - 3 \cdot \pm z - 4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Cofin. C. x})^{\pm z - 5} (\text{Sin. C. x})^5 \sqrt{-1} \&c;$$

alter

$$\frac{(L(\text{Cofin. C. x} + (\text{Sin. C. x}) \sqrt{-1}))^4 (\pm z)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{(L(\text{Cofin. C. x} + (\text{Sin. C. x}) \sqrt{-1}))^6 (\pm z)^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \&c:$$

$$\frac{(L(\text{Cofin. C. x} + (\text{Sin. C. x}) \sqrt{-1}))^3 (\pm z)^3}{1 \cdot 2 \cdot 3} + \frac{(L(\text{Cofin. C. x} + (\text{Sin. C. x}) \sqrt{-1}))^5 (\pm z)^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \&c;$$

five subro-

$$= 1 - \frac{z^2 x^2}{1 \cdot 2} + \frac{z^4 x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{z^6 x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{z^8 x^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} - \frac{z^{10} x^{10}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} + \&c:$$

$$\pm \frac{zx \sqrt{-1}}{1} \mp \frac{z^3 x^3 \sqrt{-1}}{1 \cdot 2 \cdot 3} \pm \frac{z^5 x^5 \sqrt{-1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \mp \frac{z^7 x^7 \sqrt{-1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \pm \frac{z^9 x^9 \sqrt{-1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} \mp \&c;$$

unde

dicio aut ipsis ferme oculis comprobatur evoluto Exponentialium instar Binomio; istaque demonstratio ex intima Calculi penu desumpta alteri cuilibet praestat, quae intellectum evincat, non autem erudiat.

314. Transferre iam dicta nil equidem prohibet ad Cofinus Sinusque Hyperbolicos. Namque artificium omne consistit in analogis Aequationibus percipiendis, quas summo compendio hic exhibeo.

T t t t

(Cofin. H. x)

$$(\text{Cofin. H. } x)^{\pm z} \left(1 + \frac{(\text{Sin. H. } x)}{\text{Cofin. H. } x} \right)^{\pm z} = (\text{Cofin. H. } x)^{\pm z} + \frac{(\pm z \cdot \pm z - 1)}{1 \cdot 2} (\text{Cofin. H. } x)^{\pm z - 2} \\ + \frac{(\pm z)}{1} (\text{Cofin. H. } x)^{\pm z - 1} (\text{Sin. H. } x)^1 + \frac{(\pm z \cdot \pm z - 1 \cdot \pm z - 2)}{1 \cdot 2 \cdot 3} (\text{Cofin. H. } x)^{\pm z - 3} (\text{Sin. H. } x)^2 + \dots$$

$$(\text{Cofin. H. } x)^{\pm z} \left(1 + \frac{(\text{Sin. H. } x)}{\text{Cofin. H. } x} \right)^{\pm z} = 1 + \frac{(L(\text{Cofin. H. } x + \text{Sin. H. } x))^2 (\pm z)^2}{1 \cdot 2} + \dots \\ + \frac{(L(\text{Cofin. H. } x + \text{Sin. H. } x))^4 (\pm z)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

$$= 1 + \frac{x^2 (\pm z)^2}{1 \cdot 2} + \frac{x^4 (\pm z)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^6 (\pm z)^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{x^8 (\pm z)^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \dots$$

$$+ \frac{x^1 (\pm z)^1}{1} + \frac{x^3 (\pm z)^3}{1 \cdot 2 \cdot 3} + \frac{x^5 (\pm z)^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{x^7 (\pm z)^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \dots$$

In hisce Aequationibus, quae cum iis plane consentiunt recensitis Capite V°. , facillime legitur, suadente indole Potestatum τz , esse $\text{Cofin. H.}(-zx) = \text{Cofin. H.}(+zx)$, simulque ex adverso $\text{Sin. H.}(-zx) = -\text{Sin. H.}(+zx)$, ac vicissim.

315. Non absimili methodo ratiocinantibus ceteras quoque alterutrius

Trigonometriae

$$\text{Cofin. H. } nx = v^n + \frac{n \cdot n - 1}{1 \cdot 2} v^{n-2} (v^2 - 1) + \frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3}{1 \cdot 2 \cdot 3 \cdot 4} v^{n-4} (v^4 - 2v^2 + 1) + \dots$$

$$\text{Sin. H. } nx = \frac{n}{1} y (1 + y^2)^{\frac{n-1}{2}} + \frac{n \cdot n - 1 \cdot n - 2}{1 \cdot 2 \cdot 3} y^3 (1 + y^2)^{\frac{n-3}{2}} + \frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3 \cdot n - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} y^5 (1 + y^2)^{\frac{n-5}{2}} + \dots$$

$$\text{Cofin. C. } nx = v^n - \frac{n \cdot n - 1}{1 \cdot 2} v^{n-2} (1 - v^2) + \frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3}{1 \cdot 2 \cdot 3 \cdot 4} v^{n-4} (1 - 2v^2 + v^4) - \dots$$

$$\text{Sin. C. } nx = \frac{n}{1} y (1 - y^2)^{\frac{n-1}{2}} - \frac{n \cdot n - 1 \cdot n - 2}{1 \cdot 2 \cdot 3} y^3 (1 - y^2)^{\frac{n-3}{2}} + \frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3 \cdot n - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} y^5 (1 - y^2)^{\frac{n-5}{2}} - \dots$$

$$(\text{Sin. H. } x)^2 + \frac{(\pm z \cdot \pm z - 1 \cdot \pm z - 2 \cdot \pm z - 3)}{1 \cdot 2 \cdot 3 \cdot 4} (\text{Cofin. H. } x)^{\pm z - 4} (\text{Sin. H. } x)^4 + \&c:$$

$$(\text{Cofin. H. } x)^{\pm z - 3} (\text{Sin. H. } x)^3 + \frac{(\pm z \cdot \pm z - 1 \cdot \pm z - 2 \cdot \pm z - 3 \cdot \pm z - 4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (\text{Cofin. H. } x)^{\pm z - 5} (\text{Sin. H. } x)^5 + \&c:$$

$$\frac{(L(\text{Cofin. H. } x + \text{Sin. H. } x))^4 (\pm z)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{(L(\text{Cofin. H. } x + \text{Sin. H. } x))^6 (\pm z)^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \&c:$$

$$\frac{(L(\text{Cofin. H. } x + \text{Sin. H. } x))^3 (\pm z)^3}{1 \cdot 2 \cdot 3} + \frac{(L(\text{Cofin. H. } x + \text{Sin. H. } x))^5 (\pm z)^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \&c:$$

$$= 1 + \frac{x^2 z^2}{1 \cdot 2} + \frac{x^4 z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^6 z^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{x^8 z^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \frac{x^{10} z^{10}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} + \&c:$$

$$\pm \frac{xz}{1} \pm \frac{x^3 z^3}{1 \cdot 2 \cdot 3} \pm \frac{x^5 z^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \pm \frac{x^7 z^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \pm \frac{x^9 z^9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} \pm \&c:$$

In hisce

Trigonometriæ expressiones illustrare in promptu esset; sed ulterior Cofinum Sinuumve Formulae fructus ab hac provincia diutius peragranda nos retrahit. Unica Binomii Formula duce determinatum iam fuit prostare Aequationes

Cofin. H.

$$= \frac{n-1}{2} \cdot \frac{n}{v} - \frac{n}{1} \cdot \frac{n-3}{2} \cdot \frac{n-2}{v} + \frac{n \cdot n-3}{1 \cdot 2} \cdot \frac{n-5}{2} \cdot \frac{n-4}{v} - \frac{n \cdot n-4 \cdot n-5}{1 \cdot 2 \cdot 3} \cdot \frac{n-7}{2} \cdot \frac{n-6}{v} + \&c:$$

$$= \frac{n}{1} y + \frac{n^2-1}{2 \cdot 3} ny^3 + \frac{n^2-1}{2 \cdot 3} \cdot \frac{n^2-9}{4 \cdot 5} ny^5 + \frac{n^2-1}{2 \cdot 3} \cdot \frac{n^2-9}{4 \cdot 5} \cdot \frac{n^2-25}{6 \cdot 7} ny^7 + \&c:; \text{ dum } n \text{ sit Impar};$$

$$= \frac{n-1}{2} \cdot \frac{n}{v} - \frac{n}{1} \cdot \frac{n-3}{2} \cdot \frac{n-2}{v} + \frac{n \cdot n-3}{1 \cdot 2} \cdot \frac{n-5}{2} \cdot \frac{n-4}{v} - \frac{n \cdot n-4 \cdot n-5}{1 \cdot 2 \cdot 3} \cdot \frac{n-7}{2} \cdot \frac{n-6}{v} + \&c:;$$

$$= \frac{n}{1} y - \frac{(n^2-1)}{2 \cdot 3} ny^3 + \frac{(n^2-1)}{2 \cdot 3} \cdot \frac{(n^2-9)}{4 \cdot 5} ny^5 - \frac{(n^2-1)}{2 \cdot 3} \cdot \frac{(n^2-9)}{4 \cdot 5} \cdot \frac{(n^2-25)}{6 \cdot 7} ny^7 + \&c:; \text{ dum } n \text{ sit Impar}.$$

Simplicissimum

Simplicissimum autem est absque ullius novi fundamenti subsidio postremam adipisci Aequationum commodiorem formam statim ac periodica Lex praecedentium

Cosin. H. $x = v^1$

$$2x = 2v^2 - 1$$

$$3x = 4v^3 - 3v^1$$

$$4x = 8v^4 - 8v^2 + 1$$

$$5x = 16v^5 - 20v^3 + 5v^1$$

$$6x = 32v^6 - 48v^4 + 18v^2 - 1$$

$$7x = 64v^7 - 112v^5 + 56v^3 - 7v^1$$

$$nx = 2^{n-1} v^n - \frac{n}{2} 2^{n-2} v^{n-2} + \frac{n-2}{2} v^{n-4} - \dots \&c: > 1$$

Cosin. C. $x = v^1$

$$2x = 2v^2 - 1$$

$$3x = 4v^3 - 3v^1$$

$$4x = 8v^4 - 8v^2 + 1$$

$$5x = 16v^5 - 20v^3 + 5v^1$$

$$6x = 32v^6 - 48v^4 + 18v^2 - 1$$

$$7x = 64v^7 - 112v^5 + 56v^3 - 7v^1$$

$$nx = 2^{n-1} v^n - \frac{n}{2} 2^{n-2} v^{n-2} + \frac{n-2}{2} v^{n-4} - \dots \&c: < 1$$

aut
= 1

316. Haec inter formas elegantia ceteris praecellunt eae primum a Moivreo datae in anni MDCCVII. *Philosophicis Transactionibus* Aequationes imparis Ordinis

$$\lambda y + \frac{\lambda^2 - 1^2}{2 \cdot 3} \cdot \lambda y^3 + \frac{\lambda^2 - 1^2}{2 \cdot 3} \cdot \frac{\lambda^2 - 3^2}{4 \cdot 5} \cdot \lambda y^5 + \frac{\lambda^2 - 1^2}{2 \cdot 3} \cdot \frac{\lambda^2 - 3^2}{4 \cdot 5} \cdot \frac{\lambda^2 - 5^2}{6 \cdot 7} \cdot \lambda y^7 + \frac{\lambda^2 - 1^2}{2 \cdot 3} \dots \frac{\lambda^2 - (\lambda-2)^2}{\lambda - 1 \cdot \lambda} \cdot \lambda y = m$$

$$\lambda y + \frac{1^2 - \lambda^2}{2 \cdot 3} \cdot \lambda y^3 + \frac{1^2 - \lambda^2}{2 \cdot 3} \cdot \frac{3^2 - \lambda^2}{4 \cdot 5} \cdot \lambda y^5 + \frac{1^2 - \lambda^2}{2 \cdot 3} \cdot \frac{3^2 - \lambda^2}{4 \cdot 5} \cdot \frac{5^2 - \lambda^2}{6 \cdot 7} \cdot \lambda y^7 + \frac{1^2 - \lambda^2}{2 \cdot 3} \dots \frac{(\lambda-2)^2 - \lambda^2}{\lambda - 1 \cdot \lambda} \cdot \lambda y = r,$$

quae Signorum alternatione ad unicam reduci possunt

$$\lambda y \pm \frac{(\lambda^2 - 1^2)}{2 \cdot 3} \lambda y^3 \pm \frac{\lambda^2 - 1^2}{2 \cdot 3} \cdot \frac{\lambda^2 - 3^2}{4 \cdot 5} \cdot \lambda y^5 \pm \frac{(\lambda^2 - 1^2)}{2 \cdot 3} \cdot \frac{(\lambda^2 - 3^2)}{4 \cdot 5} \cdot \frac{(\lambda^2 - 5^2)}{6 \cdot 7} \lambda y^7 \pm \frac{\lambda^2 - 1^2}{2 \cdot 3} \dots \&c: = Q,$$

ista

cedentium singularum quadam attentione inspiciatur.

Cofin. H.

Sin. H. $x = y^1$

Sin. C. $x = y^1$

$$3x = 3y^1 + 4y^3$$

$$3x = 3y^1 - 4y^3$$

$$5x = 5y^1 + 20y^3 + 16y^5$$

$$5x = 5y^1 - 20y^3 + 16y^5$$

$$7x = 7y^1 + 56y^3 + 112y^5 + 64y^7$$

$$7x = 7y^1 - 56y^3 + 112y^5 - 64y^7$$

$$9x = 9y^1 + 120y^3 + 432y^5 + 576y^7 + 256y^9$$

$$9x = 9y^1 - 120y^3 + 432y^5 - 576y^7 + 256y^9$$

$$11x = 11y^1 + 220y^3 + 1232y^5 + 2816y^7 + 2816y^9 \&c:$$

$$11x = 11y^1 - 220y^3 + 1232y^5 - 2816y^7 + 2816y^9 \&c:$$

$$13x = 13y^1 + 364y^3 + 2912y^5 + 9984y^7 + 16640y^9 \&c:$$

$$13x = 13y^1 - 364y^3 + 2912y^5 - 9984y^7 + 16640y^9 \&c:$$

$$\lambda x = \frac{\lambda}{1} y^1 + \frac{\lambda^2 - 1}{2 \cdot 3} \lambda y^3 + \&c:$$

$$\lambda x = \frac{\lambda}{1} y^1 - \frac{\lambda^2 - 1}{2 \cdot 3} \lambda y^3 + \&c:$$

aut
< 1
vel
= 1

vel
= 1

316. Hasc

ista specie Q, seu Membro *comparationis*, quemlibet Numerum positivum, negativum, Unitate maiorem, sive minorem, Unitatemve ipsam repraesentante. Aequationum latarum nativa genesis evidenter ostendit mutuam illarum similitudinem, alternantium Signorum necessitatem, et perantiquam Trigonometriae Hyperbolico-Circularis analogiam, qua omnium resolutio obtinetur puncto eodem temporis perquam facillima. Dimanat namque $Q = \text{Sin.}$

$$\text{C}_H(\lambda x), \text{ et } y = \text{Sin. } \text{C}_H\left(\frac{\lambda x}{\lambda}\right) = \text{Sin. } \text{C}_H(x) = \frac{1}{2} \left(\text{Cofin. H.}(\lambda x) + \text{Sin. H.}(\lambda x) \right)^{\frac{1}{\lambda}} - \frac{1}{2} \left(\text{Cofin. H.}(\lambda x) - \text{Sin. H.}(\lambda x) \right)^{\frac{1}{\lambda}} = \frac{1}{2} \sqrt[\lambda]{\sqrt{1+Q^2} + Q} - \frac{1}{2} \sqrt[\lambda]{\sqrt{1+Q^2} - Q}$$

Vvvv

$\frac{1}{2}$

$\frac{1}{2} \sqrt[\lambda]{\sqrt{1+Q^2}-Q}$ si valeat Signum superius, et in contraria hypothesi =

$\frac{1}{2} \sqrt[\lambda]{Q - \sqrt{Q^2-1}} = \frac{1}{2} \sqrt[\lambda]{-Q - \sqrt{Q^2-1}}$, in quam vertitur

$$\frac{1}{2} \frac{(\text{Cofin. } C.(\lambda x) + \text{Sin. } C.(\lambda x) \sqrt{-1})^{\frac{1}{\lambda}}}{\sqrt{-1}}$$

$$\frac{1}{2} \frac{(\text{Cofin. } C.(\lambda x) - \text{Sin. } C.(\lambda x) \sqrt{-1})^{\frac{1}{\lambda}}}{\sqrt{-1}}; \text{ notandumque est propter } \lambda \text{ In-}$$

dicem Imparem eandem expressionem Radicis $\frac{1}{2} \sqrt[\lambda]{Q - \sqrt{Q^2-1}} -$

$\frac{1}{2} \sqrt[\lambda]{-Q - \sqrt{Q^2-1}}$ alteri aequipollere $\frac{1}{2} \sqrt[\lambda]{Q - \sqrt{Q^2-1}} +$

$\frac{1}{2} \sqrt[\lambda]{Q + \sqrt{Q^2-1}}$.

Ad reliquas investigandas canonicae illius Aequationis

Radices vix peculiari disquisitione opus erit, quum praemissa demonstrent eas

$$\lambda y + \frac{1-\lambda^2}{2 \cdot 3} \cdot \lambda y^3 + \frac{1-\lambda^2}{2 \cdot 3} \cdot \frac{9-\lambda^2}{4 \cdot 5} \cdot \lambda y^5 + \frac{1-\lambda^2}{2 \cdot 3} \cdot \frac{9-\lambda^2}{4 \cdot 5} \cdot \frac{25-\lambda^2}{6 \cdot 7} \cdot \lambda y^7 +$$

dammodo Q sit Numerus infra Unitatem. Quod si statuatur in Limite $\tau\delta$ Membrum *Comparisonis* $Q = 1$ proveniet Formula universalis Radicum

praefatae Aequationis $1^{\frac{1}{\lambda}} \cdot \frac{1}{2} Q^{\frac{1}{\lambda}} + 1^{\frac{1}{\lambda}} \cdot \frac{1}{2} Q^{\frac{1}{\lambda}}$, vel $(\text{Cofin. } C. \frac{n\pi}{\lambda} \pm (\text{Sin.}$

$C. \frac{n\pi}{\lambda}) \sqrt{-1}) \frac{1}{2} Q^{\frac{1}{\lambda}} + (\text{Cofin. } C. \frac{n\pi}{\lambda} \mp (\text{Sin. } C. \frac{n\pi}{\lambda}) \sqrt{-1}) \frac{1}{2} Q^{\frac{1}{\lambda}}$, si-

ve ob Imaginariorum destructionem $(\text{Cofin. } C. \frac{n\pi}{\lambda}) Q^{\frac{1}{\lambda}}$, inter quas omnes

Reales

eas fore in oecumenica latentes Formula $\frac{1}{\lambda} \left(\frac{1}{2} \sqrt[2]{\frac{1-Q}{1+Q^2}} + Q - \right.$

$\left. \frac{1}{2} \sqrt[2]{\frac{1-Q}{1+Q^2}} - Q \right)$, necnon $\frac{1}{\lambda} \left(-\frac{1}{2} \sqrt[2]{\frac{1-Q}{1+Q^2}} + \right.$

$\left. \frac{1}{2} \sqrt[2]{\frac{1-Q}{1+Q^2}} - Q \right)$, nimirum $\left(\text{Cofin. C.} \pm \frac{n\Pi}{\lambda} + \left(\text{Sin. C.} \pm \frac{n\Pi}{\lambda} \right) \sqrt{-1} \right)$

$\left(\frac{1}{2} \sqrt[2]{\frac{1-Q}{1+Q^2}} + Q - \frac{1}{2} \sqrt[2]{\frac{1-Q}{1+Q^2}} - Q \right)$, et $\left(\text{Cofin. C.} \pm \frac{n\Pi}{\lambda} + \left(\text{Sin. C.} \pm \frac{n\Pi}{\lambda} \right) \sqrt{-1} \right)$

$\left(\frac{1}{2} \sqrt[2]{\frac{1-Q}{1+Q^2}} + Q - \frac{1}{2} \sqrt[2]{\frac{1-Q}{1+Q^2}} - Q \right)$. Sed hoc in

postremo Producto observandum casibus universis $Q < 1$, et idcirco $\sqrt[2]{Q^2-1}$ Imaginariis respondere, quotus est numerus λ , Radices omnes Reales ob Imaginariorum elisionem, prouti in *Casu Irreducibili* tertii Gradus, cui nempe analogi evadent infinities adnumerandi ceteri *Casus* pariter *Irreducibiles*, eiusdemque Trigonometricae *Reductionis* methodi capaces in Aequationibus Imparis Ordinis

$$\frac{1-\lambda^2}{2 \cdot 3} \cdot \frac{9-\lambda^2}{4 \cdot 5} \cdot \frac{25-\lambda^2}{6 \cdot 7} \cdot \frac{49-\lambda^2}{8 \cdot 9} \cdot \lambda y^9 + \&c: = Q,$$

dummodo

Reales Radices primum $\left(\text{Cofin. C. } 0 \right) \sqrt[2]{Q}$, scilicet 1 , habet locum, subindeque $\text{Cofin. C. } \frac{\Pi}{\lambda}$, $\text{Cofin. C. } \frac{2\Pi}{\lambda}$, $\text{Cofin. C. } \frac{3\Pi}{\lambda}$ &c: usque ad Cofin. C.

$\frac{(\lambda-1)\Pi}{\lambda}$ inclusive. Nec obstupefcat Lector Radices singulas Aequationis, de qua nunc agitur, diversimode expressas tam in Sinubus, quam in Cofinubus, quod paradoxon certe videtur, iuxta subnexam Tabellam posse exhiberi, cui maioris universalitatis gratia casum etiam $\sqrt[2]{Q} = -1$ addere libet.

$$Q =$$

$Q=1$		$Q=-1$	
$\text{Sin. C. } \frac{\Pi}{4\lambda}$	$\text{Cofin. C. } \frac{0\Pi}{\lambda} = 1$	$\text{Sin. C. } \frac{-\Pi}{4\lambda}$	$\text{Cofin. C. } \frac{0\Pi}{\lambda} = -1$
$\text{Sin. C. } \frac{\Pi + \frac{\Pi}{4}}{\lambda}$	$\text{Cofin. C. } \frac{1\Pi}{\lambda}$	$\text{Sin. C. } \frac{\Pi - \frac{\Pi}{4}}{\lambda}$	$\text{Cofin. C. } \frac{1\Pi}{\lambda}$
$\text{Sin. C. } \frac{2\Pi + \frac{\Pi}{4}}{\lambda}$	$\text{Cofin. C. } \frac{2\Pi}{\lambda}$	$\text{Sin. C. } \frac{2\Pi - \frac{\Pi}{4}}{\lambda}$	$\text{Cofin. C. } \frac{2\Pi}{\lambda}$
$\text{Sin. C. } \frac{3\Pi + \frac{\Pi}{4}}{\lambda}$	$\text{Cofin. C. } \frac{3\Pi}{\lambda}$	$\text{Sin. C. } \frac{3\Pi - \frac{\Pi}{4}}{\lambda}$	$\text{Cofin. C. } \frac{3\Pi}{\lambda}$

$\text{Sin. C. } \frac{(\lambda-1)\Pi + \frac{\Pi}{4}}{\lambda}$	$\text{Cofin. C. } \frac{(\lambda-1)\Pi}{\lambda}$	$\text{Sin. C. } \frac{(\lambda-1)\Pi - \frac{\Pi}{4}}{\lambda}$	$\text{Cofin. C. } \frac{(\lambda-1)\Pi}{\lambda}$

Duo siquidem animadverti poterunt hypotheses, nempe vel λ ita Imparis, ut $\frac{\lambda-1}{2}$ sit Numerus Par, aut ex opposito Impar, vel potius $\lambda-1$ pariter sive impariter Paris. Prima hypothesis tribuet in Serie Sinuum-Recto-

rum $\text{Sin. C. } \frac{(m\Pi + \frac{\Pi}{4})}{\lambda} = 1 = \text{Cofin. C. } \frac{0\Pi}{\lambda}$, ubi m significat Numerum Naturalis Arithmeticae Progressionis respondentem Seriei 5, 9, 13, 17, 21, 25, 29 &c: λ .
1, 2, 3, 4, 5, 6, 7 &c:

Post $\text{Sin. C. } \frac{(m\Pi + \frac{\Pi}{4})}{\lambda}$ ceteri, qui ordine consequuntur, Sinus continuata numeratione ab ultimo $\text{Sin. C. } \frac{(\lambda-1)\Pi + \frac{\Pi}{4}}{\lambda}$ ad primum $\text{Sin. C. } \frac{\Pi}{4\lambda}$

$$\frac{\Pi}{4\lambda}$$

$\frac{\Pi}{4\lambda}$ usque ad Periodi in se redeuntis complementum peraequant singulos

Cofinus nativo ordine ab Cofin. C. $\frac{0\Pi}{\lambda}$ numeratos. Et re quidem vera, dum

perventum fuerit ad punctum Peripheriae Circularis, in quo Sinus habeatur

$$\text{aequalis Radio, semper erit Sin. C. } \frac{\Pi}{4} + \frac{\Pi}{\lambda} = \text{Sin. C. } \frac{(m+1)\Pi + \frac{\Pi}{4}}{\lambda}$$

$$= \text{Cofin. C. } \frac{1\Pi}{\lambda}, \text{ Sin. C. } \frac{\Pi}{4} + \frac{2\Pi}{\lambda} = \text{Sin. C. } \frac{(m+2)\Pi + \frac{\Pi}{4}}{\lambda} =$$

Cofin. C. $\frac{2\Pi}{\lambda}$ &c.; quam perbellam affectionem mutationis Sinus in Cofi-

num, et vicissim, aliquis exemplorum delectus, atque opportuna Schematum

delineatio illustrabit. In altera autem hypothesi $\frac{\lambda-1}{2}$ Imparis, aut $\lambda-1$

$$\text{Impariter-Paris nemo non videt esse Sin. C. } \frac{(m\Pi + \frac{\Pi}{4})}{\lambda} = -1 = -$$

Cofin. C. $\frac{0\Pi}{\lambda}$, dummodo m ordine heic adscriptorum servato eligatur

3, 7, 11, 15, 19, 23, 27 &c: λ ; posteaque alii Sinus subsequentes, uti
2, 5, 8, 11, 14, 17, 20 &c: m

$$\text{Sin. C. } \frac{(m+1)\Pi + \frac{\Pi}{4}}{\lambda} = \text{Sin. C. } \frac{3\Pi}{4} + \frac{\Pi}{\lambda} = - \text{Cofin. C. } \frac{1\Pi}{\lambda},$$

$$\text{Sin. C. } \frac{(m+2)\Pi + \frac{\Pi}{4}}{\lambda} = \text{Sin. C. } \frac{3\Pi}{4} + \frac{2\Pi}{\lambda} = - \text{Cofin. C. } \frac{2\Pi}{\lambda},$$

eodemque ulteriore progressu, respectivis aequipollebunt Cofinibus, sed negative desumptis. Haec conditionis necessitas directo nunc tramite comprobatur id, quod loco citato obiter affirmavit Moivreus, et filuit in Capite XIV^o.

X x x x

prioris

prioris Voluminis *Introductionis* Eulerus, scilicet Aequationis ad examen nuper revocatae Radices negative sumi debere, veluti —

$$\frac{1}{2} \sqrt[\lambda]{Q + \sqrt{Q^2 - 1}} - \frac{1}{2} \sqrt[\lambda]{Q - \sqrt{Q^2 - 1}}, \text{ quando } \frac{\lambda - 1}{2} \text{ Impar fuerit; quo}$$

facto Sinus, ac Cofinus qualitate coincidunt, ac quantitate in hypothesi simul $Q = 1$, aliisque, et omnia mirifice conciliantur, quemadmodum cui-

vis fas est experiri. Illa itaque in suppositione $\tau \tilde{x} \frac{\lambda - 1}{2}$ Numeri Imparis

universae Radices quaesitae comprehenduntur ab unica Formula

$$\left(\text{Cofin. C.} \pm \frac{n\pi}{\lambda} + \left(\text{Sin. C.} \pm \frac{n\pi}{\lambda} \right) \sqrt{-1} \right) \times$$

$$\left(-\frac{1}{2} \sqrt[\lambda]{Q + \sqrt{Q^2 - 1}} - \frac{1}{2} \sqrt[\lambda]{Q - \sqrt{Q^2 - 1}} \right). \text{ Neque ideo infirmatur}$$

Theoria paullo superius exposita; quia, cum Radix sit $\frac{1}{2\sqrt{-1}} (\text{Cofin. C. } \lambda x +$

$$(\text{Sin. C. } \lambda x) \sqrt{-1} \Big)^{\frac{1}{\lambda}} - \frac{1}{2\sqrt{-1}} (\text{Cofin. C. } \lambda x - (\text{Sin. C. } \lambda x) \sqrt{-1} \Big)^{\frac{1}{\lambda}} =$$

$$\frac{1}{2\pm\sqrt{-1}} (\text{Cofin. C. } \lambda x \pm (\text{Sin. C. } \lambda x) \sqrt{-1} \Big)^{\frac{1}{\lambda}} - \frac{1}{2\pm\sqrt{-1}} (\text{Cofin. C. } \lambda x \pm$$

$$(\text{Sin. C. } \lambda x) \sqrt{-1} \Big)^{\frac{1}{\lambda}}, \text{ provenit tam } + \text{ quam } - \frac{1}{2} \text{ Sin. C. } x: \text{ nec possibi-}$$

li combinationi Signorum obstat mutatio Formulae Radicalis ex positiva in negativam, prouti varia exposcat Aequationum datarum natura. Quin etiam sine huiusce subtilissimi argumenti subsidio prouti idipsum ostendere animadversione facta, quod universis in casibus $\tau \tilde{x} \lambda - 1$ ab Quaternario non divisibilis debeant in Aequatione Signa omnium terminorum in contraria permutari, ut ea rite ordinetur, propter terminum ad maximam incognitae Potestatem exaltatum, qui Signo afficitur negativo. Eadem, paucis mutatis, de hypothesi $Q = -1$ praedicari aeque possent.

317. Tot inter utillima Confectaria, quae ab iis ultro dimanant, nobis omittere vitio dandum esset, tertii Gradus Aequationem *Casui Irreducibili* additam, utcunque vi numerorum 310. et proxime antecedentis per Cofinus, ac Sinus Circulares ad libitum resolvi queat, nullae ideo obnoxiam cenferi debere contradictioni, vel antilogiae. Posito etenim $Q < 1 = \pm a$ in Aequatione $y^3 - \frac{3y}{4} \mp a = 0$, aequè liberum pronumque erit assumere

$\mp a$ pro Cofinu, seu Recto-Sinu Arcus x , vel $\frac{\Pi}{4} - x$; ita, ut

$$\begin{array}{ccc} \text{Cofin. C. } \frac{x}{3} & = & \text{Sin. C. } \frac{(\frac{\Pi}{4} - x)}{3} \\ \text{Cofin. C. } \frac{\Pi - x}{3} & = & \text{Sin. C. } \frac{(\frac{5\Pi}{4} - x)}{3} \\ \text{Cofin. C. } \frac{2\Pi - x}{3} & = & \text{Sin. C. } \frac{(\frac{9\Pi}{4} - x)}{3} \end{array}$$

ordine aequalitatis mutuae ab intervacuis Lineolis demonstrato. Ipsa haec perfecta Sinuum Cofinumque aequipollentia iterum confirmat Rectos-Sinus, qui praebeant Radices Aequationis, cuius Gradus, prouti est 3, Unitate minutus det Numerum Impariter-Parem, accipi oportere negativos. Quamvis autem dubitatione careat inventorem Moivreum ad Aequationum modo Cardanico resolubilium formam Trigonometriae ope pervenisse, proptereaquod novennio ante in eisdem *Transactionibus Philosophicis* Aequationem universalem tribuerit inter M Chordam Arcus Multipli Circularis nx , atque z subtensam Arcus simplicis x ,

nimirum
$$nz^1 + \frac{n - n^3}{6 \cdot 2^2} \cdot z^3 + \frac{9n - 10n^3 + n^5}{120 \cdot 2^4} \cdot z^5 + \frac{225n - 259n^3 + 35n^5 - n^7}{5040 \cdot 2^6} \cdot z^7 + \&c = M,$$

vel alia distributione terminorum exornatam
$$nz + \frac{1^2 - n^2}{6 \cdot 2^2} \cdot z^2 A + \frac{3^2 - n^2}{20 \cdot 2^2} \cdot z^2 B + \frac{5^2 - n^2}{42 \cdot 2^2} \cdot z^2 C + \&c = M, \text{ Diametro existente } 2,$$

quae

quae expressio, posita loco Chordae z eius medietate, vel Recto-Sinu y Arcus x , vertitur in Aequationem

$$ny + \frac{1-n^2}{2.3} \cdot ny^3 + \frac{1-n^2}{2.3} \cdot \frac{9-n^2}{4.5} \cdot ny^5 + \frac{1-n^2}{2.3} \cdot \frac{9-n^2}{4.5} \cdot \frac{25-n^2}{6.7} \cdot ny^7 + \&c: = \frac{M}{2} = \text{Sin. } C \cdot nx$$

iam superius emensam; nihilo tamen minus ultra Trigonometricam facultatem se porrigunt, et servata eadem forma, quicumque sit Numerus λ , Par, vel Impar, Positivus, seu Negativus, Integer, aut Fractus, Rationalis, Irrationalis, quinimmo et Transcendens, Aequationes hinc inde derivatae eundem semper Radicum Typum admittent. Hoc Analyseos, quasi diceret incredibile, donum, quo magis, quam nativa a Trigonometriae sinu Aequationum illarum origo polliceretur, divitem aperit penum, in meliore luce locatur ab Algebrae praeceptis universalibus. Enimvero, quum Aequationi, cuius sit Paradigma oecumenicum

$$a = \frac{n}{1} x \pm \frac{n^2-1^2}{1.2.3} \cdot nx^3 \pm \frac{n^2-1^2 \cdot n^2-3^2}{1.2.3.4.5} \cdot nx^5 \pm \frac{n^2-1^2 \cdot n^2-3^2 \cdot n^2-5^2}{1.2.3.4.5.6.7} \cdot nx^7 \pm \frac{n^2-1^2 \cdot n^2-3^2 \cdot n^2-5^2 \cdot n^2-7^2}{1.2.3.4.5.6.7.8.9} \cdot nx^9 \pm \&c,$$

ipsamet etiam, quomodocunque variato Numero n , conveniat forma Radicum, et dum n fuerit Impar, et Aequatio finitis numero terminis co-

luerit Radices inventae formam habeant $1^{\frac{1}{n}} \left(\frac{1}{2} \sqrt[n]{a + \sqrt{a^2 \pm 1}} \mp \right.$

$\left. \frac{1}{2} \sqrt[n]{\mp a \pm \sqrt{a^2 \pm 1}} \right)$, idonea fiet quoque ista forma resolutioni adipiscen-

dae Aequationum Infinitinomialium, quae profluant ab hypothese π n quomodolibet ab Numero Dispari positivo vel negativo diversi. Id non parum confert ad inlustrationem superius recensitae Aequationis, quae ab Arcus Multipli Subtensa alteram Sub-Multipli, aut vicissim, deducit: quando etenim illic Numerus n recedat ab Impari negativo, aut positivo, nihilo-fecius Aequatio vera manebit, tametsi haudquaquam abrumpatur, aut nulli Limitem noscat,

318. Sed, quod in re tam insignis momenti palmarium reor, Aequatio illa Infinitinomialis

$$\text{Sin. } \frac{H}{C}(nx) = \frac{n}{1} y \pm \frac{(n^2-1^2)}{1.2.3} \cdot ny^3 \pm \frac{n^2-1^2 \cdot n^2-3^2}{1.2.3.4.5} \cdot ny^5 \pm \frac{(n^2-1^2 \cdot n^2-3^2 \cdot n^2-5^2)}{1.2.3.4.5.6.7} \cdot ny^7 \pm \frac{n^2-1^2 \cdot n^2-3^2 \cdot n^2-5^2 \cdot n^2-7^2}{1.2.3.4.5.6.7.8.9} \cdot ny^9 \pm \&c:$$

casu

casu n Paris positivo-negativi congruit cum maximopere diversa Aequationis forma.

$$\text{Sin. } \frac{H}{C}(nx) = \left(\frac{n}{1}y \pm \frac{(n^2-2^2)}{1 \cdot 2 \cdot 3}ny^3 + \frac{n^2-2^2 \cdot n^2-4^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}ny^5 \pm \frac{(n^2-2^2 \cdot n^2-4^2 \cdot n^2-6^2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}ny^7 + \frac{n^2-2^2 \cdot n^2-4^2 \cdot n^2-6^2 \cdot n^2-8^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}ny^9 \pm \&c: \right) \sqrt{1 \pm y^2},$$

quae respectu primi Factoris abrumpitur, verum ad eliminandam secundi asymmetriam Infinitis pariter conflatur numero terminis. Si alterutram velis a Serierum Infinitarum familia seiungere, aut facias oportet n Impari Numero aequalem, salva semper duarum aequipollentia, namque ea pro quolibet stat Numero n, velingas n Parem, quo in casu tribuet exaltatio ad Potestatem Quadraticam

$$\left(\text{Sin. } \frac{H}{C}(nx) \right)^2 = \left(ny \pm \frac{(n^2-1^2)}{2 \cdot 3}ny^3 + \frac{n^2-1^2}{2 \cdot 3} \cdot \frac{n^2-3^2}{4 \cdot 5}ny^5 \pm \frac{(n^2-1^2 \cdot n^2-3^2 \cdot n^2-5^2)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}ny^7 + \&c: \right)^2 =$$

$$\left(\text{Sin. } \frac{H}{C}(nx) \right)^2 = (1 \pm y^2) \left(ny \pm \frac{(n^2-2^2)}{2 \cdot 3}ny^3 + \frac{n^2-2^2}{2 \cdot 3} \cdot \frac{n^2-4^2}{4 \cdot 5}ny^5 \pm \frac{(n^2-2^2 \cdot n^2-4^2 \cdot n^2-6^2)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}ny^7 + \&c: \right)^2$$

postremo tum Membro finitos Numero terminos, prouti evidens est, complectente. Nemo dubitabit, ut arbitror, de legitima Aequationis secundo loco appositae forma ad utriusque Trigonometriae Sinum-Rectum pertinentis, quippe quae nequidem impetrato *Recurrentium Serierum* auxilio oriatur facillime ab aliquantula subsequenter Prospectus contemplatione, ex praemissis in num°. 315. efformati

$$\text{Sin. } H(2x) = (2y^1) \sqrt{1+y^2}$$

$$\text{Sin. } C(2x) = (2y^1) \sqrt{1-y^2}$$

$$(4x) = (4y^1 + 8y^3) \sqrt{1+y^2}$$

$$(4x) = (4y^1 - 8y^3) \sqrt{1-y^2}$$

$$(6x) = (6y^1 + 32y^3 + 32y^5) \sqrt{1+y^2}$$

$$(6x) = (6y^1 - 32y^3 + 32y^5) \sqrt{1-y^2}$$

$$(8x) = (8y^1 + 80y^3 + 192y^5 + 128y^7) \sqrt{1+y^2}$$

$$(8x) = (8y^1 - 80y^3 + 192y^5 - 128y^7) \sqrt{1-y^2}$$

$$(nx) = \left(\frac{n}{1}y^1 + \frac{(n^2-2^2)}{2 \cdot 3}ny^3 + \&c: \right) \sqrt{1+y^2}$$

$$(nx) = \left(\frac{n}{1}y^1 - \frac{(n^2-2^2)}{2 \cdot 3}ny^3 + \&c: \right) \sqrt{1-y^2}.$$

n Par

Y y y y

Dubium

Dubium attamen fortassis haerebit circa veram plenissimamque aequipollentiam Infinitarum Serierum

$$\frac{n}{1} \cdot y \pm \frac{(n^2-1^2)}{2 \cdot 3} ny^3 \pm \frac{n^2-1^2}{2 \cdot 3} \cdot \frac{n^2-3^2}{4 \cdot 5} ny^5 \pm \frac{(n^2-1^2)}{2 \cdot 3} \cdot \frac{n^2-3^2}{4 \cdot 5} \cdot \frac{n^2-5^2}{6 \cdot 7} ny^7 + \&c:$$

ac

$$\left(\frac{n}{1} \cdot y \pm \frac{(n^2-2^2)}{1 \cdot 2 \cdot 3} ny^3 + \frac{n^2-2^2}{1 \cdot 2 \cdot 3} \cdot \frac{n^2-4^2}{4 \cdot 5} ny^5 \pm \frac{(n^2-2^2)}{1 \cdot 2 \cdot 3} \cdot \frac{n^2-4^2}{4 \cdot 5} \cdot \frac{n^2-6^2}{6 \cdot 7} ny^7 + \&c: \right) \sqrt{1 \pm y^2},$$

quarum

$$\left(1 - \frac{y^2}{2} - \frac{y^4}{8} - \frac{y^6}{16} - \&c: \right) \left(ny - \frac{(n^2-4)}{6} ny^3 + \frac{n^2-4 \cdot n^2-16}{120} ny^5 - \frac{(n^2-4 \cdot n^2-16 \cdot n^2-36)}{5040} ny^7 + \&c: \right)$$

quod ostendendum susceperam.

319. Iste eximius certe consensus, quo Serierum prima fronte tam discrepantium semper magis emicat Harmonia, stimulos addit anatomen veluti quandam harumce Aequationum prosequendi, libenti animo iis praetermissis, quae Coronidum instar undique affluunt, proprietatibus Summas, Productave Sinuum, Cosinuumque respicientibus, de quibus multa in *Introductione* acute, uti solet, differuit Eulerus. Veritati autem quem debeo, integerrimus cultus silentium prohibet fallaciae nequaquam innocuae a Saurio in nuperrimi *Cursus* II°. Volumine insertae, ubi generaliter statuit transitum ab Imaginariis Magnitudinibus ad Reales, atque vicissim, nonnisi per Limites 0, vel ∞ fieri unquam posse. Nescius is erat huic mendo universam Aequationum cohortem hactenus illustratam, ac inferius etiam recensendam, nec non naturam ipsam compositae-Imaginarietatis, quae saepius in simpli-

ciore

quarum prior unico casu n Imparis, altera nunquam finito componitur numero terminorum. Ne oborta suspicio, vel saltem minor, quam par sit, evidētia Lectorum mentem perturbet, demonstrationem inibo harumce Formularum omnimodae aequalitatis pro quolibet Numero n , assumptis inferioribus Signis, confusionis tantum vitandae causa, cum Sinui quoque Hyperbolico non absimilis referri possit ratiocinatio. Resolvatur itaque in Seriem $\sqrt{1-y^2}$, eritque

$$\begin{aligned}
 & (1 - \\
 & = ny - \frac{(n^2-4)}{6} ny^3 + \frac{n^2-4 \cdot n^2-16}{120} ny^5 - \frac{(n^2-4 \cdot n^2-16 \cdot n^2-36)}{5040} ny^7 + \&c: \\
 & - \frac{n}{2} \cdot y^3 + \frac{n^2-4}{12} \cdot ny^5 - \frac{(n^2-4 \cdot n^2-16)}{240} \cdot ny^7 \\
 & - \frac{n}{8} \cdot y^5 + \frac{n^2-4}{48} \cdot ny^7 \\
 & - \frac{n}{16} \cdot y^7, \text{ scilicet} \\
 & = ny - \frac{(n^2-1)}{6} ny^3 + \frac{n^2-1 \cdot n^2-9}{120} ny^5 - \frac{(n^2-1 \cdot n^2-9 \cdot n^2-25)}{5040} ny^7 + \&c: ;
 \end{aligned}$$

quod

cione quoque Calculo occurrit, nempe $\tau \delta A \pm B \sqrt{-1}$, huic errori repugnare, transitumque illum cum altero Quantitatum Negativarum in Positivas, et conversim, oscitanter, ne dicam stultissime, confundebat. Supervacaneam hic nequidem duco utillimi mentionem Symptomatis, quod plerique Analystrarum *a posteriori* solummodo ratum habent, Aequationis videlicet universalissimae

$$nx \pm \frac{(n^2-1^2)}{2 \cdot 3} nx^3 + \frac{n^2-1^2}{2 \cdot 3} \cdot \frac{n^2-3^2}{4 \cdot 5} nx^5 \pm \frac{(n^2-1^2)}{2 \cdot 3} \cdot \frac{n^2-3^2}{4 \cdot 5} \cdot \frac{n^2-5^2}{6 \cdot 7} nx^7 + \&c: = \pm Q, \text{ cuiuslibet}$$

valoris fuerit Numerus n , Membrumque *Comparisonis* Q , Radices x in-

$$\text{duere formam } \frac{1}{2} \sqrt[n]{Q + \sqrt{Q^2 + 1}} - \frac{1}{2 \sqrt[n]{Q + \sqrt{Q^2 + 1}}}, \text{ nec non } \pm$$

$$\frac{1}{2}$$

$$\frac{1}{2} \sqrt[n]{Q + \sqrt{Q^2 - 1}} \pm \frac{1}{2 \sqrt[n]{Q + \sqrt{Q^2 - 1}}}, \text{ quum ex toties praenotatis fit}$$

$$\frac{1}{\sqrt[n]{Q + \sqrt{Q^2 + 1}}} = \sqrt[n]{-Q + \sqrt{Q^2 + 1}}, \text{ atque } \frac{1}{\sqrt[n]{Q + \sqrt{Q^2 - 1}}} =$$

$\sqrt[n]{Q - \sqrt{Q^2 - 1}}$, quod sine ullius etiam Theorices adiumento Multiplicatio ipsa decernit. Inde originem ducit perinsignis harumce Aequationum univ-
sarum affectio, ex qua fit, ut, si vice x substituatur $\frac{1}{2} \left(z \pm \frac{1}{z} \right)$, abeant

cae Analytica necessitate in Aequationes Trinomiales forma praeditas oecu-

menica $z^{2n} \mp 2Qz^n \mp 1 = 0$, cuius unius Typi ope dimanat, per reso-
lutionem ad instar Quadraticarum Aequationum, Radicum illarum Formula,
de quibus hucusque sermo fuit institutus. Quod miraculo pene proximum
Algebrae annatoribus non satis doctrina difficillima peritis videbitur, statim
ac memoria repetant in compendiosam eam formam redigi quoque Aequa-
tiones praefatas, tametsi ob n non-Imparem terminorum Numero gaudeant
Infinito, quin etiam iam demonstratas iisdem aequipollentes

$$\left(nx \pm \frac{(n^2-2^2)}{2 \cdot 3} nx^3 + \frac{n^2-2^2}{2 \cdot 3} \cdot \frac{n^2-4^2}{4 \cdot 5} \cdot nx^5 \pm \frac{(n^2-2^2)}{2 \cdot 3} \cdot \frac{n^2-4^2}{4 \cdot 5} \cdot \frac{n^2-6^2}{6 \cdot 7} nx^7 + \&c. \right) \sqrt{1 \pm x^2} = \pm Q$$

femper semperque terminorum Numero sine Limite compositas. Commer-
cium certe renovat, intimamque confirmat similitudinem Circuli inter
atque Hyperbolae symptomata Aequatio ipsa summaria, quae utriusque
Curvae in abscondito tenet Trigonometriam, distinctione tantum desumpta
a gemino postremi termini ± 1 & Aequationis $z^{2n} \mp 2Qz^n \mp 1 = 0$ Si-
gno, cuius moniti usus alibi perficietur.

320. Hanc Aequationum in Quadraticae formam affectae conversionem
perscrutatus est, obliqua sane methodo utens, Samuel Koenigius; male cau-
tus ille Geometra, qui Academiae Borussicae Praesidem, Regisque ipsius
a Consiliis, nondum sub illud tempus e vivis ereptum, Leibnitium de *Mi-*

nimae

nimae Actionis Principio plagii arguit acerrime. Veruntamen Aequationes ab eo perpenſae in *Berolinensis Academiae Commentariis* anni MDCCXLIX. Typum habent, dum Radius Circuli de more Unitatem peraequet,

$$x^n - \frac{n}{1} x^{n-1} + \frac{n \cdot n-1}{1 \cdot 2} x^{n-2} - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} x^{n-3} + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} x^{n-4} \&c: = \pm 2C,$$

ideoque referuntur Chordis Complementaribus $2C$ τ Arcus $\frac{\pi}{n}$ illius Circularis, cui x pro Subtenſa pariter Complementari conveniat. Evidens autem eſt Aequationem a Koenigio conſideratam, ſubrogato vice x Cofinu $\frac{x}{2} = y$, formam induere parum diverſam

$$2^{n-1} y^n - \frac{n}{1} \cdot 2^{n-2} y^{n-1} + \frac{n \cdot n-1}{1 \cdot 2} \cdot 2^{n-3} y^{n-2} - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} \cdot 2^{n-4} y^{n-3} + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 2^{n-5} y^{n-4} \&c: = \pm C = C,$$

et adamuſſim congruentem cum iam ſuperius evoluta, atque ad maiorem univerſalitatem, ut denuo conſtet de Cofinum Hyperbolico-Circularium analogia, ita ibidem expoſita

$$\text{Cofin. } \frac{H}{C} (nx) = 2^{n-1} v^n - \frac{n}{1} \cdot 2^{n-2} v^{n-1} + \frac{n \cdot n-1}{1 \cdot 2} \cdot 2^{n-3} v^{n-2} - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} \cdot 2^{n-4} v^{n-3} + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 2^{n-5} v^{n-4} \&c:.$$

Huius reſolutio facillima pendet ab ea cum primis a nobis ſalutata For-

mula Aequalitatis inter Cofin. $\frac{H}{C} (nx) = \pm C = \frac{1}{2} ((\text{Cofin. } H.x + \text{Sin. } H.x)^n + (\text{Cofin. } H.x - \text{Sin. } H.x)^n)$, ſi de priore agatur, et $= \frac{1}{2} ((\text{Cofin. } C.x + (\text{Sin. } C.x)\sqrt{-1})^n + (\text{Cofin. } C.x - (\text{Sin. } C.x)\sqrt{-1})^n)$, ſi de poſterio-

re, ex quibus effluunt $\text{Cofin. } H.x = v = \frac{1}{2} (\text{Cofin. } H.nx + \text{Sin. } H.nx)^{\frac{1}{n}}$

$+ \frac{1}{2} (\text{Cofin. } H.nx - \text{Sin. } H.nx)^{\frac{1}{n}} = \frac{1}{2} \sqrt[n]{C + \sqrt{C^2 - 1}} +$

$\frac{1}{2} \sqrt[n]{C - \sqrt{C^2 - 1}} = \frac{1}{2} \sqrt[n]{C + \sqrt{C^2 - 1}} + \frac{1}{2 \sqrt[n]{C + \sqrt{C^2 - 1}}}$, nec non

Zzzz

Cofin.

$$\text{Cofin. C. } x = v = \frac{1}{2} (\text{Cofin. C. } nx + (\text{Sin. C. } nx) \sqrt{-1})^{\frac{1}{n}} +$$

$$\frac{1}{2} (\text{Cofin. C. } nx - (\text{Sin. C. } nx) \sqrt{-1})^{\frac{1}{n}} = \frac{1}{2} \sqrt[n]{C + \sqrt{C^2 - 1}} +$$

$$\frac{1}{2} \sqrt[n]{C - \sqrt{C^2 - 1}} = \frac{1}{2} \sqrt[n]{C + \sqrt{C^2 - 1}} + \frac{1}{2 \sqrt[n]{C + \sqrt{C^2 - 1}}}. \quad \text{Formulae}$$

Radicum, tam pro Cofinu Hyperbolico, quam pro Circulari, perfecta similitudo satis aperte declarat, uti fuit toties ostensum, unicum Aequationis Typum absque diverſitate Signorum alterutrum Cofinum complecti poſſe, criteriumque discriminis conſiſtere in valore Membri *Comparisonis* C, quod, dum Unitatem ſuperet, ad Hyperbolen, dum infra ſit conſtitutum refertur ad Circulum. Hoc non ita feliciter contigit Aequationibus Sinuum, quas praecedentibus numeris enucleavimus, vidimusque diverſas Radicum formas requirere; nec quidem iniuria, ex eo quod in Cofinibus locum habeant $\text{Cofin. H. } nx \pm \text{Sin. H. } nx = \text{Cofin. H. } nx \pm \sqrt{(\text{Cofin. H. } nx)^2 - 1}$, ac $\text{Cofin. C. } nx \pm (\text{Sin. C. } nx) \sqrt{-1} = \text{Cofin. C. } nx \pm \sqrt{(\text{Cofin. C. } nx)^2 - 1}$; iſtaeque conſimiles Functiones, utpote reſpectivae Aequationis fontes, dubitare turpe eſſet quominus identicum Aequationis Typum producant.

321. Neminem ſubinde latebit in novis hiſce Aequationibus reviviſcere *Casum Irreducibilem* generico ſenſu acceptum, quando nimirum C fuerit Cofinus Circularis, adeoque Numerus minor Unitate. Patet praeterea Aequationem

$$\frac{n-1}{2} v^{\frac{n}{2}} - \frac{n}{1} \cdot 2 \frac{n-3}{2} v^{\frac{n-2}{2}} + \frac{n \cdot n-3}{1 \cdot 2} \cdot 2 \frac{n-5}{2} v^{\frac{n-4}{2}} - \frac{n \cdot n-4 \cdot n-5}{1 \cdot 2 \cdot 3} \cdot 2 \frac{n-7}{2} v^{\frac{n-6}{2}} + \&c. = \pm C,$$

quicumque ſit n, abire in Aequationis formam $z^{2n} \mp 2Cz^n + 1 = 0$ facili methodo ſubſtitutionis $z \approx \frac{1}{2} \left(z + \frac{1}{z} \right) = v$, quod Trinomium cum unius forma conſentit ex praenotatis in num^o 319, nulla certe ſuboriente anomalia, ſiquidem C heic Cofinum, et Q illic Sinum designat. Simplicem ſeſe etiam offert reſolutio ipſius Aequationis, quantum Algebrae ius poſtulat, abſolutiſſima: ſit enim

$$\text{enim vi praeostensorum } v = 1^{\frac{1}{n}} \left(\frac{1}{2} \sqrt[n]{C + \sqrt{C^2 - 1}} + \frac{1}{2} \sqrt[n]{C - \sqrt{C^2 - 1}} \right) \\ = \left(\text{Cofin. } C. \pm \frac{r\Pi}{n} + \left(\text{Sin. } C. \pm \frac{r\Pi}{n} \right) \sqrt{-1} \right) \left(\frac{1}{2} \sqrt[n]{C + \sqrt{C^2 - 1}} + \frac{1}{2} \sqrt[n]{C - \sqrt{C^2 - 1}} \right),$$

Numero r Nullitatem, vel Integrum quemlibet indicante. Eadem igitur ratiocinatione, qua supra usi sumus, in promptu est demonstrare, quod casui $C < 1$, vel $\sqrt{C^2 - 1}$ Magnitudinis Imaginariae convenient Radices omnes Reales, scilicet $\left(\text{Cofin. } C. \frac{r\Pi}{n} \pm \left(\text{Sin. } C. \frac{r\Pi}{n} \right) \sqrt{-1} \right)$

$$\left(\frac{\text{Cofin. } C. x \pm \left(\text{Sin. } C. x \right) \sqrt{-1}}{2} \right), \text{ aut, rite confecta, utrumque simul adhibendo Signum, multiplicatione per ea, quae tradidimus in num. 202., valores Numero } n, \text{ et usitatis Algebrae principiis consentientes, Cofin. } C.$$

$$\left(\frac{r\Pi}{n} + x \right), \text{ quos Reales valores hypothefi } C > 1 \text{ durius negat Analysis.}$$

Consideratio itidem Limitis $\pm C = \pm 1$, quo in puncto idem significant ambae Aequationes Trigonometricae, nulli videtur obnoxia difficultati, quia tunc abeunt Aequationis

$$\frac{1}{2} \frac{n-1}{v} - \frac{n}{1} \frac{n-3}{2} \frac{n-2}{v} + \frac{n \cdot n-3}{1 \cdot 2} \frac{n-5}{2} \frac{n-4}{v} - \frac{n \cdot n-4 \cdot n-5}{1 \cdot 2 \cdot 3} \frac{n-7}{2} \frac{n-6}{v} + \&c: = \pm 1$$

Radices in $1^{\frac{1}{n}} \left(\frac{1}{2} \sqrt[n]{C} + \frac{1}{2} \sqrt[n]{C} \right)$, five prouti in num. 316.

$$\left(\text{Cofin. } C. \frac{r\Pi}{n} \right)^n \sqrt{\pm 1}, \text{ nimirum, si } n \text{ fuerit Impar, } = \pm \text{Cofin. } C. \frac{r\Pi}{n},$$

ac, posito n Pari Indice, $\rightarrow \text{Cofin. } C. \frac{r\Pi}{n}$, aut adverso in casu $(-1)^{\frac{1}{n}} \times$

$$\frac{1}{2} + (-1)^{\frac{1}{n}} \cdot \frac{1}{2} = \text{Cofin. } C. \frac{\lambda P}{n}, \text{ et non } - \text{Cofin. } C. \frac{r\Pi}{n} \text{ quemadmo-}$$

dum accidit Impari cuilibet Exponenti. Longius, quam par esset, non immoror

moror in descriptione geminae hypotheseos valoris $\tau \tilde{n} = 1$, five $= 2$, nec in universaliore $\tau \tilde{n}$ negativi. Profecto nullus nescit finitis numero terminis constare eam Aequationem statim ac n positivus Binarium excedat, sed nusquam abrumpi quando n Unitatem, vel Binarium exaequet. Nihilo tamen minus, quum numerus 315. nos edoceat fore semper valore identicas, tametsi sub diversa specie, Aequationes

$$\frac{n-1}{2} \frac{n}{v} - \frac{n}{1} \cdot \frac{n-3}{2} \frac{n-2}{v} + \frac{n \cdot n-3}{1 \cdot 2} \cdot \frac{n-5}{2} \frac{n-4}{v} - \frac{n \cdot n-4 \cdot n-5}{1 \cdot 2 \cdot 3} \cdot \frac{n-7}{2} \frac{n-6}{v} + \&c: = C, \text{ et}$$

$$C = v^n - \frac{n \cdot n-1}{1 \cdot 2} \cdot v^{n-2} (1-v^2) + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot v^{n-4} (1-v^2)^2 - \frac{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4 \cdot n-5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} v^{n-6} (1-v^2)^3 + \&c,$$

inficiari nemo poterit esse in primis $= \left(\frac{1}{2^2 v} + \frac{1}{2^4 v^3} + \frac{2}{2^6 v^5} + \frac{5}{2^8 v^7} + \frac{14}{2^{10} v^9} + \&c: \right) = 0$, videlicet Infinitinomialum illud fore tesseram Nullitatis,

nec non $= \left(\frac{1}{2^3 v^2} + \frac{2}{2^5 v^4} + \frac{5}{2^7 v^6} + \frac{14}{2^9 v^8} + \&c: \right)$, sine Limite, pro Ni-

hili emblemate assumi debere, cuius Symboli exempla frequenter occurrunt in *falsis*, ut aiunt, Seriebus. Respectu denique habito ad Indices negativos, quomodolibet se gerant, evadantque Infinitinomiales Aequationes illae, de quibus fit mentio, eundem ex iamdudum praecognitis adservant valorem; nec earum resolutio vi numerorum 313. ac 314. diversas expostulat methodos ab Analyseos cultoribus.

322. Sunt itaque Trigonometricae originis, et idcirco Binomii Newtoniani filiae Aequationes omnes, quae diversis temporibus, ac vario Analystrarum labore resolutionem sunt natae, ceterarum quamplurimis ad summum Scientiae Magnitudinum detrimentum haecenus denegatam. In hac exornanda provincia, plusquam Algebrae datum erat, divitias sane maximas se collegisse olim vulgavit Tschirnhausius, laudem inventi captans huiusmodi, quod postea inito experimento, iuxta Formulas *Diario Eruditorum* Lipsiensi anni MDCLXXXIV. insertas, Aequationes nequidem Cubicas ad *Casum* spectantes *irreductibilem* terminis omnibus intermediis eliminatis Binomiales, ut de more immaturius Auctor ille pollicebatur, reddere valet.

Nec

Nec ad ius suum cuique tribuendum omittere licet, historicam a Saurio non parum infici veritatem quando in Praefatione *Mathematici Cursus* affirmat Vincentium Riccatum formas omnes adinvenisse Aequationum Algebraicarum methodo Cardanica resolubilium, vel substitutionis ope $x = m + n$ in Trinomia Quadraticas affectas imitantia transformabilium, ignarus praecedentium laborum, aut saltem oblitus Moivreii. Absit tamen, quod inficias eamus Riccatum ipsum inter nostrae aetatis Scriptores Aequationum utraque duce Trigonometria cultum amplificantes, non sine promerito decore, emicare. Percelebre hac de re exstat in Epistola anno labente MDCLXXVI. ad Oldemburgium missa Leibnitii effatum, de quo consuli poterit Volumen I.^{um} *Commercii Epistolici Colliniani*, illudque perlectus suspicatur Montucla Auctorem ad ea tempora iam fuisse potitum methodo quasdam Aequationes resolvendi Moivreanam aemulante, seu a Multiplicationibus, et Multisectionibus Angulorum derivata. Quidquid autem sit de huius coniecturae probabilitate vix credam serio Leibnitium adseverasse, prouti sentiunt nonnulli in eandem Epistolam iudicium ferentes, se cuiuslibet Ordinis Aequationes contrahere posse in Binomia; quod qui fecerit omnimodam illarum resolutionem adquisivisse, ac sublimius Analyseos Cartesianae punctum tulisse dicendus esset. Tanta enim facilitate antecendentium Canonum ductu Binomiales Aequationes in Factores separantur Incongnitam primo tantum Gradu complectentes, ut, nisi in Matheseos Fastis longe lateque per ora hominum volitasset fama difficultatis demonstrandi Theorematis ex Cartesianis MSS. a Roberto Smithio eruti, atque anno MDCCXXII. Cantabrigiae evulgati, facillimis Binomii Newtoniani Consectariis illud etiam adscribere non dubitarem. Radices quippe Unitatis initio huius Capitis adnumeratae nedum plenissime Theorema ipsum demonstrant, verum etiam eius ingeniosissimam ad Trinomia ampliacionem, quam in suis *Miscellaneis* praelaudatus Moivreus ad maximum Calculi commodum recensuit.

323. Esto Aequatio $x^m \pm a^m = 0$, alteri aequipollens $a^m \left(\left(\frac{x}{a} \right)^m \pm 1 \right) = 0$, vel $a^m (z^m \pm 1) = 0$ si placeat $z = \frac{x}{a}$ unica specie z indigittare. Dum superius valeat Signum, sitque m quilibet Numerus Integer po-

A a a a

sitivus

invicem aequales Signo tantummodo discrepant, inde deducitur eos abire

Factores in magis simplicem formam

$$x - a \left(\text{Cofin. C. } \frac{\Pi}{2m} + (\text{Sin. C. } \frac{\Pi}{2m}) \sqrt{-1} \right) \quad x - a \left(\text{Cofin. C. } \frac{\Pi}{2m} - (\text{Sin. C. } \frac{\Pi}{2m}) \sqrt{-1} \right)$$

$$x - a \left(\text{Cofin. C. } \frac{3\Pi}{2m} + (\text{Sin. C. } \frac{3\Pi}{2m}) \sqrt{-1} \right) \quad x - a \left(\text{Cofin. C. } \frac{3\Pi}{2m} - (\text{Sin. C. } \frac{3\Pi}{2m}) \sqrt{-1} \right)$$

$$x - a \left(\text{Cofin. C. } \frac{5\Pi}{2m} + (\text{Sin. C. } \frac{5\Pi}{2m}) \sqrt{-1} \right) \quad x - a \left(\text{Cofin. C. } \frac{5\Pi}{2m} - (\text{Sin. C. } \frac{5\Pi}{2m}) \sqrt{-1} \right)$$

$$x - a \left(\text{Cofin. C. } \frac{(m-1)\Pi}{2m} + (\text{Sin. C. } \frac{(m-1)\Pi}{2m}) \sqrt{-1} \right), \quad x - a \left(\text{Cofin. C. } \frac{(m-1)\Pi}{2m} - (\text{Sin. C. } \frac{(m-1)\Pi}{2m}) \sqrt{-1} \right).$$

Quibus peractis evadent Factores quaesiti, eliminatis Imaginariis,

$$x^2 - 2 \left(\text{Cofin. C. } \frac{\Pi}{2m} \right) ax + a^2, \quad \text{five, ut placuit Cotefio,} \quad \sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{\Pi}{2m} \right) ax + a^2}$$

$$x^2 - 2 \left(\text{Cofin. C. } \frac{3\Pi}{2m} \right) ax + a^2 \quad \sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{\Pi}{2m} \right) ax + a^2}$$

$$x^2 - 2 \left(\text{Cofin. C. } \frac{5\Pi}{2m} \right) ax + a^2 \quad \sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{3\Pi}{2m} \right) ax + a^2}$$

$$\dots \quad \sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{5\Pi}{2m} \right) ax + a^2}$$

$$\text{si } m \text{ Par, } x^2 - 2 \left(\text{Cofin. C. } \frac{(m-1)\Pi}{2m} \right) ax + a^2 \quad \sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{5\Pi}{2m} \right) ax + a^2}$$

aut ex adverfo, $x + a$, dum m Impar, et eius vice si Par

$$\sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{(m-1)\Pi}{2m} \right) ax + a^2}, \quad \sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{(m-1)\Pi}{2m} \right) ax + a^2}$$

prouti

prouti prima Theorematis pars notissimo Diagrammate inlustrata praefert.

324. Secundam aggredi partem Theorematis inter *Posthuma* Cotesiana adinventi aequae simplex videbitur. Ista supponit Aequationem resolvendam

$$x^m - a^m = 0, \text{ quae indidem vertitur in } a^m (z^m - 1) = 0, \text{ eius-}$$

que Radices, excluso Factore a^m , sunt ex praemissis $z = \text{Cofin. C. } \frac{n\Pi}{m} +$

$(\text{Sin. C. } \frac{n\Pi}{m}) \sqrt{-1}$. Ibi notandum, quod Indici Pari m convenient tam az

$= x = a$, quam $= -a$, secus ac occurrit in Impari m Exponente; tunc enim tantummodo $az = x = a$ reperitur. Usitato in systemate digestis ergo Factoribus eorum Typus sic obtinebitur.

$x - a$

$$x - a \left(\text{Cofin. C. } \frac{\Pi}{m} + (\text{Sin. C. } \frac{\Pi}{m}) \sqrt{-1} \right)$$

$$x - a \left(\text{Cofin. C. } \frac{(m-1)\Pi}{m} + (\text{Sin. C. } \frac{(m-1)\Pi}{m}) \sqrt{-1} \right)$$

$$x - a \left(\text{Cofin. C. } \frac{2\Pi}{m} + (\text{Sin. C. } \frac{2\Pi}{m}) \sqrt{-1} \right)$$

$$x - a \left(\text{Cofin. C. } \frac{(m-2)\Pi}{m} + (\text{Sin. C. } \frac{(m-2)\Pi}{m}) \sqrt{-1} \right)$$

$$x - a \left(\text{Cofin. C. } \frac{3\Pi}{m} + (\text{Sin. C. } \frac{3\Pi}{m}) \sqrt{-1} \right)$$

$$x - a \left(\text{Cofin. C. } \frac{(m-3)\Pi}{m} + (\text{Sin. C. } \frac{(m-3)\Pi}{m}) \sqrt{-1} \right)$$

et quando
m Impar

$$x - a \left(\text{Cofin. C. } \frac{(m-1)\Pi}{2m} + (\text{Sin. C. } \frac{(m-1)\Pi}{2m}) \sqrt{-1} \right) \left(x - a \left(\text{Cofin. C. } \frac{(m - \frac{m-1}{2})\Pi}{m} + (\text{Sin. C. } \frac{(m - \frac{m-1}{2})\Pi}{m}) \sqrt{-1} \right) \right)$$

sed ex adverso,
dum Par,

$$x + a = x - a \left(\text{Cofin. C. } \frac{\frac{m}{2}\Pi}{m} + (\text{Sin. C. } \frac{\frac{m}{2}\Pi}{m}) \sqrt{-1} \right).$$

Subrogatis autem aequalibus hi Factores evadunt

$$x - a,$$

$x - a$, vel etiam $x + a$

$$x - a \left(\text{Cofin. C. } \frac{\Pi}{m} + (\text{Sin. C. } \frac{\Pi}{m}) \sqrt{-1} \right) \quad x - a \left(\text{Cofin. C. } \frac{\Pi}{m} - (\text{Sin. C. } \frac{\Pi}{m}) \sqrt{-1} \right)$$

$$x - a \left(\text{Cofin. C. } \frac{2\Pi}{m} + (\text{Sin. C. } \frac{2\Pi}{m}) \sqrt{-1} \right) \quad x - a \left(\text{Cofin. C. } \frac{2\Pi}{m} - (\text{Sin. C. } \frac{2\Pi}{m}) \sqrt{-1} \right)$$

$$x - a \left(\text{Cofin. C. } \frac{3\Pi}{m} + (\text{Sin. C. } \frac{3\Pi}{m}) \sqrt{-1} \right) \quad x - a \left(\text{Cofin. C. } \frac{3\Pi}{m} - (\text{Sin. C. } \frac{3\Pi}{m}) \sqrt{-1} \right)$$

$$x - a \left(\text{Cofin. C. } \frac{(\frac{m-1}{2}\Pi)}{m} + (\text{Sin. C. } \frac{(\frac{m-1}{2}\Pi)}{m}) \sqrt{-1} \right), \quad x - a \left(\text{Cofin. C. } \frac{(\frac{m-1}{2}\Pi)}{m} - (\text{Sin. C. } \frac{(\frac{m-1}{2}\Pi)}{m}) \sqrt{-1} \right);$$

et Factoribus ideo geminis in unum coniunctis ad Imaginariarum destructionem ineundam,

$x - a$, vel $x^2 - a^2$, scilicet Cotesiano stylo, $x - a$, aut simul $x - a$ et $x + a$

$$x^2 - 2 \left(\text{Cofin. C. } \frac{\Pi}{m} \right) ax + a^2 \quad \sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{\Pi}{m} \right) ax + a^2}$$

$$\sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{\Pi}{m} \right) ax + a^2}$$

$$x^2 - 2 \left(\text{Cofin. C. } \frac{2\Pi}{m} \right) ax + a^2 \quad \sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{2\Pi}{m} \right) ax + a^2}$$

$$\sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{2\Pi}{m} \right) ax + a^2}$$

$$x^2 - 2 \left(\text{Cofin. C. } \frac{3\Pi}{m} \right) ax + a^2 \quad \sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{3\Pi}{m} \right) ax + a^2}$$

$$\sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{3\Pi}{m} \right) ax + a^2}$$

$$x^2 - 2 \left(\text{Cofin. C. } \frac{(\frac{m-1}{2}\Pi)}{m} \right) ax + a^2$$

$$\sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{(\frac{m-1}{2}\Pi)}{m} \right) ax + a^2}$$

$$\sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{(\frac{m-1}{2}\Pi)}{m} \right) ax + a^2};$$

Bbbbb

et haec

et haec postrema Factorum dispositio integram absolvit Theorematis celeberrimi, in quo contemplando sumus, Analysis.

325. Factores igitur $\tau\tilde{}$ Binomii $x^m \pm a^m$, iuxta graphicam Circuli sectionem a Cotesio elegantissime traditam, Indici semper m parem Numerum complent. Veruntamen, eliminatis Irrationalibus, tot erunt Factores Rationales secundi Ordinis, ut in casu m Paris numerum exaequent $\frac{m}{2}$, sive ipsius Indicis medietatem, at contraria in hypothese numerum $\frac{m-1}{2}$, ultra quem Binomialis Factor primi Gradus $x \pm a$ superaddetur. Multiplicatis invicem Binomiis $x^m + a^m$, et $x^m - a^m$, Binomium exurgit $x^{2m} - a^{2m}$, quod, utpote Indice $2m$ immutabiliter Pari praeditum, resolvetur praecedentium canonum ope in Factores, qui rite perpenſi veritatem confirmant Theorematis Cotesiani, cum nec hilum distent ab illorum singulis pertinentium tam ad $x^m + a^m$, quam ad $x^m - a^m$, sed simul assumptis. Sicuti vero Rationalium Indicum Limes statui poterit Infinite magnus Exponens, in abscondito non erit Binomio $x^\infty \pm a^\infty$ innumerabiles respondere Factores; hique eruentur statim ac meminerimus eos omnes pendere ab repertis Radicibus $\tau\tilde{}$ $(\mp 1)^{\frac{1}{\infty}}$, quia $x^\infty \pm a^\infty = a^\infty \left(\left(\frac{x}{a} \right)^\infty \pm 1 \right) = a^\infty \cdot (z^\infty \pm 1)$ ad praenotatorum exemplum. Habebitur ideo $x^\infty - a^\infty =$

$$\sqrt{\left(\begin{array}{c} x^2 - a^2 \\ x^2 - 2ax + a^2 \\ + \frac{a^2 \Pi^2}{\infty^2} \end{array} \right)}$$

aut solum $x - a$, subintellecto ∞ uti Limite Imparium,

$$\sqrt{\left(\begin{array}{c} x^2 - 2ax + a^2 \\ + \frac{a^2 (2\Pi)^2}{\infty^2} \end{array} \right)}$$

Bis repetendi Factores Irrationales, vel una tantum vice si eliminetur asymmetria.

$$\sqrt{(x^2 - a^2)}$$

Si ∞ pro Limite
Parium m suma-
tur, Impariter si-
mul ac Pariter-
Parium, exempli
caussa 2^∞ .

$$\sqrt{\left(x^2 - 2ax + a^2 + \frac{a^2(3\Pi)^2}{\infty^2} \right)}$$

Quoad usque Coefficiens $\tau \Pi$ sit fini-
tus. Et eo subinde verso in quamlibet
Infiniti aliquotam vel aliquantam par-

$$\sqrt{\left(x^2 - 2 \left(\text{Cofin. C. } \frac{n\Pi}{m} \right) ax + a^2 \right)}$$

tem $\frac{n \cdot \infty}{m}$, seu $\frac{r \cdot \infty}{m}$ illi Periphe-
riae divisioni relatae, a cuius conti-
nua bisectione Infinitum prodiit, puta

$$\sqrt{\left(x^2 - 2 \left(\text{Cofin. C. } \frac{n\Pi}{m} \right) ax + a^2 + \frac{a^2 \Pi^2}{\infty^2} \right)}$$

$\frac{n \cdot 2^\infty}{p}$, $\frac{r \cdot 2^\infty}{q}$ &c:

$$\sqrt{\left(x^2 - 2 \left(\text{Cofin. C. } \frac{n\Pi}{m} \right) ax + a^2 + \frac{a^2(2\Pi)^2}{\infty^2} \right)}$$

$$\sqrt{\left(x^2 - 2 \left(\text{Cofin. C. } \frac{n\Pi}{m} \right) ax + a^2 + \frac{a^2(3\Pi)^2}{\infty^2} \right)}$$

$$\sqrt{\left(x^2 - 2 \left(\text{Cofin. C. } \frac{r\Pi}{m} \right) ax + a^2 \right)}$$

Bis pariter repeten-
di, vel una vice
si vertantur in Ra-
tionales.

$$\sqrt{\left(x^2 - 2 \left(\text{Cofin. C. } \frac{r\Pi}{m} \right) ax + a^2 + \frac{a^2 \Pi^2}{\infty^2} \right)}$$

$$\sqrt{\left(x^2 - 2 \left(\text{Cofin. C. } \frac{r\Pi}{m} \right) ax + a^2 + \frac{a^2(2\Pi)^2}{\infty^2} \right)}$$

$\sqrt{}$

$$\sqrt{\left(x^2 - 2 \left(\text{Cofin. C. } \frac{r\Pi}{m} \right) ax + a^2 + \frac{a^2 (3\Pi)^2}{\infty^2} \right)}$$

usque dum fine fine cumulatis Trinomiis, quorum Lex satis patet, perven-

tum fuerit ad ultimum fine comite $\sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{\infty \cdot \Pi}{2} \right) ax + a^2}$

$= \sqrt{x^2 + 2ax + a^2}$, cuius iam rationem habuimus in $x^2 - a^2$ Factorum initium maiore elegantia aperientes. Secus equidem accidit in evolutione Factorum $x^\infty - a^\infty$ quando Infinitus Exponens Imparium classi adscriberetur, veluti $2^\infty - 1$: ipsa enim Theoria Factores investigandos hoc ordine tantum suppeditabit

$$\frac{x - a}{\sqrt{\left(x^2 - 2ax + a^2 + \frac{a^2 \Pi^2}{\infty^2} \right)}}, \text{ ut superius monuimus,}$$

$$\sqrt{\left(x^2 - 2ax + a^2 + \frac{a^2 (2\Pi)^2}{\infty^2} \right)}, \text{ bis repetendi nisi Rationalitate restituta,}$$

$$\sqrt{\left(x^2 - 2ax + a^2 + \frac{a^2 (3\Pi)^2}{\infty^2} \right)}$$

—absque ullo Limite;

et, ob Numeros Impares *primos*, Peripheria sine mutuo Punctorum remeatu secata, impossibile, vel saltem difficillimum erit in tanta Infiniti perplexitate alios Factores subiungere. Hoc unum in tuto ponere facile est, quod

quod Factor deficiat $\sqrt{x^2 - 2 \left(\text{Cofin. C. } \frac{\infty}{2} \Pi \right) ax + a^2}$, five $x \rightarrow a$,

propter Imparis Infiniti suppositionem.

326. Schema ad Cotesii mentem depictum nedum minus obscuram, quam adpareat nuda oratione, reddet investigationem adumbratam nuperime, verum etiam Analyticam commonstrabit necessitatem addendi Radici-

bus $1^{\frac{1}{\infty}}$ in numeris 293. ac 294. a generali Formula $1 \pm \frac{n\Pi}{\infty} \sqrt{-1}$ comprehensis alias ex superioribus arguendi modis deductas $\text{Cofin. C. } \frac{(n.2^{\infty-p})\Pi}{2^{\infty}}$

$\pm \left(\text{Sin. C. } \frac{(n.2^{\infty-p})\Pi}{2^{\infty}} \right) \sqrt{-1}$, vel $\text{Cofin. C. } \left(\frac{n}{2^p} \cdot \Pi \right) \pm \left(\text{Sin. C. } \left(\frac{n}{2^p} \right) \right)$

$\sqrt{-1}$). Harum veritatis comprobatio *a posteriori* potissimum constat, quip-

pe cum extra dubium habeatur fore $\left(\text{Cofin. C. } \frac{n\Pi}{2^p} \pm \left(\text{Sin. C. } \frac{n\Pi}{2^p} \right) \sqrt{-1} \right)^{2^{\infty}}$

$= \text{Cofin. C. } n(2^{\infty-p})\Pi \pm \left(\text{Sin. C. } n(2^{\infty-p})\Pi \right) \sqrt{-1} = 1$, propter

$2^{\infty-p}$, ideoque et $n(2^{\infty-p})$ Numerum integrum Infinitum. Diversos eligendo pro Indicis Infinite magni designatione Numeros Pares diversae quoque combinationes emergent, quas fusius persequi inutile duco. Semita enim detecta cum sit, quilibet proprio Marte, quae non dicta fuerint, inveniet, ac praesertim evolutionem $x^{\infty} \rightarrow a^{\infty}$ in respectivos Factores. Sed iniquus Infinito esse omittens specimen aliquod non adhuc visae Radicum

formae $(-1)^{\frac{1}{\infty}}$, quam complectitur Binomium $\text{Cofin. C. } \left(\frac{\lambda}{2^p} \right) P \pm$

$\left(\text{Sin. C. } \frac{(\lambda)}{2^p} P \right) \sqrt{-1}$, eoquod nulla operis difficultate in aperto maneat

Aequalitas $\left(\text{Cofin. C. } \frac{\lambda P}{2^p} \pm \left(\text{Sin. C. } \frac{\lambda P}{2^p} \right) \sqrt{-1} \right)^{2^{\infty}} \pm 1 = \text{Cofin. C.}$

$\lambda(2^{\infty-p} \pm 1) P \pm \left(\text{Sin. C. } \lambda(2^{\infty-p} \pm 1) P \right) \sqrt{-1} = -1$ ob

C c c c c

Coefficientem

Coefficientem Imparem $\lambda (2^{\infty} - P \pm 1)$. Altius autem, quam poterit, sit in mente repositum ideo Factores obtineri Rationales in Cotesiano Theoremate, quia, exceptis $x \pm a$, alia quaevis Radix $\Phi + \Delta \sqrt{-1}$ sociam habet ex altera Circuli Diametri parte $\Phi - \Delta \sqrt{-1}$, suntque haec Binaria numero tantum $\frac{m-1}{2}$, five $\frac{m-2}{2}$. Cumque ista conditio, sine qua non, praesupponat Peripheriae Circularis Multisectionem eiusmodi, ut eo redeat post unicam revolutionem completam unde sumpsit initium, nunquam fiet locus Theoremati si de Binomio $x^v \pm a^v$ agatur, in quo Exponens v Fractum, Irrationalem, et multo maiore ratione Transcendentem Numerum exprimat, tametsi illic etiam unicuique Radici $\Omega + Y \sqrt{-1}$, alia $\Omega - Y \sqrt{-1}$ comes adfit. Nullam denique excitabit difficultatem distinctio casus $x > a$ ab altero $x < a$ in toties citato Binomio $x^m - a^m$: namque in versis Factoribus ii pertinebunt ad $a^m - x^m$, quod demonstratione non indiget.

327. Aequatio dudum resoluta $x^m - a^m = 0$ praebet universalissimae solutionis capacem ex classe Aequalitatum, quas Analytiae vocant *Convertibiles*, hanc $x^n + ax^{n-1} + a^2x^{n-2} + a^3x^{n-3} + a^4x^{n-4} + a^5x^{n-5} + \dots + a^n = 0$, posito Indice n semper Pari, nimirum, $= m - 1$ dum m Impar, et $= m - 2$ dum Par m fuerit e contra. Neque difficultate laborat fortasse id ostendere: illa etenim canonica Aequationum forma, pro qua perficienda subtiliter valde omnem lapidem movit Gabriel Manfredius in *Litteratorum Diarii Venetiis* tunc editi *Supplementorum* Volumine II^o, ac in *Bononiensis Instituti* Commentariorum Volumine I^o, eximius scilicet ille Geometra, cui primam Itali debent tum adolescentis Infinitesimalis Algorithmi plenior notitiam ab eius Opere *de Constructione Aequationum Differentialium*, sub annum MDCCVII. in lucem aspectumque hominum prolato, doctos inter nostrates viros diffusam, nihil aliud est nisi Quotiens, persaepe reductus, ab partitione dimanans Binomii $x^m - a^m$ per $x - a$, aut $x^2 - a^2$, methodo usitata divisi. Recipiet itaque Aequatio

$$\frac{x^n}{x}$$

$x^n + ax^{n-1} + a^2x^{n-2} + a^3x^{n-3} + a^4x^{n-4} + a^5x^{n-5} + \dots + a^n = 0$, in qua
 n semper Par, Radices omnes Imaginarias

$$x = a \left(\text{Cofin. C. } \frac{\Pi}{n+1} + (\text{Sin. C. } \frac{\Pi}{n+1}) \sqrt{-1} \right), \quad \text{five Aequatio ipsa erit Productum Fa-}$$

ctorum Trinomialium

$$x = a \left(\text{Cofin. C. } \frac{\Pi}{n+1} - (\text{Sin. C. } \frac{\Pi}{n+1}) \sqrt{-1} \right)$$

$$x^2 - 2 \left(\text{Cofin. C. } \frac{\Pi}{n+1} \right) ax + a^2$$

$$x = a \left(\text{Cofin. C. } \frac{2\Pi}{n+1} + (\text{Sin. C. } \frac{2\Pi}{n+1}) \sqrt{-1} \right)$$

$$x^2 - 2 \left(\text{Cofin. C. } \frac{2\Pi}{n+1} \right) ax + a^2$$

$$x = a \left(\text{Cofin. C. } \frac{2\Pi}{n+1} - (\text{Sin. C. } \frac{2\Pi}{n+1}) \sqrt{-1} \right)$$

$$x^2 - 2 \left(\text{Cofin. C. } \frac{3\Pi}{n+1} \right) ax + a^2$$

$$x = a \left(\text{Cofin. C. } \frac{3\Pi}{n+1} + (\text{Sin. C. } \frac{3\Pi}{n+1}) \sqrt{-1} \right)$$

$$x = a \left(\text{Cofin. C. } \frac{3\Pi}{n+1} - (\text{Sin. C. } \frac{3\Pi}{n+1}) \sqrt{-1} \right)$$

$$x^2 - 2 \left(\text{Cofin. C. } \frac{n}{2} \frac{\Pi}{n+1} \right) ax + a^2$$

$$x = a \left(\text{Cofin. C. } \frac{\frac{n}{2} \Pi}{n+1} + (\text{Sin. C. } \frac{\frac{n}{2} \Pi}{n+1}) \sqrt{-1} \right)$$

$$x^2 - 2 \left(\text{Cofin. C. } \frac{\frac{n}{2} \Pi}{n+1} \right) ax + a^2$$

$$x = a \left(\text{Cofin. C. } \frac{\frac{n}{2} \Pi}{n+1} - (\text{Sin. C. } \frac{\frac{n}{2} \Pi}{n+1}) \sqrt{-1} \right)$$

Quae cum ita sint, dum in animo nobis esset Factores illos graphice in Circulo, prouti Cotesius fecit, depingere, voto plenissime satisfaceret quilibet secans Peripheriam in Arcus aequales $n+1$, initio desumpto ab uno Radii extremo, super quem iacet x ; namque ab huius x extremitate veluti centro Rectae omnes digressae, quae coniungant praefatarum Sectionum

Puncta, earum minima excepta, invicem multiplicatae producent $x^n + ax^{n-1}$

$+ a^2x^{n-2} + a^3x^{n-3} + a^4x^{n-4} + a^5x^{n-5} + \dots + a^n$, Polynomium

nimirum

nimirum *Convertibile*, atque $n \rightarrow 1$ terminis coacervatum. Si ergo Index $n = \infty$, Aequatio, quae oritur ex Infinitis numero terminis composita x^∞

$$+ ax^{\infty-1} + a^2 x^{\infty-2} + a^3 x^{\infty-3} + a^4 x^{\infty-4} + a^5 x^{\infty-5} +$$

$$a^6 x^{\infty-6} + \dots + a^\infty = 0, \text{ gaudebit Radicibus innumeris}$$

$$x = a \pm a \frac{(n\pi)}{\infty + 1} \sqrt{-1} = a \left(1 \pm \frac{n\pi}{\infty} \sqrt{-1} \right), \text{ et omnibus Imaginariis, ut}$$

fileam de ceteris paullisper in num°. 326. memoratis.

328. Confimili pacto Aequatio pariter *Convertibilis*, formam induens

Signorum alternatione ab priore discrepantem $x^n - ax^{n-1} + a^2 x^{n-2} - a^3 x^{n-3} + a^4 x^{n-4} - a^5 x^{n-5} + \dots + a^n = 0$, ideoque Ordinis Paris, resolvetur ope Radicum Imaginariarum ab Circulari Trigonometria dependentium, nempe

$$x = a \left(\text{Cofin. C. } \frac{P}{n+1} \pm \left(\text{Sin. C. } \frac{P}{n+1} \right) \sqrt{-1} \right), \text{ quae tribuunt Factores } x^2 - 2 \left(\text{Cofin. C. } \frac{P}{n+1} \right) ax + a^2$$

$$x = a \left(\text{Cofin. C. } \frac{3P}{n+1} \pm \left(\text{Sin. C. } \frac{3P}{n+1} \right) \sqrt{-1} \right) \quad x^2 - 2 \left(\text{Cofin. C. } \frac{3P}{n+1} \right) ax + a^2$$

$$x = a \left(\text{Cofin. C. } \frac{5P}{n+1} \pm \left(\text{Sin. C. } \frac{5P}{n+1} \right) \sqrt{-1} \right) \quad x^2 - 2 \left(\text{Cofin. C. } \frac{5P}{n+1} \right) ax + a^2$$

$$x = a \left(\text{Cofin. C. } \frac{n-1}{n+1} P \pm \left(\text{Sin. C. } \frac{n-1}{n+1} P \sqrt{-1} \right) \right) \quad x^2 - 2 \left(\text{Cofin. C. } \frac{n-1}{n+1} P \right) ax + a^2$$

ex quorum Multiplicatione Polynomium $x^n - ax^{n-1} + a^2 x^{n-2} - a^3 x^{n-3} + a^4 x^{n-4} - \&c: + a^n$ superioris ad instar conflatur. Est enim Polyno-

mium istud Quotiens $\frac{x^{n+1} + a^{n+1}}{x + a}$, quemadmodum actualis Divisio patefacit; illaque etiam hoc idem ostenditur intentata, eoquod vi numeri

praecedentis

praecedentis Multinomium $x^n + ax^{n-1} + a^2x^{n-2} + a^3x^{n-3} + a^4x^{n-4}$

$+ \&c: + a^n = \frac{x^{n+1} - a^{n+1}}{x - a}$, et idcirco permutando $- a$ in $+ a$

esse debeat $\frac{x^{n+1} + a^{n+1}}{x + a} = x^n - ax^{n-1} + a^2x^{n-2} - a^3x^{n-3}$

$+ a^4x^{n-4} - \&c: + a^n$, prouti decernunt Elementa Analyseos. Quibus sic stantibus nullus dubito, quominus per se constet Aequationem etiam Infinitinomiam

$$x^\infty - ax^{\infty-1} + a^2x^{\infty-2} - a^3x^{\infty-3} + a^4x^{\infty-4} - a^5x^{\infty-5} + a^6x^{\infty-6} - \dots + a^\infty = 0$$

resolutionem universalem admittere praesidio innumerarum Radicum, quas

Formula suppeditat $x = a \left(1 \pm \frac{\lambda P}{\infty + 1} \sqrt{-1} \right) = a \left(1 \pm \frac{\lambda P}{\infty} \sqrt{-1} \right)$, nec

non primum Membrum Aequationis illius Polynomialis aequae recipere graphicam ac Cotesiano Theoremati similem evolutionem, divisa Circuli Circumferentia in Arcus inter se pares $2n + 2$, Rectisque more solito ductis, dummodo ipsarum maxima ceteris non adnumeretur.

329. Instat autem a Moivreo promotum Theorema, ut extremam manum imponamus pervestigationi, in qua longius versati. Trinomium $z^m \mp 2Cz^m + 1 = 0$ Aequationum Quadraticarum stylo tractatum praebet $z = (\pm C \pm$

$\sqrt{C^2 - 1})^{\frac{1}{m}}$, quemadmodum ab Algebrae deducitur Institutionibus. Esto primum $C < 1$, five $= \text{Cofin. } C. A$, quod indicat fieri $z =$

$$\sqrt[m]{\text{Cofin. } C. A \pm (\text{Sin. } C. A) \sqrt{-1}} = i^{\frac{1}{m}} \left(\text{Cofin. } C. \frac{A}{m} \pm (\text{Sin. } C. \frac{A}{m}) \sqrt{-1} \right) =$$

$$\left(\text{Cofin. } C. \frac{n\pi}{m} \pm (\text{Sin. } C. \frac{n\pi}{m}) \sqrt{-1} \right) \left(\text{Cofin. } C. \frac{A}{m} \pm (\text{Sin. } C. \frac{A}{m}) \sqrt{-1} \right).$$

$$\text{Igitur } z = \left(\text{Cofin. } C. (A + n\pi) \pm (\text{Sin. } C. (A + n\pi)) \sqrt{-1} \right)^{\frac{1}{m}} =$$

$$\text{Cofin. } C. \frac{(A + n\pi)}{m} \pm \left(\text{Sin. } C. \frac{(A + n\pi)}{m} \right) \sqrt{-1} \text{ iuxta ea, quae toties de-}$$

D d d d d

monstravi,

monstravi, et haftenus adhibui pro inveniendis Unitatis Radicibus. Resolvitur inde oecumenica Aequatio $z^{2m} - 2 (\text{Cofin. C. } A) z^m + 1 = 0$, vel potius $z^{2m} - 2 (\text{Cofin. C. } A) a^m z^m + a^{2m} = 0$ Radicum heic ordine dispositarum subsidio, nimirum

$$z = a \left(\text{Cofin. C. } \frac{A}{m} \pm \left(\text{Sin. C. } \frac{A}{m} \right) \sqrt{-1} \right)$$

existen-
te earum
Numero
 $2m$

$$z = a \left(\text{Cofin. C. } \frac{\Pi + A}{m} \pm \left(\text{Sin. C. } \frac{\Pi + A}{m} \right) \sqrt{-1} \right), \text{ omnibus existen-}$$

$$z = a \left(\text{Cofin. C. } \frac{2\Pi + A}{m} \pm \left(\text{Sin. C. } \frac{2\Pi + A}{m} \right) \sqrt{-1} \right) \text{ tibus Imaginariis e-}$$

$$z = a \left(\text{Cofin. C. } \frac{3\Pi + A}{m} \pm \left(\text{Sin. C. } \frac{3\Pi + A}{m} \right) \sqrt{-1} \right) \text{ tiamfi Cofin. C. (A)}$$

evanescat, aut Tri-
nomium abeat in Bi-
nomium $z^{2m} + a^{2m}$
 $= 0$,

$$z = a \left(\text{Cofin. C. } \frac{(m-1)\Pi + A}{m} \pm \left(\text{Sin. C. } \frac{(m-1)\Pi + A}{m} \right) \sqrt{-1} \right);$$

ita, ut $z^{2m} - 2 (\text{Cofin. C. } A) a^m z^m + a^{2m} = z^2 - 2 \left(\text{Cofin. C. } \frac{A}{m} \right) az + a^2 \times$

$$z^2 - 2 \left(\text{Cofin. C. } \frac{\Pi + A}{m} \right) az + a^2 \times$$

$$z^2 - 2 \left(\text{Cofin. C. } \frac{2\Pi + A}{m} \right) az + a^2 \times$$

$$z^2 - 2 \left(\text{Cofin. C. } \frac{3\Pi + A}{m} \right) az + a^2 \times$$

$$z^2 - 2 \left(\text{Cofin. C. } \frac{(m-1)\Pi + A}{m} \right) az + a^2$$

Factoribus Trinomialibus invicem multiplicatis, ex quorum Producto Trinomium coalescit, uti graphicis delineavit Moivreus Circularis Arcus Multisectionibus.

330. Nulla insuper singulari eget mentione hypothesi $\tau\tilde{C} = 1$, nisi Theorema Cotesii denuo contemplari mens foret. Tunc etenim Trinomium $z^{2m} \mp 2a^m z^m + a^{2m}$ vertitur in Quadratum $(z^m \mp a^m)^2$, cuius idcirco Factores, verso A in $n\Pi$, ac λP respective, non secus ac supra dispositi evadent

$(z^m - a^m)^2 =$ $z^2 - 2az + a^2 \times$ $z^2 - 2\left(\text{Cofin. C. } \frac{\Pi}{m}\right) az + a^2 \times$ $z^2 - 2\left(\text{Cofin. C. } \frac{2\Pi}{m}\right) az + a^2 \times$ $z^2 - 2\left(\text{Cofin. C. } \frac{3\Pi}{m}\right) az + a^2 \times$ ----- $z^2 - 2\left(\text{Cofin. C. } \frac{(m-1)\Pi}{m}\right) az + a^2$	$(z^m + a^m)^2 =$ $z^2 - 2\left(\text{Cofin. C. } \frac{P}{m}\right) az + a^2 \times$ $z^2 - 2\left(\text{Cofin. C. } \frac{3P}{m}\right) az + a^2 \times$ $z^2 - 2\left(\text{Cofin. C. } \frac{5P}{m}\right) az + a^2 \times$ $z^2 - 2\left(\text{Cofin. C. } \frac{7P}{m}\right) az + a^2 \times$ ----- $z^2 - 2\left(\text{Cofin. C. } \frac{(2m-1)P}{m}\right) az + a^2.$
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Hisce ad superiorum normam consideratis universa apte inter se cohaerere, et adamussim congruere cum iis, quae in numeris 323. ac 324. exstant, patebit.

331. Adeunda denique erit postrema suppositio $\tau\tilde{C} > 1$, scilicet $= \text{Cofin. H. A}$, ad quam perventi Analystae filum usque nunc expositae resolutionis abrumpere solent, ac separare Trinomium in duo Realia Binomia, quibus discriminatim adplicant Cotesiani Inventi Theorice. Profecto admiratione digna res est eloquentissimam neglexisse Geometras huius in Trinomii evolvendi doctrina Hyperbolae, et Circuli Harmoniam, cui in integrum restituendae Paradigma Calculi maioris evidentiae causa subnectere placet.

$$z^{2m} \mp 2(\text{Cofin. H. A}) a^m z^m + a^{2m} = \frac{z^{2m}}{a^{2m}} \mp 2(\text{Cofin. H. A}) \frac{z^m}{a^m} + 1$$

$1 = x^{2m} \mp 2 (\text{Cofin. H. A}) x^m + 1 = 0$ praebet Radicum Formulam $x =$

$$(\pm \text{Cofin. H. A} \pm \text{Sin. H. A})^{\frac{1}{m}} = 1^{\frac{1}{m}} \left(\sqrt[m]{\pm \text{Cofin. H. A} \pm \text{Sin. H. A}} \right), \text{ vel } z$$

$$= 1^{\frac{1}{m}} \cdot a \sqrt[m]{\text{Cofin. H. A} \pm \text{Sin. H. A}} = 1^{\frac{1}{m}} a \left(\text{Cofin. H. } \frac{A}{m} \pm \text{Sin. H. } \frac{A}{m} \right),$$

nimirum dempta, ut unicum methodi facillime prosequendae specimen exhibeam, Signi Cofinus ambiguitate

$$z = a \sqrt[m]{\text{Cofin. H. A} + \text{Sin. H. A}} = aB$$

$$z = a \sqrt[m]{\text{Cofin. H. A} - \text{Sin. H. A}} = aC$$

$$z = aB \left(\text{Cofin. C. } \frac{\Pi}{m} + \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1} \right)$$

$$z = aB \left(\text{Cofin. C. } \frac{\Pi}{m} - \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1} \right)$$

$$z = aC \left(\text{Cofin. C. } \frac{\Pi}{m} + \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1} \right)$$

$$z = aC \left(\text{Cofin. C. } \frac{\Pi}{m} - \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1} \right) \text{ , quae Ra-}$$

dices tri-
buunt m
Factores

$$z = aB \left(\text{Cofin. C. } \frac{2\Pi}{m} + \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1} \right)$$

$$z = aB \left(\text{Cofin. C. } \frac{2\Pi}{m} - \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1} \right)$$

$$z = aC \left(\text{Cofin. C. } \frac{2\Pi}{m} + \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1} \right)$$

$$z = aC \left(\text{Cofin. C. } \frac{2\Pi}{m} - \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1} \right)$$

*dum m Impar
fit ;*

*si vero Par , erunt
postremae Radices*

$$z = -aB$$

$$z = -aC$$

Quoad B et C simul.

$$z^2 - 2 \left(\text{Cofin. H. } \frac{A}{m} \right) az + a^2 \begin{cases} m \text{ Impar} \\ m \text{ Par} \end{cases} \left| \begin{matrix} z^2 - B^2 a^2 \end{matrix} \right.$$

$$z^2 - 2 \left(\text{Cofin. C. } \frac{\Pi}{m} \right) Baz + B^2 a^2$$

$$z^2 - 2 \left(\text{Cofin. C. } \frac{\Pi}{m} \right) Caz + C^2 a^2$$

$$z^2 - 2 \left(\text{Cofin. C. } \frac{2\Pi}{m} \right) Baz + B^2 a^2$$

$$z^2 - 2 \left(\text{Cofin. C. } \frac{2\Pi}{m} \right) Caz + C^2 a^2$$

$$\begin{matrix} \text{si } m \text{ Par} \\ \left| \begin{matrix} z^2 - C^2 a^2 \end{matrix} \right. \end{matrix}$$

&c: usque ad Radicum Numerum m, tam pro B, quam pro C.

, ordine multiplicationis servato, qualem praeseferrunt e-regione descriptae Radices.

332. Huic

332. Huic etiam Trigonometriae Hyperbolico-Circularis systemati evo-

lutio se praebet Trinomialis Aequationis $z^{2m} \pm 2F \cdot a^m z^m - a^{2m} = 0$,
quam nescio quo fato pene omnes minus complent Analystae. Ad geminum
enim intermedii termini Signum oblitterandum, subintellecto coefficiente F
positivo, seu negativo, nanciscimur $z = a(F \pm \sqrt{F^2 + 1})^{\frac{1}{m}} = 1^{\frac{1}{m}} \cdot a(\text{Sin.}$

H. A $\rightarrow$ Cofin. H. A $)^{\frac{1}{m}}$, et $(-1)^{\frac{1}{m}} \cdot a(-\text{Sin. H. A} \rightarrow \text{Cofin. H. A})^{\frac{1}{m}} = a$

$(\text{Cofin. C. } \frac{n\Pi}{m} \pm (\text{Sin. C. } \frac{n\Pi}{m})\sqrt{-1})(\text{Sin. H. } \frac{A}{m} \rightarrow \text{Cofin. H. } \frac{A}{m})$, nec

non $= a(\text{Cofin. C. } \frac{\lambda P}{m} \pm (\text{Sin. C. } \frac{\lambda P}{m})\sqrt{-1})(-\text{Sin. H. } \frac{A}{m} \rightarrow \text{Cofin. H. } \frac{A}{m})$:

quamobrem Factores primi, ac secundi Ordinis ita se habebunt, prouti bre-
vissima methodo sequens Diagramma in elenchi formam patefacit.

Eeeee

^m
Par

$$\begin{cases} z - a \left(\text{Sin. H. } \frac{A}{m} + \text{Cofin. H. } \frac{A}{m} \right) = z - aD \\ z + a \left(\text{Sin. H. } \frac{A}{m} + \text{Cofin. H. } \frac{A}{m} \right) = z + aD \end{cases} \begin{matrix} m \\ \text{Par} \\ \\ m \\ \text{Impar} \end{matrix}$$

Unica ergo multiformium Unitatis Radicum confecta disquisitione, nedum Cotesii, Moivreii, latiusque Taylorig Theoremata demonstrantur, quin etiam elegans Circuli atque Hyperbolae rursus menti obversatur *Analogia*, qua fit, ut in diversis Coefficientium, Signorumque hypothefibus, servata nihiloſecius Factorum forma, unius Lineae Trigonometricae in Lineas ad alteram Curvam ſpectantes traducantur; hoc autem discrimine, ut, dum Circulares in ſimpli-

Factores Binomiales $2m$ Factores Trinomiales m

$$\left\{ \begin{array}{l} z - a \left(\text{Sin. H. } \frac{A}{m} + \text{Cofin. H. } \frac{A}{m} \right) = z - aD \\ z + a \left(-\text{Sin. H. } \frac{A}{m} + \text{Cofin. H. } \frac{A}{m} \right) = z + aE \end{array} \right. \quad z^2 - 2 \left(\text{Sin. H. } \frac{A}{m} \right) az - a^2 \mid z^2 - D^2 a^2$$

$$z - a \left(\text{Cofin. C. } \frac{\Pi}{m} + \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1} \right) (D)$$

$$z^2 - 2 \left(\text{Cofin. C. } \frac{\Pi}{m} \right) Daz + D^2 a^2$$

$$z - a \left(\text{Cofin. C. } \frac{\Pi}{m} - \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1} \right) (D)$$

$$z - a \left(\text{Cofin. C. } \frac{\Pi}{2m} + \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1} \right) (E)$$

$$z^2 - 2 \left(\text{Cofin. C. } \frac{\Pi}{2m} \right) Eaz + E^2 a^2$$

$$z - a \left(\text{Cofin. C. } \frac{\Pi}{2m} - \left(\text{Sin. C. } \frac{\Pi}{m} \right) \sqrt{-1} \right) (E)$$

$$z - a \left(\text{Cofin. C. } \frac{2\Pi}{m} + \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1} \right) (D)$$

$$z^2 - 2 \left(\text{Cofin. C. } \frac{2\Pi}{m} \right) Daz + D^2 a^2$$

$$z - a \left(\text{Cofin. C. } \frac{2\Pi}{m} - \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1} \right) (D)$$

$$z - a \left(\text{Cofin. C. } \frac{3\Pi}{2m} + \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1} \right) (E)$$

$$z^2 - 2 \left(\text{Cofin. C. } \frac{3\Pi}{2m} \right) Eaz + E^2 a^2$$

$$z - a \left(\text{Cofin. C. } \frac{3\Pi}{2m} - \left(\text{Sin. C. } \frac{2\Pi}{m} \right) \sqrt{-1} \right) (E)$$

Unica
in simpliciore coeunt Summam, Hyperbolicae cum Circularibus commixtae
illam perquam maximam respuant simplicitatem, eoquod Hyperbola Peri-
metro in se redeunte non gaudeat.

333. Reminiscor adhuc tribus abhinc lustris Amicorum sollicitationi-
bus excitatum me conscripsisse summariam iuxta principia superius tradita
Theorematis Cotesiani enodationem, huiusque specimen Cl. Canterzania Bo-
noniensis

noniensis Academiae a Secretis misisse. Pudet nunc dicere, quanta tum diligentia intimam prope metaphysicam ipsius Theorematis enucleare conatus fuerim, quanto eruditionis adparatu historicam susceperim demonstrationum ab aliis Geometris latarum narrationem, et quomodo eas invicem singillatim comparans vel nimiam obscuritatem, vel ordinem minus nativum, vel naevos aliquos castigaverim. Ea namque tempestate iam persenseram Theorema a Cotesio repertum, et occasione Problematis universalioris a Taylora Mathematicis non-Anglis anno MDCCXIX. peculiari Programmate edito subodoratum primitivis Circuli affectionibus, et simpliciori Algebrae restitui posse ope multiplicatis Radicum ex Unitate. Celebriorum siquidem novarum demonstrationum methodos, nimirum Pembertonii in *Epistola de Cotesii inventis*, Moivreii in *Miscellaneis de Seriebus et Quadraturis*, Ioannis Bernoullii in *Anecdosis*, Iacobi Hermanni, ac Leonardi Euleri in veteribus *Petropolitanæ Academiae Commentariis*, et signanter Volumine VI^o, Gabrielis Manfredii loco præcitato, Samuelis Koenigii tam in *Diario Lipsiensi* anni MDCCXLI., quam in novis *Berolinensis Societatis Memorabilibus* ad annum MDCCXLIX., Valmesleyi in *Analyti Mensurarum Rationum atque Angulorum*, eiusque hac in re minus felicitis Interpretis Bougainvillii in *Prolegomenis ad Calculi Integralis Tractatum*, Vincentii Riccati in *Opusculorum*, et *Institutionum Analyticarum* Volumine I^o, Gregorii Fontanae inter *Analyseos sublimioris Opuscula* &c., sparsim completi elementa, quae collecta simul faciliorem inlustrationem illam conficerent, cui supra omnes in *Introductione* nusquam satis laudibus extollenda adpropinquavit Eulerus. Verum in ea evolutionis Theorematis qualicunque novitate nimium olim absumpti laboris sero nimis me poenitet: potius enim decebat, curis tum molestiisque vacuae mentis favore, quae Cotesius in Circulo statuerat, ea reddidisse generaliora pro aliis Curvis in se redeuntibus, ac suos pariter Cosinus, Sinusque respondentes determinato cuilibet Sectori, Summisque Areae Sectoris et indefiniti Multipli Areae totius Curvae-Ovalis, instar Circuli, habentibus.

C A P U T VIII.

APPLICATIO CALCULI INFINITESIMALIS IUXTA VULGAREM
 POTESTATUM METHODUM AD QUANTITATES LOGARITHMICAS
 EXPONENTIALIALES ET TRIGONOMETRICAS.

334.  Itulus huic Capiti adpositus suspicor, ne intempestivus nonnullis occurrat perinde ac si actum agere videar. Antequam autem de dicendorum opportunitate ii ferant iudicium meminerint, quaeso, Logarithmorum Differentiationem haecenus vel ab Hyperbolae Tetragonismo, vel a Geometricarum Progressionum natura in re meri Calculi minus apte deductam fuisse, nec non Trigonometricarum Magnitudinum Differentia lia Circularis Schematis consideratione atque affectionibus sustineri. Fluxiones etiam Exponentialium indirecta methodo substitutionum inventas nemo est qui ignoret; unde fit, ut perraro huc illuc in recentioribus Mathematicorum Scriptis quaedam directae Differentiationis spicilegia, exempli magis, quam Scientiae progressus causa, offendantur. Ille itaque, cui Differentialia haec omnia ad unicum, atque evidentius principium, veluti cardinem totius rei Infinitesimalis, revocare curae fit, scilicet ad Variabilium invicem multiplicatarum Fluxiones sine ulla Figurarum contemplatione, non equidem magnus erit Apollo, at meo sensu inique disruptam inter Potestates ac Logarithmos Exponentialiaque societatem coniungere nedum valebit, verum etiam aliquantulam novitatem praeferebat. Ea etenim arte, qua potuerunt postremae Magnitudinum praedictarum species ita concinnari, ut ipsarum finitus Algorithmus Calculo Potestatum, primum a grandiloquo Cartesio Indicum affixione praecclusum, subiiceretur, Infinitesimalia quoque earundem Magnitudinum usitato Potestatum canoni subdere absque utilitate non apparebit. Quanto in hisce tractandis fructu laborem impenderim is, qui subsequencia perleget, metiri poterit; sed frustra quaeret aliquam Integralium mentionem, quandoquidem praeter regulas

Fffff

notissimas

notissimas a Potestatum Fluxionibus ad Fluentia regrediendi nihil habeam novum pro nunc adiiciendum in tam imperfecta Calculi Summatorii seu Tetragonismici, ac praesertim Inversae Tangentium Methodi, ut aiunt, Doctrina.

335. A Logarithmis initio sumpto, ratum firmumque sit Siglam X in posterum significaturam quamcunque Variabilium Functionem. Quo subintellecto oecumenicam consequentem Formulam nanciscemur

$$d(L(X)) = d\left(\frac{\overline{X-1}^1}{1} - \frac{\overline{X-1}^2}{2} + \frac{\overline{X-1}^3}{3} - \frac{\overline{X-1}^4}{4} + \frac{\overline{X-1}^5}{5} - \frac{\overline{X-1}^6}{6} + \frac{\overline{X-1}^7}{7} - \frac{\overline{X-1}^8}{8} + \frac{\overline{X-1}^9}{9} \&c: \right)$$

$$= dX \left(1 - (X-1) + (X-1)^2 - (X-1)^3 + (X-1)^4 - (X-1)^5 + (X-1)^6 - (X-1)^7 + (X-1)^8 - \&c: \right) =$$

$$\frac{dX}{1 + (X-1)} = \frac{dX}{X}. \text{ Sed admodum universalius } d(LX) = d(M(L(X))) =$$

$$\frac{MdX}{X}; \text{ adeo, ut implicitus quoque Moduli, sive Unitatis valor Numerato-}$$

rem Fluxionis antea dicti Logarithmi Hyperbolici afficiat ad instar alterius

$$\text{cuiusvis Systematis, nimirum, servato Analytico rigore, } d(LX) = \frac{1 \cdot dX}{X}$$

debeat conscribi. Alia etiam semita ab usitatis Potestatum Differentialibus non deflectens eligi tuto poterit ad hoc ipsum conficiendum. Habetur enim ex toties iam dictis post numerum 59., neglecta Unitatis Radice Infiniti

$$\text{Ordinis, quae nunc nihil ad rem facit, } LX = \infty \left(X^{\frac{1}{\infty}} - 1 \right), \text{ ideoque}$$

$$d(LX) = \infty \cdot \frac{1}{\infty} \cdot X^{\frac{1}{\infty}-1} dX = \frac{dX}{X}; \text{ quod aeque ceteris Logometricis}$$

Systematibus adplicare fas erit. Superiorum Graduum Fluxiones $d^n(LX)$,

$$\text{prima cognita, mox colligentur facillime, quum } d^2(LX) = d\left(\frac{dX}{X}\right) =$$

$$d\left(X^{\frac{1}{\infty}-1} dX\right) = \left(\frac{1}{\infty} - 1\right) X^{\frac{1}{\infty}-2} dX^2 + X^{\frac{1}{\infty}-1} d^2 X = -\frac{dX^2}{X^2} + \frac{d^2 X}{X},$$

ac, eadem Potestatum Differentiationis methodo praeunte, $d^3(LX) =$

$$d($$

$$d(d^2(LX)) = \frac{2dX^3}{X^3} - \frac{3dXd^2X}{X^2} + \frac{d^3X}{X}, \text{ vel ad tuendam consideratione Mo-}$$

duli terminorum Homogeneitatem

$$d^2(L(X)) = -\frac{1.dX^2}{X^2} + \frac{1.d^2X}{X}$$

$$d^2(LX) = -\frac{MdX^2}{X^2} + \frac{Md^2X}{X}$$

, propterea quod
generalius

$$d^3(L(X)) = \frac{1.2dX^3}{X^3} - \frac{1.3dXd^2X}{X^2} + \frac{1.d^3X}{X}$$

$$d^3(LX) = \frac{2MdX^3}{X^3} - \frac{3MdXd^2X}{X^2} + \frac{Md^3X}{X};$$

nullusque hisce Fluxionibus continuandis aderit finis, uti proprium est omnium Transcendentium Functionum. Notandum hic autem occurrit, quod, constante supposita, vel uniformiter fluente prima Fluxione Variabilis X , aliae omnes ita sese component, ut $d^2(L(X))$, $d^3(L(X))$,

$d^4(L(X))$ &c: aequales sint respective 1.2, 1.2.3, 1.2.3.4 ^{cuplo} &c: homologarum Fluxionum, quae oriantur ab evolutione $\tau \tilde{L}(X + dX) = L(X)$

$$+ L\left(1 + \frac{dX}{X}\right) = L(X) + \frac{dX}{X} - \frac{dX^2}{2X^2} + \frac{dX^3}{3X^3} - \frac{dX^4}{4X^4} + \&c: \text{ ex}$$

demonstratis in Capite III°. Hoc mirum equidem non videbitur eis, qui id symptomatis convenire perspexerint, post emendationem levis lapsus Newtoni ab Ioanne Bernoullio traditam, universis singulisque Variabilium Potestatibus, in quas evolvitur Logarithmica Functio.

336. Eodem ritu Differentiae Finitae $\tau \tilde{L}(X)$ adinveniri possent, quippe

$$\text{quae sint } L(X + \omega) - L(X) = L(X) + L\left(1 + \frac{\omega}{X}\right) - L(X) = \frac{\omega}{X} -$$

$$\frac{\omega^2}{2X^2} + \frac{\omega^3}{3X^3} - \frac{\omega^4}{4X^4} + \frac{\omega^5}{5X^5} - \&c:; \text{ et haec Formula habet eo magis per-}$$

infignem utilitatem, quod ad Infinitesimas Differentias investigandas translata omnes cuiusvis Gradus Fluxiones innumeras eam primi Ordinis simul integrantes, nulla nova operatione superveniente, praeseferat; idque mutando ω in dX protinus obtinebitur. Ipsamet ergo Logarithmica Series, cui adipiscendae soli canones Finitorum suffecere additi Binomio Neutonia-

no (*), nos perducit ad demonstrandam sublimioris Theorematis in tota Logometria veritatem, nempe

$$LX \rightarrow D(L(X)) = L(X) + \frac{\omega d(L(X))}{1.dX} + \frac{\omega^2 d^2(L(X))}{1.2.dX^2} + \frac{\omega^3 d^3(L(X))}{1.2.3.dX^3} + \frac{\omega^4 d^4(L(X))}{1.2.3.4.dX^4} + \frac{\omega^5 d^5(L(X))}{1.2.3.4.5.dX^5} + \&c.,$$

in quo Theoremate D Differentiam Finitam denotet $\tau L(X)$, dum X ver-
tatur in $X + \omega$, et dX constans fiat. Namque, rite peractis substitutionibus
opportunis, proveniet

$$LX \rightarrow D(L(X)) = L(X) + \frac{\omega}{X} + \frac{\omega^2}{2X^2} + \frac{\omega^3}{3X^3} + \frac{\omega^4}{4X^4} + \frac{\omega^5}{5X^5} + \&c.,$$

quemadmodum supra. Tam singularis affectio nonnisi parvulum specimen
exstat universalioris Theorematis Functionem quamlibet comitantis $F(x)$
 $= v$, pro qua adferunt, ac diversimode ostendunt Analystae esse, nulla hy-
pothesi excepta,

$$v \rightarrow Dv = F(x + \omega) = v + \frac{\omega dv}{1.dx} + \frac{\omega^2 d^2v}{1.2.dx^2} + \frac{\omega^3 d^3v}{1.2.3.dx^3} + \frac{\omega^4 d^4v}{1.2.3.4.dx^4} + \frac{\omega^5 d^5v}{1.2.3.4.5.dx^5} + \&c..$$

Istam foecundissimam Aequationem, cuius mirabili sane adiumento Diffe-
rentiae Finitae ab Infinitesimis praecognitis derivantur, si ostendere tantum
liceret in Variabilium Potestatibus ope unius nuper inventae Logarithmi
Differentiae Finitae, videlicet ope Neutoniani Binomii, nedum Potestatibus
ipsis, verum etiam Polynomiis, atque Infinitis Seriebus, et ideo Transcen-
denti cuivis Magnitudini non parum nitoris, ac novae simplicitatis accede-
ret. Omnem movebo lapidem, ut argumenti exitus plenissime voto respon-
deat.

$$337. \text{ Priusquam id experiar praemittere necesse est } \tau d(L(X))^n = n \cdot (L(X))^{n-1} \tau d(L(X))$$

(*) Congruunt omnia cum Eulerianis in *Institutionum Calculi Differentialis* &c:
prioris Partis Capite I°. sed inibi Auctor praesupponit evolutionem Logarithmorum,
et Exponentialium e Sinu Infiniti, ac minus universalia persequitur. Eodem loco obi-
ter aggressus est etiam acutissimus ille Geometra in Capite VI°. Fluxionem Logarithmi
ex Aequalitate $dLp = d\left(\frac{p^0 - 1^0}{0}\right)$.

$(L(X))^{n-1} d(L(X))$, vi communis Regulae Potestatum, seu $= n.$

$(L(X))^{n-1} \frac{dX}{X}$, atque, si dX constans fuerit, $d^2 (L(X))^n =$

$\left(\frac{n \cdot n - 1}{X^2} (L(X))^{n-2} - n \cdot (L(X))^{n-1} \right) \frac{dX^2}{X^2}$; ac praeterea

$d^3 (L(X))^n = \left(n \cdot n - 1 \cdot n - 2 (L(X))^{n-3} - 2n \cdot n - 1 (L(X))^{n-2} + \right.$

$\left. 2n (L(X))^{n-1} \right) \frac{dX^3}{X^3}$; et sic de singulis in Infinitum. Quin etiam ad ma-

ius rei complementum $d(LZ^n)^m = m \cdot (LZ^n)^{m-1} d(LZ^n) = mn \times$

$(LZ^n)^{m-1} \cdot \frac{nZ^{n-1} dZ}{Z^n} = mn (LZ^n)^{m-1} \cdot \frac{dZ}{Z}$, absque substitutionum ad-

iuimento; aliaeque subinde Fluxiones proprio Marte deduci poterunt. Ani-
madversio mihi interea dignum videtur fore iuxta Capitis IIⁱ. Theorice-

$$d(Z^m) = d\left(1 + \frac{(LZ)^m}{1} + \frac{(LZ)^2 m^2}{1 \cdot 2} + \frac{(LZ)^3 m^3}{1 \cdot 2 \cdot 3} + \frac{(LZ)^4 m^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{(LZ)^5 m^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{(LZ)^6 m^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \&c\right)$$

$$= d\left(1 + \frac{(LZ)^m}{\infty}\right), \text{ ac simul ex aliis Regulis inconcussis } = mZ^{m-1} dZ.$$

Ne in re tanti momenti obscuritas aliqua lateat, identitatem primum Diffe-
rentialium $mZ^{m-1} dZ$, et $d\left(1 + \frac{(LZ)^m}{\infty}\right)$ demonstrare satago; quod, quam

facillime contingat, subsequens statuit Aequatio

$$d\left(1 + \frac{(LZ)^m}{\infty}\right) = \infty \left(1 + \frac{(LZ)^m}{\infty}\right)^{\infty-1} d\frac{(LZ)^m}{\infty} = \frac{\infty}{\infty} \left(1 + \frac{(LZ)^m}{\infty}\right)^{\infty} \cdot \frac{mdZ}{Z} = 1 \cdot Z^m \cdot \frac{dZ}{Z} = mZ^{m-1} dZ,$$

nullibi praetergresso Algorithmi Potestatum, earumque Fluxionum limite.

Consentiunt igitur utraque ipsius Differentiae expressiones, idemque con-
sensus manu Calculo admota denuo nitebit. Profecto

$$d\left(1 + \frac{(LZ)^m}{1} + \frac{(LZ)^2 m^2}{1 \cdot 2} + \frac{(LZ)^3 m^3}{1 \cdot 2 \cdot 3} + \frac{(LZ)^4 m^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{(LZ)^5 m^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{(LZ)^6 m^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \&c\right) =$$

G g g g g

$$\frac{mdZ}{Z}$$

$$\frac{mdZ}{Z} + \frac{m^2}{1.2} \cdot \frac{2(LZ)dZ}{Z} + \frac{m^3}{1.2.3} \cdot \frac{3(LZ)^2 dZ}{Z} + \frac{m^4}{1.2.3.4} \cdot \frac{4(LZ)^3 dZ}{Z} + \frac{m^5}{1.2.3.4.5} \cdot \frac{5(LZ)^4 dZ}{Z} + \&c. =$$

$$\frac{mdZ}{Z} \left(1 + \frac{m(LZ)}{1} + \frac{m^2(LZ)^2}{1.2} + \frac{m^3(LZ)^3}{1.2.3} + \frac{m^4(LZ)^4}{1.2.3.4} + \frac{m^5(LZ)^5}{1.2.3.4.5} + \&c. \right) = \frac{mdZ}{Z} \cdot Z^m = mZ^{m-1} dZ.$$

Seriei ergo Infinitae illius Fluxiones ita se habent moderanturque, ut in eandem recidant denique simpliciore Differentiam oecumenicam $mZ^{m-1} dZ$; et idcirco gemina Differentialia, quomodocunque adparenter dissita ob variam eorum expositionem, idem adamussim significant, prouti Potestatibus universis contingere visum est num°. 47. aut usitato more, aut *exponentialiter* evolutis.

338. Quibus positis, cum passim omnibus innotescat versio Potestatum quarumvis in Exponentialium Serierum formam, habebitur, compendio Calculi pro viribus inito,

$$D(X^m), \text{ five } (X + \omega)^m - X^m = X^m \left(1 + \frac{\omega}{X} - 1 \right) = X^m \times$$

$$\left(\frac{m^1 \left(L \left(1 + \frac{\omega}{X} \right) \right)^1}{1} + \frac{m^2 \left(L \left(1 + \frac{\omega}{X} \right) \right)^2}{1.2} + \frac{m^3 \left(L \left(1 + \frac{\omega}{X} \right) \right)^3}{1.2.3} + \right.$$

$$\left. \frac{m^4 \left(L \left(1 + \frac{\omega}{X} \right) \right)^4}{1.2.3.4} + \frac{m^5 \left(L \left(1 + \frac{\omega}{X} \right) \right)^5}{1.2.3.4.5} + \frac{m^6 \left(L \left(1 + \frac{\omega}{X} \right) \right)^6}{1.2.3.4.5.6} + \&c. \right)$$

nimirum, Differentia Finita cuiuslibet Potestatis, ope unius praecognitae Differentiae pariter Finitae Logarithmi Hyperbolici Radicis, quam numerus tribuit 336., elicietur. Uberior huiusce affectionis usus mox elucescet tum, quum rem ipsam inferius perficientes transferre quoque moliemur argumenti nobilitatem ad Potestates Exponente variabili praeditas. Interea prodest monere, quod, dum $D(X^m)$ abeat in $d(X^m)$, haec inveniatur aequalis X^m .

$\frac{mdX}{X}$, scilicet, uti supra, $m \cdot X^{m-1} dX$, nec non $d^2(X^m) = (m^2 - m)$

$\frac{dX}{X}$

$\frac{dX}{X^2} \cdot X^m = m \cdot m - 1 \cdot X^{m-2} dX^2$, quae Differentiae Infinitesimae a Finitis deductae communi Fluxionum Calculo consonant. Id, ut universalius aggrediar, dudum tradita Formula evolvenda erit in Figuram Trigoni Analytici

$$D(X^m) = X^m \left(\frac{mD(X)}{X} - \frac{m(D(X))^2}{2X^2} + \frac{m(D(X))^3}{3X^3} - \frac{m(D(X))^4}{4X^4} + \frac{m(D(X))^5}{5X^5} - \frac{m(D(X))^6}{6X^6} + \frac{m(D(X))^7}{7X^7} - \&c. \right),$$

$$+ \frac{m^2(D(X))^2}{2X^2} - \frac{m^2(D(X))^3}{2X^3} + \frac{11m^2(D(X))^4}{24X^4} - \frac{5m^2(D(X))^5}{12X^5} + \frac{137m^2(D(X))^6}{360X^6} - \frac{7m^2(D(X))^7}{20X^7} + \&c.$$

$$+ \frac{m^3(D(X))^3}{6X^3} - \frac{m^3(D(X))^4}{4X^4} + \frac{7m^3(D(X))^5}{24X^5} - \frac{5m^3(D(X))^6}{16X^6} + \frac{29m^3(D(X))^7}{90X^7} - \&c.$$

$$+ \frac{m^4(D(X))^4}{24X^4} - \frac{m^4(D(X))^5}{12X^5} + \frac{17m^4(D(X))^6}{144X^6} - \frac{7m^4(D(X))^7}{48X^7} + \&c.$$

$$+ \frac{m^5(D(X))^5}{120X^5} - \frac{m^5(D(X))^6}{48X^6} + \frac{5m^5(D(X))^7}{144X^7} - \&c.$$

$$+ \frac{m^6(D(X))^6}{720X^6} - \frac{m^6(D(X))^7}{240X^7} + \&c.$$

$$+ \frac{m^7(D(X))^7}{5040X^7} - \&c.,$$

in qua exornanda nihil adlaborare oportebit, proptereaquod numerici Coefficientes coincidere debeant cum iis diagonalibus in numero 44. prolatis. Nedum itaque toti simul Finitas Differentias $\tau \tilde{X}^m$ componentes termini hoc in Diagrammate ordinatim disponuntur pro quovis Radicis X decremento vel incremento in locum generalis speciei $D(X)$ substituendo, verum etiam Infinitesimarum plenissima habetur ratio. Fiet enim, Coefficientibus eius Se-

$$\text{riei rite evolutis, } d(X^m) = \frac{m}{1} X^{m-1} dX + \frac{(m \cdot m - 1)}{1 \cdot 2} X^{m-2} dX^2 +$$

$$\frac{(m \cdot m - 1 \cdot m - 2)}{1 \cdot 2 \cdot 3} X^{m-3} dX^3 + \frac{(m \cdot m - 1 \cdot m - 2 \cdot m - 3)}{1 \cdot 2 \cdot 3 \cdot 4} X^{m-4} dX^4 +$$

$$\frac{(m \cdot m - 1 \cdot m - 2 \cdot m - 3 \cdot m - 4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} X^{m-5} dX^5 + \frac{(m \cdot m - 1 \cdot m - 2 \cdot m - 3 \cdot m - 4 \cdot m - 5)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} X^{m-6} dX^6 + \&c.,$$

quae

quae Series mirifice a remotioribus derivata principiis cum Neutoniana consentit, ac eodem in termino $\overline{m}^{\text{esimo}}$ sistit, quando Index m integris positivis Numeris adscribatur. Provenient ideo cuiuslibet Ordinis Fluxiones, manente dX invariabili, $d(X^m) = mX^{m-1}dX$; $d^2(X^m) = m.m-1.X^{m-2}dX^2$; $d^3(X^m) = m.m-1.m-2.X^{m-3}dX^3$; $d^4(X^m) = m.m-1.m-2.m-3.X^{m-4}dX^4$ &c.;, uti Bernoullius Neutoni Formulam usque ab anno MDCCXIII. in *Commentariis Lipsiensibus* emendavit. Perpulchrum est autem, Differentias pariter Finitas Potestatis X^m , eadem hypothese facta, non secus ac Infinitesimae post m terminorum Numerum abrumpi, quum ipsamet Formula eas repraesentet

$$D(X^m) = \frac{m}{1} X^{m-1} D(X) + \frac{(m.m-1)}{1.2} X^{m-2} (D(X))^2 + \frac{(m.m-1.m-2)}{1.2.3} X^{m-3} (D(X))^3 + \frac{(m.m-1.m-2.m-3)}{1.2.3.4} X^{m-4} (D(X))^4 + \frac{(m.m-1.m-2.m-3.m-4)}{1.2.3.4.5} X^{m-5} (D(X))^5 + \frac{(m.m-1.m-2.m-3.m-4.m-5)}{1.2.3.4.5.6} X^{m-6} (D(X))^6 + \&c.;$$

scilicet in usitata $\tau \approx dX$ constantis suppositione,

$$D(X^m) = D(V) = \frac{dV.D(X)}{1.dX} + \frac{ddV.(D(X))^2}{1.2.dX^2} + \frac{dddV.(D(X))^3}{1.2.3.dX^3} + \frac{ddddV.(D(X))^4}{1.2.3.4.dX^4} + \frac{dddddV.(D(X))^5}{1.2.3.4.5.dX^5} + \&c.;$$

aut, si mavis, dum X evadat $X \pm \omega$,

$$V \pm D(V) = V \pm \frac{\omega dV}{dX} + \frac{\omega^2 d^2 V}{2.dX^2} \pm \frac{\omega^3 d^3 V}{2.3.dX^3} + \frac{\omega^4 d^4 V}{2.3.4.dX^4} \pm \frac{\omega^5 d^5 V}{2.3.4.5.dX^5} + \frac{\omega^6 d^6 V}{2.3.4.5.6.dX^6} \pm \&c.;$$

atque in re Infinitesimali, quando X abeat in $X \pm dX$,

$$V \pm dV = V \pm \frac{dV}{1} + \frac{d^2 V}{1.2} \pm \frac{d^3 V}{1.2.3} + \frac{d^4 V}{1.2.3.4} \pm \frac{d^5 V}{1.2.3.4.5} + \frac{d^6 V}{1.2.3.4.5.6} \pm \frac{d^7 V}{1.2.3.4.5.6.7} + \&c.;$$

quod est MacLaurinii Theorema.

339. Post haec dicta resolventur prope per iocum sequentia Fluxionum Logometricarum Problemata. Ac primum

$$d(LL(X))$$

$$d(LL(x)) = d\left(\frac{(L(x)-1)^1}{1} - \frac{(L(x)-1)^2}{2} + \frac{(L(x)-1)^3}{3} - \frac{(L(x)-1)^4}{4} + \frac{(L(x)-1)^5}{5} - \frac{(L(x)-1)^6}{6} + \frac{(L(x)-1)^7}{7} - \&c:\right)$$

$$= \frac{dx}{x} \left(1 - (L(x)-1)^1 + (L(x)-1)^2 - (L(x)-1)^3 + (L(x)-1)^4 - (L(x)-1)^5 + (L(x)-1)^6 - (L(x)-1)^7 + \&c:\right)$$

$$= \frac{dx}{x} \left(\frac{1}{1+(L(x)-1)}\right) = \frac{dx}{xL(x)} : eademque methodo duce $d(LL(x)) =$$$

$$\frac{dx}{x(L(x))(LL(x))}, d(LLLL(x)) = \frac{dx}{x(L(x))(LL(x))(LLL(x))} \text{ sine ullo Limite,}$$

quorum Differentialium Lex satis emicat. Fluxiones secundas, tertias, quartas &c: omitto, cum eae sponte fluant a prioribus directo tramite inventis; neque Formulas ex Logarithmis superiorum horumce Graduum varie existentes curam habeo differentiandi, quia ab iam conscriptis Fluxio-

nibus omnia pendent. Ita $d(LL(x))^n = n(LL(x))^{n-1} dLL(x) =$

$n(LL(x))^{n-1} \frac{dx}{xL(x)}$. Nec ulla difficultas obstat, quominus idem contin-

gat alteri cuilibet Logarithmorum Systemati ad Modulum M comparato,

vel m: namque $d(l(x)) = \frac{Mdx}{xL(x)}$; et $d(Ll(x)) = \frac{dx}{xL(x)} = d(LL(x))$;

$d(lL(x)) = \frac{Mdx}{xL(x)} = d(l(x))$; $d(Ll(x)) = \frac{mdx}{xL(x)}$, quae expressiones in-

finitimode componi, misceri, ac variari possunt, iisdem tamen principiis inconcussis manentibus.

340. Fluxiones Magnitudinum Exponentialium, servato meliori ordine synthetico, investigandae nunc proponantur. Erit inprimis

$$d(a^x) = a^{x+\frac{dx}{x}} - a^x = d\left(1 + \frac{(La)^1 x^1}{1} + \frac{(La)^2 x^2}{1.2} + \frac{(La)^3 x^3}{1.2.3} + \frac{(La)^4 x^4}{1.2.3.4} + \frac{(La)^5 x^5}{1.2.3.4.5} + \frac{(La)^6 x^6}{1.2.3.4.5.6} + \&c:\right) =$$

$$(La) dx \left(1 + \frac{(La)^1 x^1}{1} + \frac{(La)^2 x^2}{1.2} + \frac{(La)^3 x^3}{1.2.3} + \frac{(La)^4 x^4}{1.2.3.4} + \frac{(La)^5 x^5}{1.2.3.4.5} + \frac{(La)^6 x^6}{1.2.3.4.5.6} + \&c:\right) = a^x \cdot (La) dx.$$

H h h h h

Magis

Magis autem affinis Potestatum vulgari Differentiationi subnexus modus apparet.

$$d(a^x) = d\left(1 + \frac{(La)x}{\infty}\right)^{\infty} = \infty \left(1 + \frac{(La)x}{\infty}\right)^{\infty-1} \cdot \frac{(La)dx}{\infty} = \frac{\infty}{\infty} \left(1 + \frac{(La)x}{\infty}\right)^{\infty} \cdot (La)dx = a^x (La)dx.$$

Alterutra methodus non abfimiliter detegit Differentialia Exponentialium z^x , Basi, atque Indice simul variabilibus affectorum. Et re quidem vera primum extat

$$d(z^x) = d\left(1 + \frac{(Lz)x}{\infty}\right)^{\infty} = \infty \left(1 + \frac{(Lz)x}{\infty}\right)^{\infty-1} \cdot d\frac{(Lz)x}{\infty} = \frac{\infty}{\infty} \cdot z^x \left((Lz)dx + x \frac{dz}{z}\right) = z^x (Lz)dx + z^{x-1} \times$$

$x dz$. Haec Formula universalior comprobatur veritatem Differentiationis superius expositae: facta etenim Basi $z = a$ evanescet dz , ac idcirco etiam

$z^{x-1} x dz$, fietque tunc $d(z^x) = d(a^x) = a^x (La)dx$. Ad haec, si velles Fluxionem $\tau\theta$ z^x ab Differentiali simplicioris a^x derivare, nihil morae oporteret, cum statim quisque perspiciat ope $d(a^x) = a^x (La)dx$ obtineri $d(C^x) = C^x dx$, propter $LC = 1$, necnon ex notissima Aequalitate $z^x =$

C^{xLz} profluere $d(z^x) = d(C^{xLz}) = C^{xLz} \cdot d(xLz) = z^x \left(dx(Lz) + \frac{x dz}{z}\right) = z^x (Lz)dx + z^{x-1} x dz$, nimirum eandem Fluxionis formam ante

tea repertam. Subinde consequuntur $d(z^{x^v}) = z^{x^v} \cdot (Lz) d(x^v) +$

$$z^{(x^v)-1} \cdot x^v dz = z^{x^v} \cdot x^v \frac{dz}{z} + z^{x^v} \cdot x^v (Lx)(Lz)dv + z^{x^v} \cdot x^{v-1} v(Lz) \cdot$$

dx ; eiusque Differentiae universalis Coronides sunt $d(a^{x^v}) = a^{x^v} (x^v(La) \cdot$

$$(Lx)dv + x^{v-1} (La) v dx) = a^{x^v} \cdot x^v La \left((Lx)dv + \frac{v dx}{x}\right), d(C^{x^v}) =$$

$$C^{x^v} \cdot x^v \left((Lx)dv + \frac{v dx}{x}\right), d(z^{a^v}) = z^{a^v} \cdot a^v \left(\frac{dz}{z} + (La)(Lz)dv\right),$$

$$d(z^{C^v})$$

$$d(z^{C^v}) = z^{C^v} \cdot C^v \left(\frac{dz}{z} + (Lz) dv \right), \quad d(z^{x^a}) = z^{x^a} \cdot x^a \left(\frac{dz}{z} + a(Lz) \cdot \frac{dx}{x} \right),$$

ceteris omnibus praetermissis, quae adeo dimanant facillime, ut recte adseruerit Matheos Historicus in II°. Volumine, totum Exponentialium Algorithmum ad gemina haec elementa $L(x^n) = nL(x)$, et $d(L(x)) = \frac{dx}{x}$ summatim referri.

341. Nec vero putes Fluxiones nuperrime inventas a Seriebus Exponentialibus, prouti in a^x experti fausto omine sumus, obtineri non posse: periculum enim, quod nunc paramus, methodi illius in z^x , scito ad altiora pariter Exponentialia sese porrigere. Typum Calculi adiicio absque orationis necessitate.

$$d(z^x) = d\left(1 + \frac{(Lz)^1 x^1}{1} + \frac{(Lz)^2 x^2}{1 \cdot 2} + \frac{(Lz)^3 x^3}{1 \cdot 2 \cdot 3} + \frac{(Lz)^4 x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{(Lz)^5 x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{(Lz)^6 x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{(Lz)^7 x^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \&c. \right)$$

$$= d((Lz)x) \left(1 + \frac{(Lz)^1 x^1}{1} + \frac{(Lz)^2 x^2}{1 \cdot 2} + \frac{(Lz)^3 x^3}{1 \cdot 2 \cdot 3} + \frac{(Lz)^4 x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{(Lz)^5 x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{(Lz)^6 x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \&c. \right) =$$

$$z^x \cdot d((Lz)x) = z^x (Lz) dx + z^{x-1} x dz.$$

Haec ultima expressio coincidit cum ea numeri praecedentis, ultroque determinat communium Potestatum z^m Differentiam esse $z^{m-1} m dz$, five $m z^{m-1} dz$, quemadmodum placuit Analystis singularem ipsum Exponentialium oecumenicae Fluxionis casum disponere. Sed, quod insignius existimari debet in Algorithmo Infinitesimali, nonnisi ope Fluxionis repertae Magnitudinum Exponentialium formae z^x , et idcirco non sine universa hucusque tradita Quantitatum *Percurrentium* Theorice asseri tuto potuisset Aequatio

$$d(z^{\sqrt[n]{m}}) = \frac{n}{\sqrt[n]{m}} \cdot z^{\sqrt[n]{m}-1} dz; \quad \text{nam usitatae Differentiationis regulae, quae in Potestatibus Indice Rationali praeditis valent, et Multiplicationi innituntur, impares sunt Potestatibus Interscendentibus apta demonstratione}$$

different-

differentiandis. Nullus denique dubito, quin evidentiam omnem praefere-
rant consequentes Differentiae Infinitesimae Magnitudinum Logarithmico-
Exponentialium, nempe $(Lz)^x$, z^{Lx} , $(LL(z))^x$, hisque similium magis
minusve compositarum. Fiet enim, vi praemissorum, $d(Lz)^x = d(C^{x(LLz)})$
 $= C^{x(LLz)} d(x(LLz)) = (Lz)^x \left((LLz) dx + \frac{x dz}{zLz} \right)$, $d(z^{Lx}) = z^{Lx} \times$
 $(Lz) \frac{dx}{x} + z^{(Lx)-1} (Lx) dz$, $d(LL(z))^x = d(C^{x(LLz)}) = C^{x(LLz)} \times$
 $d(x(LLz)) = (LL(z))^x \left(dxLLz + \frac{x dz}{zLzLLz} \right)$.

342. Quum τa^x idem sit ac C^{xLa} , five $C^{L(a^x)}$, Differentiae Finitae
prioris Quantitatis Exponentialis ab illis secundae pendeant. Sit igitur
universaliter quaerenda $D(C^x)$; haec peraequabit $C^{x \pm Dx} - C^x$, scili-

cet elicietur subsidio $\tau C^{x \pm Dx} = C^{x \pm \omega} =$

$$1 + \frac{(x \pm \omega)^1}{1} + \frac{(x \pm \omega)^2}{1.2} + \frac{(x \pm \omega)^3}{1.2.3} + \frac{(x \pm \omega)^4}{1.2.3.4} + \frac{(x \pm \omega)^5}{1.2.3.4.5} + \frac{(x \pm \omega)^6}{1.2.3.4.5.6} + \&c. =$$

$$\begin{aligned}
 & 1 + \frac{x^1}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} + \frac{x^7}{1.2.3.4.5.6.7} + \&c: , \\
 & \pm \frac{\omega}{1} + \frac{\omega^2}{1.2} \pm \frac{\omega^3}{1.2.3} + \frac{\omega^4}{1.2.3.4} \pm \frac{\omega^5}{1.2.3.4.5} + \frac{\omega^6}{1.2.3.4.5.6} \pm \frac{\omega^7}{1.2.3.4.5.6.7} + \&c: \\
 & \pm \frac{\omega x^1}{1} + \frac{\omega^2 x^1}{1.2} \pm \frac{\omega^3 x^1}{1.2.3} + \frac{\omega^4 x^1}{1.2.3.4} \pm \frac{\omega^5 x^1}{1.2.3.4.5} + \frac{\omega^6 x^1}{1.2.3.4.5.6} \pm \frac{\omega^7 x^1}{1.2.3.4.5.6.7} + \&c: \\
 & \pm \frac{\omega x^2}{1.2} + \frac{\omega^2 x^2}{1.2.2} \pm \frac{\omega^3 x^2}{1.2.2.3} + \frac{\omega^4 x^2}{1.2.2.3.4} \pm \frac{\omega^5 x^2}{1.2.2.3.4.5} + \frac{\omega^6 x^2}{1.2.2.3.4.5.6} + \&c: \\
 & \pm \frac{\omega x^3}{1.2.3} + \frac{\omega^2 x^3}{1.2.2.3} \pm \frac{\omega^3 x^3}{1.2.2.3.3} + \frac{\omega^4 x^3}{1.2.2.3.3.4} \pm \frac{\omega^5 x^3}{1.2.2.3.3.4.5} + \&c: \\
 & \pm \frac{\omega x^4}{1.2.3.4} + \frac{\omega^2 x^4}{1.2.2.3.4} \pm \frac{\omega^3 x^4}{1.2.2.3.3.4} + \frac{\omega^4 x^4}{1.2.2.3.3.4.4} + \&c: \\
 & \pm \frac{\omega x^5}{1.2.3.4.5} + \frac{\omega^2 x^5}{1.2.2.3.4.5} \pm \frac{\omega^3 x^5}{1.2.2.3.3.4.5} + \&c: \\
 & \pm \frac{\omega x^6}{1.2.3.4.5.6} + \frac{\omega^2 x^6}{1.2.2.3.4.5.6} + \&c: \\
 & \pm \frac{\omega x^7}{1.2.3.4.5.6.7} + \&c: \\
 & \&c:
 \end{aligned}$$

nimirum, collectis, quod fieri tuto potest, Seriebus Numero terminorum Infinitis Columellarum omnium Verticalium, sed immutata manente suprema Linea Horizontali,

$$C^x \pm \frac{C^x \omega^1}{1} + \frac{C^x \omega^2}{2} \pm \frac{C^x \omega^3}{2.3} + \frac{C^x \omega^4}{2.3.4} \pm \frac{C^x \omega^5}{2.3.4.5} + \frac{C^x \omega^6}{2.3.4.5.6} \pm \frac{C^x \omega^7}{2.3.4.5.6.7} + \frac{C^x \omega^8}{2.3.4.5.6.7.8} \pm \&c: .$$

Hoc idem confirmatur breviori, uti confecit Eulerus, at minus eloquenti alia methodo, quum per se constet esse

liiii

$C^x \pm \omega$

$$C^{x \pm \omega} = C^x \cdot C^{\pm \omega} = C^x \left(1 \pm \frac{\omega^1}{1} \pm \frac{\omega^2}{1.2} \pm \frac{\omega^3}{1.2.3} \pm \frac{\omega^4}{1.2.3.4} \pm \frac{\omega^5}{1.2.3.4.5} \pm \frac{\omega^6}{1.2.3.4.5.6} \pm \frac{\omega^7}{1.2.3.4.5.6.7} \pm \&c. \right)$$

in Infinitum. Quo viso nanciscimur

$$D(C^x) = C^x \left(\pm \frac{(Dx)}{1} \pm \frac{(Dx)^2}{1.2} \pm \frac{(Dx)^3}{1.2.3} \pm \frac{(Dx)^4}{1.2.3.4} \pm \frac{(Dx)^5}{1.2.3.4.5} \pm \frac{(Dx)^6}{1.2.3.4.5.6} \pm \frac{(Dx)^7}{1.2.3.4.5.6.7} \pm \&c. \right),$$

$$D(a^x) = a^x \left(\pm \frac{(La)(Dx)}{1} \pm \frac{(La)^2(Dx)^2}{1.2} \pm \frac{(La)^3(Dx)^3}{1.2.3} \pm \frac{(La)^4(Dx)^4}{1.2.3.4} \pm \frac{(La)^5(Dx)^5}{1.2.3.4.5} \pm \frac{(La)^6(Dx)^6}{1.2.3.4.5.6} \pm \&c. \right),$$

$$z^x \pm D(z^x) = (z \pm \omega')^x \times$$

$$\left(1 \pm \frac{(L(z \pm \omega'))^1(Dx)}{1} \pm \frac{(L(z \pm \omega'))^2(Dx)^2}{1.2} \pm \frac{(L(z \pm \omega'))^3(Dx)^3}{1.2.3} \pm \frac{(L(z \pm \omega'))^4(Dx)^4}{1.2.3.4} \pm \frac{(L(z \pm \omega'))^5(Dx)^5}{1.2.3.4.5} \pm \&c. \right);$$

et in posteriore ista Formula oecumenica tam Basis z , quam Index x variationem patiuntur. Altero etiam modo Trigonum illud Analyticum contrahi in Summam potest, suprema Linea Horizontali non amplius seclusa, veruntamen omnibus Horizontalibus singillatim collectis absque ulla Signorum consideratione, videlicet in $C^x \pm (C^\omega - 1) \pm x(C^\omega - 1) \pm \frac{x^2}{1.2} \pm \frac{x^3}{1.2.3} (C^\omega - 1) \pm \frac{x^4}{1.2.3.4} (C^\omega - 1) \pm \frac{x^5}{1.2.3.4.5} (C^\omega - 1) \pm \frac{x^6}{1.2.3.4.5.6} (C^\omega - 1) \pm \&c. = C^x \pm C^x(C^\omega - 1) = C^x \cdot C^\omega$; ideoque

Potestatum Calculo deducitur consona Aequalitas.

343. Inter plurima, quae adfatim confluunt, principem locum merentur Infinitesimae omnium Ordinum Differentiae deliberato animo haecenus praetermissae. Eae proveniunt a Formula Finitarum Differentiarum iuxta vulgatum Canonem, de cuius veritate fidem facit universa Fluxionum doctrina.

$$\text{Sicuti enim } d(C^x) = C^{x+dx} - C^x = C^x \left(dx \pm \frac{dx^2}{2} \pm \frac{dx^3}{2.3} \pm \frac{dx^4}{2.3.4} \pm \frac{dx^5}{2.3.4.5} \pm \frac{dx^6}{2.3.4.5.6} \pm \frac{dx^7}{2.3.4.5.6.7} \pm \&c. \right), \text{ exinde oriuntur Leibnitiano}$$

$$\text{more } d^2 C^x = C^x dx^2, \quad d^3 C^x = C^x dx^3, \quad d^4 C^x = C^x dx^4, \quad d^5 C^x = C^x dx^5$$

&c.;

&c.; quoniam $d^2 a^x = d^2 C^{xLa} = a^x (La)^2 dx^2$, $d^3 a^x = a^x (La)^3 dx^3$,
 $d^4 a^x = a^x (La)^4 dx^4$, $d^5 a^x = a^x (La)^5 dx^5$ &c: in praefata hypothefi dx

constantis. Hac ipsa manente evadunt praeterea $d^2 z^x = d^2 C^{xLz} = z^x$.

$((d(xLz))^2 + d^2(xLz))$, $d^3 z^x = d^3 C^{xLz} = z^x ((d(xLz))^3 + 3d(xLz).$
 $d^2(xLz) + d^3(xLz))$ &c.; dummodo in condendis non evolutis Differentialibus regulae antea latae observentur. Eadem ad altiorum Graduum Fluxiones adipiscendas conducunt; nec iniucundum scitu est, magis directam etiam patere viam, ad examen denuo vocatis Seriebus ipsis Exponentialibus. Specimen praebeat C^x , a qua Potestate Gradus indefiniti dimanant

$$d(C^x) = dx + \frac{xdx}{1} + \frac{x^2 dx}{1.2} + \frac{x^3 dx}{1.2.3} + \frac{x^4 dx}{1.2.3.4} + \frac{x^5 dx}{1.2.3.4.5} + \frac{x^6 dx}{1.2.3.4.5.6} + \frac{x^7 dx}{1.2.3.4.5.6.7} \&c: = C^x dx;$$

$$d^2(C^x) = dx^2 + \frac{xdx^2}{1} + \frac{x^2 dx^2}{1.2} + \frac{x^3 dx^2}{1.2.3} + \frac{x^4 dx^2}{1.2.3.4} + \frac{x^5 dx^2}{1.2.3.4.5} + \frac{x^6 dx^2}{1.2.3.4.5.6} \&c: = C^x dx^2;$$

$$d^3(C^x) = dx^3 + \frac{xdx^3}{1} + \frac{x^2 dx^3}{1.2} + \frac{x^3 dx^3}{1.2.3} + \frac{x^4 dx^3}{1.2.3.4} + \frac{x^5 dx^3}{1.2.3.4.5} \&c: = C^x dx^3;$$

$$d^4(C^x) = dx^4 + \frac{xdx^4}{1} + \frac{x^2 dx^4}{1.2} + \frac{x^3 dx^4}{1.2.3} + \frac{x^4 dx^4}{1.2.3.4} \&c: = C^x dx^4;$$

ac sic de singulis consequentibus sine Limite. Occurrit etiam notandum, quod Differentiae Finitae cuiusvis Magnitudinis Exponentialis a^x adiumento Infinitesimalium nosci facile queant, cum aequipollentes sint ex praepositis geminae Formulae

$$a^x + D(a^x) = a^x \left(1 + \frac{(La)Dx}{1} + \frac{(La)^2 Dx^2}{1.2} + \frac{(La)^3 Dx^3}{1.2.3} + \frac{(La)^4 Dx^4}{1.2.3.4} + \frac{(La)^5 Dx^5}{1.2.3.4.5} + \&c: \right), \text{ et}$$

$$a^x + D(a^x) = a^x \pm \frac{d(a^x)}{1.dx} Dx + \frac{d^2(a^x)}{1.2.dx^2} Dx^2 \pm \frac{d^3(a^x)}{1.2.3.dx^3} Dx^3 + \frac{d^4(a^x)}{1.2.3.4.dx^4} Dx^4 \pm \frac{d^5(a^x)}{1.2.3.4.5.dx^5} Dx^5 + \&c:.$$

Harumce posterior integrum Differentiale τa^x , nullo Fluxionum neglecto Ordine, nempe ad mentem MacLaurinii, tribuit ab Aequatione deductum

$$a^x + d(a^x)$$

$$a^x + d(a^x) = a^x \pm d(a^x) + \frac{d^2(a^x)}{2} \pm \frac{d^3(a^x)}{2 \cdot 3} + \frac{d^4(a^x)}{2 \cdot 3 \cdot 4} \pm \frac{d^5(a^x)}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{d^6(a^x)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \&c.; \text{ scilicet}$$

$$d(a^x) = \pm \frac{dV}{1} + \frac{ddV}{1 \cdot 2} \pm \frac{dddV}{1 \cdot 2 \cdot 3} + \frac{ddddV}{1 \cdot 2 \cdot 3 \cdot 4} \pm \frac{dddddV}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{ddddddV}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \pm \frac{dddddddV}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \&c.; \text{ vel potius}$$

$$d(a^x) = a^x \left(\pm \frac{(La)dx}{1} + \frac{(La)^2 dx^2}{1 \cdot 2} \pm \frac{(La)^3 dx^3}{1 \cdot 2 \cdot 3} + \frac{(La)^4 dx^4}{1 \cdot 2 \cdot 3 \cdot 4} \pm \frac{(La)^5 dx^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{(La)^6 dx^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \pm \&c. \right);$$

adeo, ut Differentiae Finitae, plenariaeque Infinitesimae in earum forma neutiquam discriminentur.

344. Logarithmicae, aut Logisticae Curvae affectiones, quas omnium primus ad calcem Operis *de Causa Gravitatis*, anno MDCXCI. in lucem editi, synthetico ordine, sed indemonstratas, protulit Christianus Hugenus, supervacaneum nunc esset ex praemissis ostendere. Veruntamen, dum abstinere libentissime a repetita crambe recoquenda, non male audiam a Lectoribus, si nonnulla attigerim, quae novitatem aliquam pollicentur. Quam oecumenica Aequatio de more Geometrarum Locum ad Logisticam continens

$a^x = y$ eodem recidat atque altera $C^{xLa} = y$, inde consequitur Logarithmicas universas ad singularem Basin C referri posse, ac, factis aequalibus xLa et z , formam $C^z = y$ induere. Quidquid igitur de prima Aequationis specie differatur, ceteris omnibus aequae adiudicandum fore non est qui non sentiat. Nec peculiarem mentionem exposcunt Logarithmicae Lineae, quae Ordinatum-Applicatis oblique super Asymptotum dispositis gaudeant, qualem semi-recto Ordinarum ad Axem Angulo distinctam Florimundi Beaunii celeberrimo Problemati satisfaciendam ante dimidium praeteriti saeculi contemplari fueverunt Geometrae. Prodigii autem similis plerisque videbitur enunciatio, Logisticam Curvam toto iure Parabolis adscribi debere, quod ut ex sententia cedat, quanta facilitate symptomata Logisticae ab iis Parabolarum colligere liceat inamoenum futurum esse non censeo. Praecognitis igitur Parabolarum diversi Ordinis affectionibus, quae Cavalerio, Torricellio, Robervallio, Vallisio, Fermatio, Barrovio, aliisque Infinitesimali Calculum parantibus iamdudum innotuere, exoriuntur Hugeniiana omnia Theoremata, quorum non pauca Grandio postmodum ea demonstranti anno

MDCCL.

MDCCL. multum adtulere laboris. Est quippe Aequatio Localis ad Logarithmicam C^z , vel $C^x = y$ eadem cum altera $(1 + \frac{x}{\infty})^\infty = \frac{y}{1^x}$, five,

quum de uno Curvae Ramo Reali sit sermo, posita Multiformis valoris Potestate $1^x = 1$, et in locum $\infty + x$ substituta specie v , ipsamet abit Aequatio in $v^\infty = \frac{\infty}{1} \cdot y$ pertinentem ad Parabolarum Familiam, scilicet

ad Parabolarum $(1 + \frac{x}{m})^m = y$, seu $(m + x)^m = \frac{m^m}{1} \cdot y$, aut tan-

dem brevius $v^m = y$ Aequatione gaudentium ultimum inassignabilem Limitem.

345. Rebus sic stantibus praecipuas Parabolarum omnium proprietates meliori ordine locabo, atque e-regione conscribam analogas Logarithmicae respondentes, et a Parabolicis derivatas.

Verior Parameter $\frac{m}{1}$ PARABOLA,

in qua $v^m = y$

Parameter

communis

LOGISTICA,

in qua $v^\infty = y$

Verior Parameter $\frac{\infty}{1}$

Ad Axem relata Subtangens $= \frac{v}{m} = \frac{m+x}{m} = 1 + \frac{x}{m}$.

Trilinei integri Area inter Ordinas y datam, ac y = 0 $= \frac{vy}{m+1} = \frac{(m+x)y}{m+1}$.

Trapezoidis Area inter duas quasvis Ordinas $= \frac{(m+x)y - (m+x')y'}{m+1}$.

Rotundi Corporis Soliditas circa Axem eiusdem Curvae revolutione geniti ad illam Cylindri, quem Circumscriptum vocant, uti $1 : 2m + 1$.

Ipfius Corporis Soliditas erit ideo $= \frac{P.y^2(m+x)}{2m+1} = \frac{P.y^2 v}{2m+1}$.

Corporis alterius Soliditas rotatione Trilinei circa Ordinatam producti $= \frac{2}{m+1 \cdot m+2} P v^2 y$, aut $= \frac{2}{(m+1)(m+2)} P(m+x)^2 y$.

Subtangens $= 1 + \frac{x}{\infty} = 1$, nempe Constans.

Trilineum Infinite protensum, sed Areae finitae $= \frac{(\infty+x)y}{\infty+1} = \frac{\infty}{\infty} \cdot y = 1 \cdot y$.

Quadrilinei Area inter y, ac y' quaslibet $= \frac{\infty \cdot y - \infty \cdot y'}{\infty} = 1(y - y')$.

Solidum Infinites longum circa Asymptoton ad Infinites magnum Cylindrum Circumscriptum, ut $1 : 2\infty + 1$, vel $1 : \infty$, quae Ratio decernit esse Mensurae illud Solidum finitae,

vel $= \frac{P.y^2(\infty+x)}{2 \cdot \infty + 1} = \frac{P \cdot \infty \cdot y^2}{2 \cdot \infty} = \frac{P \cdot 1 \cdot y^2}{2} = P \cdot y^2 \cdot \frac{1}{2} = \frac{3}{2} (P y^2 \cdot \frac{1}{3})$, ut placuit Hugenio.

Solidum Rotundum Infinites longum revolutione ortum Trilinei circa Ordinatam quamlibet $= \frac{2}{(\infty+1)(\infty+2)} P(\infty+x)^2 y = \frac{2 \cdot \infty^2}{\infty^2} \cdot P y = 2(1 \cdot P y) = 6(\frac{1 \cdot P \cdot y}{3})$.

K k k k k

Distantia

Distantia Centri Gravitatis Trilinei, super
quamvis Ordinatum insistentis, ab Axe $= \left(\frac{m+1}{4m+2} \right) y$.

Centri ipsius Gravitatis Distantia
ab Ordinatum-Applicata $= \frac{v}{m+2} = \frac{m+x}{m+2}$.

Distantia Centri Gravitatis I.
Solidi a propria Basi $= \frac{v}{2m+2} = \frac{m+x}{2m+2}$.

In Solido II°. a propria Basi Centri
antea dicti Distantia $= \left(\frac{m^2 + 3m + 2}{8m^2 + 12m + 4} \right) y$.

Co-Radius Osculi in
directione Ordinatae,
uti habent Hospitalius,
aliique $= \frac{y}{\frac{1}{m} - 1} \left(\frac{1 + m^2 v^{2m-2}}{m^2 v^{2m-2}} \right)$.

Distantia Centri Gravitatis ab
Asymptoto Infinite longi Trilinei $= \frac{\infty + 1}{4\infty + 2} y = \frac{y}{4}$.

Praedicti Centri Distantia ab Ordinata $= \frac{\infty + x}{\infty + 2} = \frac{\infty}{\infty} = 1$.

Distantia Cen. Grav. I. Solidi a Basi $= \frac{\infty + x}{2\infty + 2} = \frac{\infty}{2\infty} = \frac{1}{2}$.

Itidem ea Distantia in II°. Solido $= \frac{\infty^2}{8\infty^2} \cdot y = \frac{y}{8}$.

Co-Radius pariter Osculi $= \frac{y}{\frac{1}{\infty} - 1} \left(\frac{1 + \frac{\infty^2 y^2}{(\infty + x)^2}}{\frac{\infty^2 y^2}{(\infty + x)^2}} \right) =$
 $- y \left(\frac{1 + \frac{\infty^2 y^2}{\infty^2}}{\frac{\infty^2 y^2}{\infty^2}} \right) = - \left(\frac{1 + y^2}{y} \right) \cdot (*)$

346. Contrañior haec methodus omnia fere complectitur Logarithmicae
Curvae theoretica Elementa, cuius viribus cedunt etiam sublimiora a Cote-
fio de Logisticae ipsius Perimetri dimensione, de Superficie Solidorum de-
finita mensura, ideoque de Centri Gravitatis Perimetri Infinite longi nulla
ab Asymptoto Distantia &c: in I°. ac II°. suae *Logometriae* Partibus tam
sagaciter adinventae. Familiaris, maximeque Geometrarum manibus trita
haec ipsamet Curva Schematibus haecenus carere sine obscuritate equidem
potuit: veruntamen Diagrammati XXVII°. parcendum esse non arbitror ad
cognitionem Figurae Parabolae, Logarithmicaeque intimius contemplan-
dam. Tota etenim pendet adhaecusque inobservata Curvarum harumce so-
cietas ab Aequationum $\left(1 + \frac{x}{m} \right)^m = y$, atque $\left(1 + \frac{x}{\infty} \right)^\infty = y$ com-

muni origine, ex qua fit, ut eodem manente in Parabolis omnibus initio
Abscissarum x , videlicet Puncto B, ac principe Ordinata $BC = 1$, Vertex
A, A', A'' &c: eo magis in longinquum abeat sine ullo Limite, quo plus
excreseat

(*) Adde disserta in calce numeri 387. ad Theoriae complementum.

excreſcat Index m Unitate maior, diverſum Gradum designans Parabolarum. Profecto y tum evaneſcit, quum $x = -m$, ac tandem converſo Exponente m in ∞ Vertex A''' Infinite diſtans a BC Axem Abſciſſarum, ſeu Verticalem Tangentem in Aſymptoton vertit $A'''BD$; quae transformatio nihil aberrans a Curvarum Geometria poſtremum Parabolarum, Gradu ſefe elevantium, ac Vertice ſemper ſemperque retro fluentium, Limitem Logarithmicam reddit. Et ſicuti Vertex omnium Parabolarum Punctum eſt *Singulare*, praeter Apollonianam ubi ſimplex habetur Contactus, inde dimanat Infinites *Singulare* fore illud Punctum extremum Logiſticae A''' , quod in Infinitatis ſinu latet obſcurum, et cum Aſymptoto, tunc Tangente, confunditur. Punctorum Curvae cuiuslibet *Singularium* ſymptomata, vel ſint Regreſſus ut A , vel Inflexionis ad inſtar A'' , vel Serpentina, Multiplae Cuſpidis, et Regreſſus alterius Speciei, Maupertuiſii ſtylo, vel Multiplicis Inflexionis, viſibilis, aut inviſibilis, iuxta vocabula a Bragelongnio, et Cramero uſitata (*), qualis eſt A' , communis affectio criteriumve a Geometris paſſim ſtatuitur in eo, quod Circulum reſpuant Oſculatorem, gaudeantque $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, &c: $\frac{d^ny}{dx^n} = 0$ aut ∞ , proportionem habita ad maiorem minoremve Curvae Ordinem $n \rightarrow 1$, ac *Singularis* Puncti diverſam in Gradu compositionem. Punctum igitur A''' Logarithmicae *Singulare* erit, ſive caſtigatiore expreſſione *Singularium* poſſibilium extremus Limes, ſcilicet *Singulare* Infinites-Serpentinum-inviſibile, in quo $\frac{d^\infty y}{dx^\infty} = 0$, Radiusque Oſculi Infinite-Magnus, prouti confirmant Radii, et Co-Radii Formulae $\frac{(1+y^2)^{\frac{3}{2}}}{y}$, ac ſuperius expoſita $\left(\frac{1+y^2}{y}\right)$, quum nulleſcente y evadant $\frac{1}{0}$, nempe Magnitudinis Infinitae. Elementum Logiſticae Curvae prope A''' Infinite diſſitum

(*) Conſulantur *Memorabilia Scientiarum Academiae Pariſienſis* relata ad annos MDCCXXX. ac XXXI., nec non eximia *Introductio in Curvarum Analyſin* anno MDCCL. in lucem edita.

difficilius a B Abscissarum initio fit ergo Punctum Contactus innumeris Intersectionibus aequipollens, perinde ac si Curva ex innumeris Rectis Lineolis in eadem directione, atque super Asymptoton iacentibus, et idcirco haudquam ad modum Polygoni usitato sensu concepti, illic existeret. Crus autem Logarithmicae ipsius GCFA''' est Parabolici Generis, sed ad Parabolen Gradus Infinitecupli pertinentis; ita, ut quemadmodum, excepta Apolloniana, Parabolae omnes munere funguntur simpliciorum variae Speciei Curvarum Osculatricium in *Singularibus* Curvilinearum Punctis, non secus Logistica in Transcendentium Curvarum Punctis Infinitecupla *Singularitate* praeditis peculiarem Curvaturae earundem Speciem, quam apte diceret *Logarithmico-eiden*, ibidem toto iure osculari, ac metiri posset. Nec obstat Asymptoti existentia in Logistica, quae notissimo Canone duce Hyperbolicos, non Parabolicos, Curvarum Ramos distinguere in Geometricis solet Lineis, cum tam

ex depicto Schemate, tam ex Aequatione $\left(\frac{\infty+x}{\infty}\right)^{\infty} = y$ luculenter pateat,

quod necessaria conditio, Aequationem illam comitans, Verticis A''' Infinites remoti, nimirum nullibi existentis, aliam inferat, Curvae Parabolicae Axem Abscissarum sive Tangentem Verticalem, metamorphosi eiusdem facta in Logisticam, affectiones Asymptoti induere, et Hyperbolicam mentiri Asymptoton. Sed a vocis, non rei, disceptatione supersedendum.

347. Absit tamen quisquis ab iudicando adscriptam Parabolarum Familiae Logisticam ex Transcendentium Curvarum classe subtrahi; quod qui sibi polliceretur laterem certe lavaret. Enimvero Infinitecuplus Ordo Parabolae-Logarithmicae Aequationem Localem finitis Numero ac Magnitudine terminis coalescentem, scilicet Algebraicam, excludit. Id in contracta emicat

Aequatione $\left(1 + \frac{x}{\infty}\right) = y$; et eo clarius in expansa Infinitinomiali y

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} + \frac{x^6}{2.3.4.5.6} + \frac{x^7}{2.3.4.5.6.7}$$

+ &c: perlegitur. Porro haec Infiniti Gradus Aequatio Curvam rursus denotat Perimetrum Generis Parabolici, commonstratque altiore Potestatem $\propto x$ Indice Infinito seu Inassignabili praeditam; Puncto A''' evanescentis Ordinatae y aequales sine Numero respondere Abscissas $= \infty$; ad Punctum illud innumeras invisibiles Intersectiones Axis, et Curvae in unum confundi; ac

di; ac denique in *Singulari* eodem Puncto Ordinis, si liceret, $\infty - 2$ Infinite proximis Sectionibus numero ∞ .^{is} aequipollente repositam esse Legis illius oecumenicae observantiam, quae iubet Intersectionum Rectae, et Curvae mutuarum Numerum Lineae cuiuslibet Ordini aequalem, sancitque, dum sint numero Infinitae, Transcendentem earundem Linearum naturam. Elegans praeterea in Logistica Theorema percipitur, quod, si ad mentem Newtoni Fluxionum-Fluxiones nusquam abruptae nec evanescentes cuiusvis Ordinatis Applicatae y in Logistica Ordinatis alterius Curvae repraesentarentur, hae Fluxiones Genitrici Logarithmicae aequipollentem Logarithmicam perpetuo producant. Hanc singularem ab unica data Logistica innumerarum eidem aequalium per Differentias quasi regenerationem sine aliqua animi iucunditate speculari unquam non possum. Et re quidem vera

$$\frac{dy}{dx} = z = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} + \frac{x^6}{2.3.4.5.6} + \&c.;$$

$$\text{aut } \frac{dy}{dx} = \infty \left(1 + \frac{x}{\infty} \right)^{\infty-1} \frac{dx}{\infty \cdot dx} = \left(1 + \frac{x}{\infty} \right)^{\infty};$$

$$\frac{d^2 y}{dx^2} = \frac{dz}{dx} = v = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} + \&c.;$$

$$\text{vel } \frac{d^2 y}{dx^2} = \infty (\infty - 1) \left(1 + \frac{x}{\infty} \right)^{\infty-2} \frac{dx^2}{\infty^2 \cdot dx^2} = \left(1 + \frac{x}{\infty} \right)^{\infty};$$

$$\frac{d^3 y}{dx^3} = \frac{d^2 z}{dx^2} = \frac{dv}{dx} = t = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \&c.;$$

$$\text{five } \frac{d^3 y}{dx^3} = \infty \cdot \infty - 1 \cdot \infty - 2 \left(1 + \frac{x}{\infty} \right)^{\infty-3} \frac{dx^3}{\infty^3 \cdot dx^3} = \left(1 + \frac{x}{\infty} \right)^{\infty};$$

neque his Differentio-Differentialibus prosequendis confinia posuit Analysis Naturae. Nihilo tamen minus in suscepta Logarithmicae anatome adhuc incertum manet, an Curva ista Transcendens ex unico, aut gemino circum Asymptoton Crure existat: namque inter innumeras Parabolas, quarum

Limes aequae concipi potest ea pertinens ad Aequationem $\left(1 + \frac{x}{\infty} \right)^{\infty} =$

y, nedum reperitur Cuspidata in morem ECAL seu duplicis Rami ad unam alteramque Logisticae Asymptoti partem sese aequaliter componentis, ve-

rum etiam conformata instar CA'P, CA''H &c:; quarum utraque Figura Punctum *Regressus* iisdem Abscissis additum, idest geminum Logarithmicæ Ramum negaret. Suspenso in tot Figurarum discrimine iudicio, dum Infinitati fidere tuto liceret, ac nisi vera Logarithmicæ Aequatio Localis fuisset $1^x \left(1 + \frac{x}{\infty}\right)^\infty = y$, ita proferri sententia posset, ut omnis fortasse cederet dubitatio, ratioque magis minusve probabilia librandi. Cyphra siquidem ∞ in Aequatione ad Logisticam $\left(1 + \frac{x}{\infty}\right)^\infty = y$ non Infinitum generaliter conceptum significat, sed ex toties iam dictis, ac praesertim num°. 110., speciale Infinitum Harmonicum. Inde consequitur valde plausibilem prae ceteris sese offerre Aequationis formam $\left(1 + \frac{x}{L_\infty}\right)^{L_\infty} = C^x = y$, quae, ut in num°. praecitato 110., ultro dimanat ab Aequatione *conditionali* $\left(1 + \frac{x}{\infty}\right)^\infty = y = 0$, quando $x = L_0 = -\infty = -L_\infty$, et idcirco in ea hypothese fore Logarithmicæ Aequationem Localem huiusmodi

$$\left(1 + \frac{x}{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} \&c:}\right)^{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \&c:} = y, \text{ nimirum typum ha-}$$

bentem $v^{2m} = y$. Id subnexa patefacit Numerica Tabula

$$1 + \frac{1}{2} = \frac{3}{2}$$

$$1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{137}{60}$$

, absque ullo Limite continuanda.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{147}{60}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{363}{140}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{761}{280}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} = \frac{7129}{2520}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} = \frac{7381}{2520}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} = \frac{83711}{23720}$$

&c:

&c:

Exponens etenim ∞ , vel Summa Infinite magna terminorum omnium illius Harmonicae Progressionis adeo est comparata, ut a Transcendente et inaffignabili Fractionario Numero repraesentetur, cuius Denominator Par Infinitus, at Infinite maior Numerator Impar. Aequatio vero $v^{\frac{2n-1}{2m}} = y$ respondet Parabolae, quae universaliter Cuspide aut Regressus Puncto gaudeat ad instar percelebris Nelianae, vel Semi-Cubicae, primum inter Geometricas Curvas anno MDCLVII. Rectificationis honore donatae (*); et idcirco Cuspidata

(*) Videantur *Philosophicae Transactiones* anni MDCLXXIII., ubi exstat Vallisii Epistola ad Oldenburgium inventionis primatum auferens Van-Heuraetio Hollando.

spidata etiam esset Logistica, duobusque ex consimilibus Ramis existens circa Axem positis A''BD, ac Vertice, Cuspide, Regressusve Puncto in Infiniti latebris veluti abscondito. Huic coniecturae obstat *Exponentialitas* 1^x , quae ex praemissis observationibus Localem inficit Logisticae Aequationem, necnon Coefficiens ille r Indeterminatus, et in num°. 110. expositus. Attamen gemini Logisticae Cruris opinioni novum quiddam robur accedet tum, cum in Capite subsequente celebriora collegerim argumenta, quae simul iuncta unum et alterum convenire Ramum Logarithmicæ, Asymptotonque eius esse Diametrum adseverant.

348. Mirabilis equidem Infiniti in Serie Harmonica repositi non pauca symptomata superius attigi, sed nullum mea sententia nuperrimo perpulchrius, subtilius, atque ad rem opportunius. Dum in Analyticarum veritatum numerum cooptari iure posset, quod superius propter Infiniti obscuritatem coniecturae loco fuit, videlicet Aequatio $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$$+ \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c: = \frac{2.^\infty'' - 1}{2.^\infty'}, \text{ nonnulla perpolire}$$

fas esset Theoremata, ac certiore significatione exornare, quae in numeris 39. et 73. circa Potestates $(1 - x)^\infty$ versantur, Signo ∞ Infinitum illud Harmonicum repraesentante. Foret enim, vix dum $x > 1$, perpetuo

$$(1-x)^\infty \text{ Harm.} = (1-x) \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \&c: \right)$$

$$= (1-x) \frac{2.^\infty'' - 1}{2.^\infty'}$$

Quantitas Imaginaria, non amplius *Reale Infinitum æquivocum*, quippe quum hoc proprium exstet Magnitudinum omnium Negativarum, si ad Potestates Indicibus $\frac{2n-1}{2m}$ praeditas eleventur. Sed ab his-

ce desistam nimis fallacibus divinationum prope modis, melioraque delibabo. Promovendae universalis quo ad diversas Curvas Logometricae facultati valde proderit speculatio combinationum omnium possibilium in Continuo Arithmeticae simul, ac Geometricae Progressionis. Communis Logistica, et in Curvis, quarum Ordinatae instar Radiorum ab unico Centro seu Foco digrediantur iuxta considerationem primum inductam ab excellentibus Viris Desarguesio, ac Leibnitio, Logarithmica - Spiralibus Progressionum illarum simplicissi-

simplicissimo satisfaciunt Systemati, cum Abscissae super rectilineum Axem iacentes in priore, atque Anguli-Abscissae in altera sint respondentium Ordinarum, vel aequidistantium, vel divergentium, Logarithmi. Modulus Systematis cuiusvis Logometrici in Logarithmica Subtangentem exaequat, uti constat ex Prolegomenis in num°. XXV., et compendiosius ex Differentia Finita Logarithmi cuiuslibetunque Magnitudinis paullo ante in num°. 336. supputata. Fit enim Aequatio ad Logisticam, translato Abscissarum initio ad Ordinatam $y = a$ punctum intersectionis cum Curva, numeratisque Abscissis iisdem

$$x \text{ super Rectam Axi parallelam, } D(ly) = Dx = v = M \left(\frac{(Dy)}{y} - \frac{(Dy)^2}{2y^2} + \frac{(Dy)^3}{3y^3} - \frac{(Dy)^4}{4y^4} + \frac{(Dy)^5}{5y^5} - \frac{(Dy)^6}{6y^6} + \frac{(Dy)^7}{7y^7} - \&c. \right) = M \left(\frac{z}{a} - \frac{z^2}{2a^2} + \frac{z^3}{3a^3} - \frac{z^4}{4a^4} + \frac{z^5}{5a^5} - \frac{z^6}{6a^6} + \frac{z^7}{7a^7} - \&c. \right), \text{ interea dum Ae-}$$

quatio ad Lineam-Rectam, ipsis Co-Ordinatis manentibus, est $v = \frac{N(Dy)}{a} = \frac{Nz}{a}$: quamobrem ad hoc, ut Recta Curvae Tangens evadat, oportet

aequipollentia expressionum $\frac{Mz}{a}$ et $\frac{Nz}{a}$, nimirum $M = N = \frac{av}{z} =$ Sub-

tangenti quaesitae. Huiusce methodi exempla consulto paravimus in Prolegomenis ad numerum XXVII.^{um}; satisque nunc elucescit, ob terminum sub-

sequentem negativum $-\frac{Mz^2}{2a^2}$, Logisticam vertere suae Perimetri con-

vexitatem ad Tangentem, et ideo ad Axem seu Asymptoton. Coefficientis

etiam ille numericus $k = \frac{1}{M}$, sive Logometrici Systematis *Characteristica*,

cuius toties in Prolegomenis, et in Capitibus II°. ac III°. mentio recurrit, Proportionis Indicem aut Mensuram exprimet inter elementa Numeri nascentis supra Unitatem, atque Logarithmi correspondentis; quippe cum valeat ex supra positis Proportio $M:1::d(ly):dy::d(ly):kd(ly)$, a qua dimanat $Mk \cdot d(ly) = 1 \cdot d(ly)$, nimirum $Mk = 1$, facto Numero y inassignabiliter ab Unitate diverso. Modulus autem in Helice-Logarithmica diversimode se

M m m m m habet,

habet, quum sit Tangens Anguli vel Arcus Radio-Unitate descripti, mensurantisque constantem Angulum, quem Ordinata quaelibet Polaris, seu Recta a Foco Centrove innumerarum circumvolutionum educta, et Curvae Tangens simul constituunt. Itaque, si super quamlibet Ordinatum-Applicatam locetur, initio sumpto a Logarithmicae-Spiralis Perimetro, Recta Semi-Diametro aequalis illius Circuli Circumferentiae, in qua Logarithmi supputentur, invariabilem servat longitudinem; Modulumque Systematis praebet non vera Curvae Subtangens, sed eius *Vicaria*, nempe verae Subtangenti parallela, aut ea Ordinatae Normalis inter Tangentem, Punctumque in Ordinata locati Semi-Diametri extremum disposita. Systema igitur Logarithmorum Hyperbolicorum Helix-Logistica tribuet dum eius Ordinatae Angulo semi-recto ad Tangentes Curvae inclinentur, cui nomen *Semi-Rectangulae*, et Tabularium vel Briggianorum tum, quum Helix illa Angulo proxime $23^{\circ} 28' 30''$ gaudeat. Cetera insuper sine numero, rite recteque permutatis iis Angulis, repraesentari poterunt in elegantissima huiusce formae Spirali Systemata Logometrica, quae Limitibus interiaceant infinities diffitis Anguli vel Tangentis nullius dum Curva contrahitur in Focum, et Anguli-Recti Tangentisve Infinitae dum universae sine fine Spirae in Circularem abeunt Peripheriam innumeris vicibus repetitam. Hoc demum accidit sane iucundum, quod, cum alibi ostendatur in Capite IX°. Logisticam communem duobus constare Ramis, ita etiam Spiralis-Logarithmica gemino Spirarum ordine coalescere debeat, ac, si totam exhaustire velis Localem eiusmodi Curvae vim, conformetur propemodum circa unicum Polum, qualiter

XXVIII.^{um} Schema demonstrat, primo nimirum Ramo seu Crure Spiriformi per semi-Peripheriam revoluto aut promoto, ut alterius similis, et aequalis positio habeatur. Effatum ergo illud celeberrimi Iacobi Bernoullii *Eadem mutata resurgo* (*), quod praecipue respexit Logisticam-Helicoidem, quia sui ipsius Evolutione genitam, aliquantisper ditare fas erit: namque, omni *Resurrectionis* similitudine sublata, duplicatur ob geminum Ramum regenera-

(*) De hoc egit eximius Auctor in *Commentariis Lipsiensibus* anni MDCXCII.; fama-que est, Battierio eius Biographo adfirmante, Spiralem-Logarithmicam, aut *Spiram mirabilem* Sepulchrali suo in Lapide veluti *Resurrectionis* Emblema Bernoullium ipsum incidi iussisse cum Epigraphe praemissa, quae supra.

nerationum numerus Spirae mirabilis; ac, dum foret Rectangula, accedit Evolutam unius Rami in medio stare utrorumque Curvae Ramorum. Hoc palmarium equidem, novumque in Curvilinearum Theorice symptoma, Ramis scilicet Curvae interferi paribus Angulis aequalem, quae Evolutae munere fungatur, Geometrarum considerationem sollicitat perquam maximam; quinimmo et stimulos addit idem inter Curvas Regressu secundae Speciei, Ordinatisque parallelis praeditas investigandi, si non aequaliter quadripartito ut in Logistica Rectangula, bisecto saltem ab Evoluta Spatio Ramis Curvae interposito.

349. Logarithmos omnigenos geometrice contemplari nedum in Curvarum Abscissis, verum etiam in Arcubus, Areis, Solidis-Rotundis, eorumve Superficiebus, aliisque Coordinatarum Functionibus Algebraice aut Transcendenter quomodocunque compositis, licet. Eo enim pacto, quo potuerunt

ab Aequatione $\frac{Mdy}{y} = \frac{dx}{1}$ elici Curvae Logisticae expansae, aut in Spiram contortae, Ordinatis y Numeros exhibentibus, Abscissisque x nunc Axis Segmenta, nunc Arcus eiusdem Circularis Peripheriae denotantibus, consimiliter

$\frac{Mdy}{y} = ds = \sqrt{dx^2 + dy^2}$, vel $= \sqrt{y^2 dx^2 + dy^2}$ Curvam, cuius Arcus s sint Ordinatarum Aequidistantium, aut a Polo exeuntium Logarithmi ad Modulum M .

Aequalitates $\frac{Mdy}{y} = ydx$, vel $= \frac{y^2 dx}{2}$ designabunt Curvam Ordinatis parallelis praeditam, aut Polarem, cuius Areae praebeant Ordinatarum Logarithmos ad Modulum M .

$\frac{Mdy}{y} = Py^2 dx$ Curvam, cuius Solidi Rotundi Frustra dent Logarithmos Ordinatarum ad Modulum M .

$$\frac{Mdy}{y}$$

$$\frac{Mdy}{y} = \Pi y ds = \Pi y \sqrt{dx^2 + dy^2}$$

Curvam, in qua Frustra, seu
Zonae Superficie Solidi
pariter Rotundi sint invi-
cem uti Ordinatarum Lo-
garithmi ad Modulum M.

$$\frac{Mdy}{y} = d\Phi(x, y)$$

Curvam, ubi quaeliber-
cunque Functio $\Phi(x, y)$
sit Logarithmus τy ad
Modulum M comparatus.

Delibabo pauca, fortassis elegantiora, quum universa nequeam complecti, ur-
gente rerum, quae differenda supersunt, necessitate. Sciant autem ii, qui
hisce studiis navaverint operam, forma varias, sed cum superioribus coin-

cidentes fore Aequationes $C^{\frac{x}{M}} = y$, $C^{\frac{s}{M}} = y$, $C^{\frac{S(ydx)}{M}}$ seu $\frac{S\frac{y^2dx}{2}}{M} = y$,
 $C^{\frac{S(Py^2dx)}{M}} = y$, $C^{\frac{S(\Pi y ds)}{M}} = y$, $C^{\frac{\Phi(x,y)}{M}} = y$, Basinque B, conso-

nam diversis Modulis M, aut Logarithmorum Systematibus, esse $C^{\frac{1}{M}}$
iuxta adserta in num°. 95; unde Aequationes praefatae formam etiam Expo-

nentialem $B^x = y$, $B^s = y$, $B^{\frac{S(ydx)}{2}}$ seu $B^{\frac{S(\frac{y^2dx}{2})}{2}} = y$, $B^{\frac{S(Py^2dx)}{2}} = y$,
 $B^{\frac{S(\Pi y ds)}{2}} = y$, $B^{\frac{\Phi(x,y)}{2}} = y$ &c: adipiscuntur.

350. Aequatio $\frac{Mdy}{y} = (dx^2 + dy^2)^{\frac{1}{2}}$ usitatae est tessera Curvae, cui
Traëtricis aut Traëtoriae nomen impositum, habetque antiquitus hoc sin-
gulare, ut Tangens $\frac{y(dx^2 + dy^2)^{\frac{1}{2}}}{dy} = M$ sit semper constans, et Modulum

statuat definiti cuiusvis Systematis, in quo Ordinatum-Adplicatae officio
Numerorum fungantur, Arcusque respectivi Logarithmorum. Nec diffici-
lius

lius intellectu est, Aequationem alteram $\frac{Mdy}{y} = \sqrt{y^2 dx^2 + dy^2}$, sive $\frac{dx}{1}$

$= \frac{\sqrt{M^2 - y^2} dy}{y^2}$ repraesentare Spiralem - Traatricem, vel Helicis a Circuli

Evolutione ortae Reciprocam, Tangente pariter constanti exornatam

$\frac{y(y^2 dx^2 + dy^2)^{\frac{1}{2}}}{dy} = M$, quae fit Logometrici Systematis Modulus Arcus in-

ter, et Radius. Nonnulla maioris momenti de Spiralis nunc obiter memoratae affectionibus commentatur Cotesius in III^a. *Harmoniae Mensurarum* Parte; atque ibidem Helicen insuper Lituiformem considerat Asymptoto gaudentem, cuius Polarium Sectorum Areae Radiorum exprimant Logarithmos. Praecipua Curvae symptomata, a Cotesio indemonstrata, alienum abs

re non existimo heic recensere. Quum $\frac{Mdy}{y} = \frac{y^2 dx}{2}$ fit Aequatio ex prae-

missis ipsam referens Curvam, habebitur $\frac{dx}{1} = 2My^{-3} dy$, nimirum $\frac{x}{1} = -$

My^{-2} ; vel Anguli Radiorum cum Asymptoto erunt inverse ut istorum Quadrata, et Gravitatis Coelestium Corporum quasi symbolum. Sectores praeterea Circulares Lituo inscripti, ad instar illorum, qui Vallisii stylo Spiralem *Spuriam* dum Infinitesimi sejunctique forent composuissent, in Asymptoton communiter desinentes aequalem Aream concludent, ex eo quod Area

cuiusque $= \frac{y^2 \cdot \frac{x}{1}}{2} = \frac{My^2}{2y^2} = \frac{M}{2}$; ita, ut Logometrici Systematis Modu-

lus duplum cuiuslibet cunq; Sectoris peraequet, vel Triangulum illum ab Ordinata Polari, Tangente, ac SubTangente circumseptum, quippe cum

SubTangens $\frac{y^2 dx}{dy}$ ducta in $\frac{y}{2}$ praebeat spatium Trigonum $\frac{y^3 \cdot \frac{dx}{1}}{2dy} = M$. Huius

igitur mirificae Helicis Modulus perinde se habet, ac ille Hyperbolae inter Asymptotos: obvium est enim Radium, Tangentem, Asymptotique partem Tangenti ac Centro interpositam, quae SubTangentis *analogae* vicem gerit, Potentiam Aequilaterae Hyperbolae, aut inscriptum Parallelogrammum cuiusvis Scalенаe, scilicet Modulam, exaequare.

351. Areas Quadrilnearum, Sectorumve Hyperbolae Logarithmos Abscissarum delineare neminem latet, absque eo quod deduci haec debeat affe-

ctio a Summatoria Aequationis propositae, nimirum ex $S\left(\frac{Mdy}{y^2}\right) = x$, aut

$xy = M. 1 = M$, Signo, prouti saepius occurrit, neglecto. Monitum etiam fuit, sancitumque in Prolegomenis, praesertim num°. XXV°. Parallelogrammi Aream Hyperbola atque Asymptotis inscripti recte assumi debere pro Modulo Systematis Logarithmici; id, quod denuo evincit superius tradita Aequatio. Nihilo tamen minus sedulo est advertendum non omnia Scalenis Hyperbolis imitari nos posse Systemata Logometrica, sed ea dumtaxat Modulum habentia Unitate minorem, quae conditio deficit in Tabularium Reciproci, in iis, qui a Basi 2 pendeant, aliisve non difficili ratione effingendis. Simul autem iunguntur in eadem Aequilatera Hyperbola Systemata tam affinia Naturalium, et Antinaturalium Logarithmorum; namque illi referuntur ab Abscissas, Modulumque positivum 1, hi vero ad Ordinatas, ac Modulum — 1 a Potentia exhibitum negative desumpta. Nec difficultatem pariet invisentibus hoc duorum Systematum in ipsamet Hyperbola foedus, propter quod Logarithmus Infinitus Numeri Infinite Magni in Abscissa AB, uti Schema XXIX.^{um} patefacit, ab Area Quadrilinei sine Limite producti repraesentetur CEBD, dum ex adverso Numeri pariter Infinite-Magni AF in Ordinatum Asymptoto adplicata a Quadrilineo CGFAD referri debeat. Sunt etenim haec duo Spatia Infinita ad invicem aequalia, et eorum unumquodque Infinito-Harmonico par 1 +

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13}$$

+ &c.; nihilque Infiniti interest Differentia Finita CDAH = 1, quae, inductis adparenter diversis Infinitorum Quadrilnearum valoribus, frustra ratiocinationis fallaciam excitare molitur. Iste inter ludos Infinitatis certe non minimus, Geometricae Syntheseos rigore impetrato, evanescit, ex eo quod aequaliter hinc inde ad SemiLatus Transversum AC inclinatis Radiis innumeris AL, AN, AB, et AK, AM, AF proveniant

Aequalitates

$$\text{Aequalitates Arearum } CDQL = CAL = CAK = CDPK$$

$$LQNS = LAN = KAM = KPRM$$

-----,

$$\text{ideoque tandem } CDBE = CABE = CAFG = CDAFG.$$

352. Summa praeterea res est in hac Naturalium vel Anti-Naturalium Logarithmorum ab Hyperbolicis Areis repraesentatione, Abscissis $AR', AP', AQ', AS' \dots \dots \dots$, AB' , atque Ordinatis $R'M', P'K', Q'L', S'N' \dots \dots \dots$, $B'E'$, scilicet Numeris negativis, aequales admodum Areas aut Logarithmos eosdem respondere, qui aequis singillatim Numeris $AR, AP, AQ, AS \dots \dots \dots$, AB , five $RM, PK, QL, SN \dots \dots \dots$, BE positivis conveniunt. Namque Areae negativae $CDPK, CDRM \dots \dots$ progrediuntur usque ad Infinitam-Harmonicam $CDAFG = -1 - \frac{1}{2} - \frac{1}{3}$

$$- \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9} - \frac{1}{10} - \frac{1}{11} - \frac{1}{12} - \frac{1}{13} - \&c;$$

quo perventae, Continuitatis Lege facta testata, ab Infinito-Harmonico negativo regredi incipiunt statim ac ex parte U opposita vacuo Spatio T Quadrilinea compareant positiva, quae inchoant ab Areae nullitate, seu Asymptoto AF' , subindeque per gradus $AF'G'M'R', AF'G'K'P', AF'G'L'Q', AF'G'N'S' \dots \dots$ ita temperant Spatium illud Infinitum negativum ab eadem Recta manente CD semper supputatum $- CDAFG$, ut inita rationis compensatione, vel reductione facta proveniant

$$- R'M'C'D' = L(AR') = L(-AR) = -RMCD = L(AR)$$

$$- P'K'C'D' = L(AP') = L(-AP) = -PKCD = L(AP)$$

$$+ D'C'Q'L' = L(AQ') = L(-AQ) = +CDQL = L(AQ)$$

$$+ D'C'N'S' = L(AS') = L(-AS) = +DCNS = L(AS)$$

$$- CDAFG + AF'G'C'D' = 0 = L(AD') = L(-1) = L(+1)$$

$$+ C'D'B'E' = L(AB') = L(-\infty) = +CDBE = L(AB) = L(+\infty);$$

ipsaque

ipsaque ratiocinatio ad Hyperbolen quamlibet Scalenam transferri consimiliter posset. Huius argumenti synthetici pondere controversiam pene omnem de Logarithmis Numerorum negativorum feliciter sustulit Ioannes Bernoullius in Epistola ad Leibnitium anno MDCCXII. exarata (*), nec vim eius labefactare potis fuit Arearum Hyperbolae in transitu unius Cruris ad oppositum *Discontinuitas* a Davieto Foncenexio ingeniose propugnata, sed fallaci expressionis $\frac{Mdx}{x}$ interpretationi subnixae. Auctor etenim ille in priore

Miscellaneorum, scilicet anni MDCCCLIX., *Societatis Taurinensis* Volumine veram Formulam Analyticam Arearum Hyperbolae Apollonianae inter Asym-

ptotos, nempe $\int \frac{Mdx}{x} = M(Lx - L_0) = M(Lx + \infty \text{ Harm.}) = y$, Ab-

scissis x a Curvae Centro computatis, ac posita eius Potentia $= M$, nescio qua mentis celeritate non vidit, cui Formulae omnimoda adest Continuitas, et Constructionis Geometricae sancitis Legibus obsecundatio (**).

353. Logarithmos in Quadrilineis, aut Sectoribus Hyperbolicis perspicientes nonnulla addemus Geometrica, consulto in retroactis omissa, de percelebri, utillimoque Systemate, quod ad Sinus, ac Cosinus Hyperbolicos, et Circulares refertur. Sit in Schemate XXX.^{mo} Hyperbola Aequilatera Apolloniana ABFC, cuius Abscissa AL, ideoque etiam Potentia ALFE $= 1$ ad hoc, ut Areae FLGC &c: Logarithmos tribuant Hyperbolicos Numerorum AG &c:, circumscribaturque alia Hyperbole concentrica, pariter Aequilatera, HDK, in qua Semi-Axis Transversus AD $= AL = 1$. Ductis Verticali Tangente DM, ac Normali NSP ad Axem primarium habebitur DM, $= 1 = \text{Sinui-toti}$, vel Radio Hyperbolico, nec non PN $= QR = PS + SN = AS + SN = AS + SQ = \text{Summae Cosinus}$, Sinusque Hyperbolicorum. Agatur a puncto Q Recta QT perpendicularis Asymptoto, fiatque AG: AT seu VQ $:: \sqrt{2} : 1 :: FA: AD :: LA: AY :: RQ: VQ$ vel AT, nimirum AG $= RQ = PN$. Quo facto erit Logarithmus Hyperbolicus Numeri PN, vel Summae Cosinus, Sinusque Parilaterae Hyperbolae HDK aequalis Quadrilineo FLGC, aut, propter Hyperbolas *Similes*, et AG: AT $:: AL: AY$, sive AG: AL $::$

(*) Consulatur Volumen II.^{um} *Commercii Philosophici et Mathematici* pag. 293. 294.

(**) Vide numerum 386.

AL::AT:AY, aequalis duplo Quadrilinei DYTQ, seu duplo Sectoris DAQ, vel Quadrilineo, aut Sectori per $\frac{AD}{2} = \frac{1}{2}$ divisi, quum satis pateat Pro-

portionis Geometricae FLGC:DYTQ::FEAL:DZAY::2:1 integra veritas. Recte igitur riteque ad calcem IIⁱ Capitis, atque in toto fere Capite V^o. Analysis eo nos perduxit, quo nunc simpliciorē Synthesi adfirmante pervenimus, scilicet ad definiendam Localem Hyperbolae Apollonianae Aequilaterae Aequationem, cuius Semi-Axis sit Unitas, et idcirco Potentia $\frac{1}{2}$,

ope Formulae Exponentialis $\frac{C^x + C^{-x}}{2} = \text{Cofin. H. (x)}$; a qua dimanat et altera $\frac{C^x - C^{-x}}{2} = \text{Sin. H. (x)}$, sive contractiore typo $C^{\pm x} = \text{Cofin. H. (x)}$

$\pm \text{Sin. H. (x)}$, ubi x duplum Areae significet Sectoris, aut Quadrilinei Hyperbolici illi Cofinui, vel Sinui eadem in Curva respondentis. Namque habebitur vi ipsius Aequationis resolutae $x = L(\text{Cofin. H. (x)} + \text{Sin. H. (x)})$, nimirum FLGC, vel $2\text{DYTQ} = \frac{\text{DYTQ}}{\frac{1}{2} AD} = \frac{\text{DAQ}}{\frac{1}{2} AD} = 2\text{DAQ} = \text{Log. H}$

(PN) = Log. H(AS + QS), prouti superius ostensum. Systema ergo Logarithmorum, quos Vincentius Riccatus vocavit *Analogos* (*) ob eximiam cum Logarithmis Imaginariis ad Circulum pertinentibus Analogiam, est mere Hyperbolicum si in Hyperbola BFC ipsi considerentur, quo ritu perquam maxime illustratur translatio Logarithmorum ad Circulum eodem Radio AD = 1 descriptum. Educta siquidem a Puncto quovis Φ Normali $\Delta\Phi\Psi\Lambda$ ad Axem utrique Curvae communem ADF, habebitur $\Delta\Lambda = \Delta\Psi + \Psi\Lambda = A\Psi$

$+ \Psi\Phi = \text{Cofin. C}(\text{Arc. D}\Phi) + \text{Sin. C}(\text{Arc. D}\Phi) = \text{Cofin. C}\left(\frac{\text{DA}\Phi}{\frac{1}{2} AD}\right) +$

Ooooo

Sin. C

(*) Haec Theoria apodictice comprobatur, clariusque evoluit id, quod in numeris praecedentibus 170. 171. 258. 281. &c: paravi huc respiciens. Acute satis, sed non integre, hoc Logarithmorum Systema perspexit Cotescius, uti habent paginae 12. 13. *Harmoniae suae Mensurarum*.

$$\text{Sin. } C \left(\frac{DA\Phi}{\frac{1}{2}AD} \right) = \text{Cofin. } C(2DA\Phi) \rightarrow \text{Sin. } C(2DA\Phi) \text{ instar Hyperbolae;}$$

$$\text{et ideo } C^{\pm x \sqrt{-1}} = \text{Cofin. } C(x) \pm (\text{Sin. } C(x)) \sqrt{-1} \text{ idem nunciat atque}$$

$$(2DA\Phi) \sqrt{-1} = \left(\frac{DA\Phi}{\frac{1}{2}AD} \right) \sqrt{-1} = (\text{Arc. } (D\Phi)) \sqrt{-1} = \text{Log. H.}$$

$(A\psi \rightarrow (\Phi\psi) \sqrt{-1})$; tantaque emicat Hyperbolae ac Circuli Logometrica similitudo, ut unicam Curvam Continuum prope imitentur. Nativa haec Logarithmos Analogos contemplandi methodus perplexitate aliqua obruitur in Riccati Operibus pluries antea citatis, qui non in Hyperbola *comparationis* BFC, Semi-Axe aut Radio $\sqrt{2}$, Potentiaque 1 exornata, sed in ipsa HDK,

cui Semi-Axis 1, ac Potentia $\frac{1}{2}$ adscribuntur, elegantissimam legit Logometriam.

Tametsi systema iam proditum mihi magis arrideat, facilis attamen demonstratio est omnium Elementorum, quae Riccatiano stylo Logarithmos distinguunt Analogos, sive tali ordine excusos, ut divisi per $\frac{1}{2}$,

seu per Semissem Radii, aut Transversi-Lateris Areas reddant Sectorum Hyperbolicorum, vicemque ideo gerant Pseudo-Arcuum Hyperbolae. Paucis verbis adhibitis, ipsorum adiiciam perbreve Theoriae elenchum.

$$\text{Potentia Hyperbolae AZDY} = \frac{1}{2}.$$

$$\text{Numerus habens 0 Logarithmum, sive Protonumerus exaequat AY, vel DY} = \frac{1}{\sqrt{2}}.$$

$$2(\text{DYTQ}) = \text{Log. Anal. (AT)} = \text{Log. Hyp. (AG)} = \text{Log. Hyp. } (\sqrt{2} (AT)) = \text{Log. H(AT)} + \text{Log. H}(\sqrt{2});$$

$$\text{ideft Log. Anal. (AT), sive Anti-Log. Anal (TQ)} = \text{Log. H(AT)} + \text{Log. H}(\sqrt{2}) = \text{Log. H(AT)} + \frac{1}{2} \text{Log. H(2)}.$$

$$\begin{array}{l} \text{Ergo} \quad \text{Log. Anal. (N)} - \text{Log. H}(\sqrt{2}) = \text{Log. H(N)}, \text{ ac vicissim,} \\ \quad \text{Log. H(N)} + \text{Log. H}(\sqrt{2}) = \text{Log. Anal. (N)} \end{array} \quad \left. \vphantom{\begin{array}{l} \text{Log. Anal. (N)} - \text{Log. H}(\sqrt{2}) = \text{Log. H(N)}, \\ \text{Log. H(N)} + \text{Log. H}(\sqrt{2}) = \text{Log. Anal. (N)} \end{array}} \right\} \text{Aequationes Fundamentales; nimirum,}$$

Log. Anal.

$$\text{Log. Anal.}(N) = \text{Log. H}(N) + 2 \left(\frac{(\sqrt{2}-1)}{(\sqrt{2}+1)} + \frac{(\sqrt{2}-1)^3}{3(\sqrt{2}+1)^3} + \frac{(\sqrt{2}-1)^5}{5(\sqrt{2}+1)^5} + \frac{(\sqrt{2}-1)^7}{7(\sqrt{2}+1)^7} + \frac{(\sqrt{2}-1)^9}{9(\sqrt{2}+1)^9} \&c: \right)$$

ex num°. 102., aut

$$\text{Log. Anal.}(N) = \text{Log. H}(N) + \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \frac{1}{18} - \frac{1}{20} + \frac{1}{22} - \frac{1}{24} \&c: \right) \text{ ex num°. 90.}$$

Hinc $\frac{C^x}{\sqrt{2}} = y$, aut $C^{x + L\left(\frac{1}{\sqrt{2}}\right)} = y$, vel $C^{x - \frac{L_2}{2}} = y$ erit Aequatio inter Numeros y ,

ac eorum Logarithmos Analogos x .

Sub-Tangens $= 1$, seu Modulus idem ac ille Systematis Hyperbolici.

Et demum Numerus, cuius Log. Anal $= \frac{1}{\sqrt{2}}$, seu Basis peraequabit $\frac{C^{\frac{1}{\sqrt{2}}}}{\sqrt{2}}$

$$= \frac{1}{\sqrt{2}} \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{2 \cdot 2} + \frac{1}{2 \sqrt{2} \cdot 2 \cdot 3} + \frac{1}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 4} \&c: \right) \text{ ex num°. 61.,}$$

quae universa, si loco τ 1 subrogetur r , congruunt cum Riccatianis, nostro tamen obiecto parum utilibus.

354. Eruditioni Geometricae melius quam praxi inservientia ea Systemata Logometrica, quae vel a Solidis, vel Solidorum Superficiebus exprimentur, ut de postrema universalissima Aequatione inter eas numeri 349. taceam, pauca desiderant. Dudum enim in Prolegomenis, ac signanter num°. XIV°, Formulam nacti consimilem animadvertimus Frustra Solidi Acuti Hyperbolici ab Hyperbolae, Lineis tertii Ordinis adnumeratae, versatione circa Asymptoton geniti Logarithmos Abscissarum a Centro Hyperbolicos vel

Naturales repraesentare; et duntaxat addendum censeo ab Aequatione $\frac{Mdy}{y}$

$$= Py^2 dx, \text{ five } xy^2 = \frac{M}{2P} \text{ deduci, quod Modulus Systematis } M \text{ par sit}$$

$2Pxy^2$, scilicet duplo Rotundi Solidi, vel potius Cylindri a circumvoluto

Rectangulo

Rectangulo xy super eandem Asymptoton veluti Axem producti. Pendet demum a Curvarum Quadraturis Integratio Aequationis $\frac{Mdy}{y} =$

$\Pi y \sqrt{dx^2 + dy^2}$: namque, constructa Linea Ordinis sexti $y^4 + y^4 z^2 = \frac{M^2}{\Pi^2}$, vel $(1 + z^2) y^4 = a$, erit $x = S(zdy)$, seu $x = S \frac{\pm \sqrt{a - y^4} dy}{\pm y^2}$; nec

vacat, praeter Indeterminatarum adeptam separationem, alias perambulare nunc semitas simpliciori fortassis Summationi obtinendae non idoneas. Sed, antequam ad Trigonometricas me vertam Formulas, minus expertos monebo, ne in errores impingant, crassius dormitasse Varignonium tum, quum in *Illustrationibus Analyseos Hospitalianae infinite-parvorum* de Calculo agens Exponentiali protulit esse (*)

$$L(x^x) = x; L(x^{x+P}) = x + S \frac{Pdx}{x}; L(x^x + ax^{x-1}) = x - S \frac{adx}{x^2 - ax},$$

qui Pseudo-Logarithmi sunt ita caute emendandi,

$$L(x^x) = xLx; L(x^{x+P}) = (x+P)Lx; L(x^x + ax^{x-1}) = \overline{x-1}.Lx + L(x+a).$$

Vitium quandoquidem latet in falsa Differentiola $x \cdot x^{x-1} dx$ Exponentialium x^x , qua Fluxione restituta iuxta numerum 341. systema ipsum Auctoris, tametsi multo, quam par fuisset, obliquius. praepositas castigationum Formulas ratas haberet. Profecto

$$L(x^x) = S \frac{d(x^x)}{x^x} = S \frac{x^x dx + x^x Lx dx}{x^x} = S(dx + Lx \cdot dx) = xLx;$$

$$L(x^{x+P}) = S \frac{d(x^{x+P})}{x^{x+P}} = S \frac{x^{x+P} Lx dx + x^{x+P} (\frac{x+P}{x}) dx}{x^{x+P}} = S dx (Lx + \frac{x+P}{x}) = (x+P)Lx;$$

$$L(x^x + ax^{x-1}) = S \frac{d(x^x + ax^{x-1})}{x^x + ax^{x-1}} = S(dx(Lx) + \frac{dx}{x}(x-1) + \frac{dx}{x+a}) = (x-1)Lx + L(x+a);$$

haeque

(*) Videatur Corollarium secundum Lemmatis pariter II.ⁱ ad pag.^{am} 110. praenotati Operis Posthumi anno MDCCXXV. in lucem editi.

haeque legitimae expressiones immane ab iis a Varignonio datis recedunt.

355. Formularum Differentialia ad Trigonometriam Hyperbolico-Circularem pertinentium determinanda obviam veniunt; nec novam penitus methodum venditare mihi est in animo, cum non pauca, nec levia, huiusce inveniantur specimina in recentissimis etiam Analyseos, quam vocant Leibnitianam, Disquisitionibus, uti in Bougainvillii, Simpsonii, Bezoutii, Iacquerii ac LeSeurii, Riccati &c: Elementorum Synopsis, sed plerumque innixa Regressui Exponentialium ad Logarithmos, et nuspiam ab unico Neutoniano Binomio, quod supra feliciter contigit, derivata. Quantumque vero argumenti futura sit novitas, pretiumve rerum interferendarum, quae in Calculi taedium quandoque moderandum huc illuc sparsa adiicientur, non iniuria verebar, ne, si Trigonometrica praetermissem, quibus ad instar elenchi heic locum concedam, Exercitatio ista mutila videretur.

TRIGONOMETRIA HYPERBOLICA.

$$\frac{C^x + C^{-x}}{2} = \text{Cofin. H. x} \text{ reddit } d(\text{Cofin. H. x}) = \frac{C^x dx - C^{-x} dx}{2} = (\text{Sin. H. x}) dx;$$

$$\frac{C^x - C^{-x}}{2} = \text{Sin. H. x} \text{ tribuit } d(\text{Sin. H. x}) = \frac{C^x dx + C^{-x} dx}{2} = (\text{Cofin. H. x}) dx;$$

$$\frac{C^{2x} - 1}{C^{2x} + 1} = \text{Tang. H. x} \text{ exhibet } d(\text{Tang. H. x}) = \frac{4C^{2x} dx}{(C^{2x} + 1)^2} = \frac{dx}{\left(\frac{C^x + C^{-x}}{2}\right)^2} = \frac{dx}{(\text{Cofin. H. x})^2}, \text{ vel}$$

$$d \text{Tang. H. x} = d\left(\frac{\text{Sin. H. x}}{\text{Cofin. H. x}}\right) = \frac{(\text{Cofin. H. x}) d \text{Sin. H. x} - (\text{Sin. H. x}) d \text{Cofin. H. x}}{(\text{Cofin. H. x})^2} = \frac{dx}{(\text{Cofin. H. x})^2}, \text{ exindeque}$$

$$d \text{Cotang. H. x} = d\left(\frac{1}{\text{Tang. H. x}}\right) = \frac{-d \text{Tang. H. x}}{(\text{Tang. H. x})^2}, \text{ five potius } = \frac{-dx}{(\text{Cofin. H. x})^2 (\text{Tang. H. x})^2} = \frac{-dx}{(\text{Sin. H. x})^2};$$

$$d \text{Sin. Verf. H. x} = d(\text{Cofin. H. x} - 1) = (\text{Sin. H. x}) dx.$$

Altero etiam modo eadem Differentialia obtineri possent, prouti selecta exempla demonstrent.

P p p p p

$d \text{Sin. H. x} =$

$$d \text{ Sin. H. } x = d \left(\frac{x}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \frac{x^7}{1.2.3.4.5.6.7} \&c. \right) = dx \left(1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} \&c. \right) = dx (\text{Cofin. H. } x);$$

$$d \text{ Cofin. H. } x = d \left(1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} \&c. \right) = dx \left(\frac{x}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \&c. \right) = dx (\text{Sin. H. } x);$$

$$d \text{ Tang. H. } x = d \left(\frac{\frac{x}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \frac{x^7}{1.2.3.4.5.6.7} \&c.}{1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} \&c.} \right) = \frac{dx}{\left(1 + \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} \&c. \right)^2} = \frac{dx}{(\text{Cofin. H. } x)^2};$$

et sic quoque ceteris subrogatis Seriebus. Quinimmo alia vice, diversoque itinere

$$d \text{ Sin. H. } x = d \left(\frac{\left(1 + \frac{x}{\infty} \right)^{\infty} - \left(1 - \frac{x}{\infty} \right)^{\infty}}{2} \right) = \frac{dx}{\infty} \left(\frac{\infty \left(1 + \frac{x}{\infty} \right)^{\infty-1} + \infty \left(1 - \frac{x}{\infty} \right)^{\infty-1}}{2} \right)$$

$$= dx \left(\frac{\left(1 + \frac{x}{\infty} \right)^{\infty} + \left(1 - \frac{x}{\infty} \right)^{\infty}}{2} \right) = dx (\text{Cofin. H. } x);$$

$$d \text{ Cofin. H. } x = d \left(\frac{\left(1 + \frac{x}{\infty} \right)^{\infty} + \left(1 - \frac{x}{\infty} \right)^{\infty}}{2} \right) = \frac{dx}{\infty} \left(\frac{\infty \left(1 + \frac{x}{\infty} \right)^{\infty-1} - \infty \left(1 - \frac{x}{\infty} \right)^{\infty-1}}{2} \right) = dx (\text{Sin. H. } x);$$

quod artificium Exponentialia Indice variabili praedita in Potestates Indice constante, tametsi Infinito, distinctas permutandi pluries iam vidimus fructu semper uberrimo opimum.

356. Dum Hyperbolae Apollonianae Aequilaterae Cofinus nuncupetur

z , evadet eius Sinus $= (z^2 - 1)^{\frac{1}{2}}$, unde profilit

$$C^x = z + \sqrt{z^2 - 1}, \text{ vel } x = L(z + \sqrt{z^2 - 1}); \text{ ac, propter } dx = \frac{d \text{ Cofin. H. } x}{\text{Sin. H. } x} =$$

$$\frac{dz}{\sqrt{z^2 - 1}}, \text{ erit } dx = dL(z + \sqrt{z^2 - 1}) = \frac{dz}{\sqrt{z^2 - 1}}; \text{ ideoque } S \frac{dz}{\sqrt{z^2 - 1}} = L(z +$$

$$\sqrt{z^2 - 1}), \text{ nec non, ob geminum Radicalis Signum, } S \frac{dz}{\pm \sqrt{z^2 - 1}} = L(z \pm \sqrt{z^2 - 1}),$$

cui

cui innitur canonicae expressioni universus Analystarum labor, qui ab Infinitesimali Calculo Realem hauserunt et Imaginariam Logometriam. Elicitur quoque

$$\int \frac{dz \sqrt{-1}}{\pm \sqrt{z^2 - 1}} = \sqrt{-1} \cdot L(z \pm \sqrt{z^2 - 1}), \text{ cuius recta interpretatio}$$

adparebit in Trigonometria Circulari. Negligitur, cum de Integratione qualibet heic sermo fit, Magnitudinis *Constantis* additio, quam semper implicitam velim Lectorum mentibus obversantem. Interea subiungere non gravabor, quod, vice Cofinus z substituto Recto-Sinu v , habebitur $z^2 = 1 + v^2$,

$$dz = \frac{v dv}{\sqrt{1+v^2}}, \text{ scilicet } \frac{dz}{\sqrt{z^2-1}} = \frac{v dv}{\sqrt{1+v^2} \cdot v} = \frac{dv}{\sqrt{1+v^2}} = dx. \text{ Ad haec, si Tan-}$$

gens Hyperbolae ipsius dicatur y , proveniet Cofinus $= \frac{1}{\pm \sqrt{1-y^2}}$, Sinusque

$$= \frac{y}{\pm \sqrt{1-y^2}}: \text{ quamobrem ab Aequatione } d \text{ Tang. H. } x = \frac{dx}{(\text{Cofin. H. } x)^2} \text{ di-}$$

manabit altera $dx = \frac{dy}{1-y^2}$, atque insuper $x = \int \frac{dy}{1-y^2}$, necnon $x \sqrt{-1}$

$$= \int \frac{dy \sqrt{-1}}{1-y^2}; \text{ et ista Formula congruit cum ea a Valmesleyo in Diatriba}$$

de Logarithmis Imaginariis praesertim adhibita. Sit demum Differentiola quaerenda Formularum eiusmodi, quae consequuntur, aliarumve non valde abfimilium in Infinitum.

$$dL(\text{Sin. H. } x) = \frac{dx (\text{Cofin. H. } x)}{\text{Sin. H. } x} = dx (\text{Cot. H. } x) = dx \left(\frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \frac{x^7}{4725} + \frac{2x^9}{93555} - \&c. \right);$$

$$dL(\text{Cofin. H. } x) = \frac{dx (\text{Sin. H. } x)}{\text{Cofin. H. } x} = dx (\text{Tang. H. } x) = dx \left(x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \frac{62x^9}{2835} - \&c. \right);$$

$$dL(\text{Tang. H. } x) = \frac{dx}{(\text{Cofin. H. } x)^2 (\text{Tang. H. } x)} = \frac{dx (\text{Cofin. H. } x)}{(\text{Cofin. H. } x)^2 (\text{Sin. H. } x)} = \frac{dx}{(\text{Cofin. H. } x) (\text{Sin. H. } x)};$$

$$dL(\text{Cot. H. } x) = \frac{-dx}{(\text{Sin. H. } x)^2 (\text{Cot. H. } x)} = \frac{-dx (\text{Sin. H. } x)}{(\text{Sin. H. } x)^2 (\text{Cofin. H. } x)} = \frac{-dx}{(\text{Cofin. H. } x) (\text{Sin. H. } x)};$$

$$d(\text{Cofin. H. } x)^n = n (\text{Cofin. H. } x)^{n-1} (\text{Sin. H. } x) dx;$$

$$d(\text{Sin.}$$

$$d(\text{Sin. H. } x)^z = (\text{Sin. H. } x)^z \cdot L(\text{Sin. H. } x) dz + (\text{Sin. H. } x)^{z-1} z (\text{Cofin. H. } x) dx;$$

$$d(\text{Tang. H. } x)^{(\text{Sin. H. } x)} = (\text{Tang. H. } x)^{(\text{Sin. H. } x)} L(\text{Tang. H. } x) \cdot (\text{Cofin. H. } x) dx +$$

$$(\text{Tang. H. } x)^{(\text{Sin. H. } x)-1} \frac{(\text{Sin. H. } x)}{(\text{Cofin. H. } x)^2} \cdot dx, \text{ ubi } dx \text{ generaliter designat Du-}$$

plam Differentiolam, vel Elementum Duplum nascens five evanescens illius Quadrilinei, aut Sectoris, ad quem referantur Trigonometricae Lineae, sci-

licet $d\left(\frac{V}{\frac{1}{2}}\right)$, aut $2dV$, si Specie alphabetica V Quadrilineum ipsum, seu

Sectorem indicare libuerit. Ex hisce principiis facile derivantur nonnullae Aequalitates elegantissimae, nec usu carentes, nimirum,

$$dL(\text{Sin. H. } x) \times dL(\text{Cofin. H. } x) = dx^2 (\text{Cot. H. } x) (\text{Tang. H. } x) = dx^2;$$

$$dL(\text{Tang. H. } x) = -dL(\text{Cot. H. } x), \text{ ac vicissim; unde } L(\text{Tang. H. } x) =$$

$$-L\left(\frac{1}{(\text{Tang. H. } x)}\right) = -L(\text{Cot. H. } x), \text{ idque directa etiam arguendi me-}$$

thodo constat;

$$dL(\text{Sin. H. } x) \times dL(\text{Tang. H. } x) = \frac{dx^2}{(\text{Sin. H. } x)^2};$$

$$dL(\text{Cofin. H. } x) \times dL(\text{Cot. H. } x) = \frac{-dx^2}{(\text{Cofin. H. } x)^2};$$

$$dL(\text{Sin. H. } x) \times dL(\text{Cofin. H. } x) + dL(\text{Sin. H. } x) \times dL(\text{Tang. H. } x) = (\text{Cot. H. } x)^2 dx^2;$$

$$dL(\text{Sin. H. } x) \times dL(\text{Cofin. H. } x) + dL(\text{Cofin. H. } x) \times dL(\text{Cot. H. } x) = (\text{Tang. H. } x)^2 dx^2;$$

$$S \frac{dx}{\text{Sin. H. } x} = S \frac{\frac{1}{2} dx}{(\text{Cofin. H. } \frac{x}{2})(\text{Sin. H. } \frac{x}{2})} = S dL(\text{Tang. H. } \frac{x}{2}) = L(\text{Tang. H. } \frac{1}{2} x); \text{ et}$$

$$S \frac{(\text{Cofin. H. } x) dx}{\text{Sin. H. } x} = S dL(\text{Sin. H. } x) = L(\text{Sin. H. } x), \text{ veluti habet Seiou-}$$

rius (*) in Circulo nosque ad Hyperbolen nunc primum eodem ritu transferimus.

357. Ni-

(*) Consuli possunt pag.^{ae} 80. 85. praelaudati Operis Cometici, ac subsequens numerus 363.

357. Nihil vero ambigendum de assumpta aequalitate expressionum

$$\text{Sin. H. } x = 2 \left(\text{Cofin. H. } \frac{x}{2} \right) \left(\text{Sin. H. } \frac{x}{2} \right), \text{ a qua penultima Formulae ve-}$$

$$\text{ritas pendet: namque } 2 \left(\text{Cofin. H. } \frac{x}{2} \right) \left(\text{Sin. H. } \frac{x}{2} \right) =$$

$$2 \left(\frac{C^{\frac{x}{2}} + C^{-\frac{x}{2}}}{2} \right) \left(\frac{C^{\frac{x}{2}} - C^{-\frac{x}{2}}}{2} \right) = \frac{C^x - C^{-x}}{2} = \text{Sin. H. } x, \text{ quemadmo-}$$

dum Theoremata docent in num°. 202. fusius exposita. Dum haec recensentur, summam facere rei Trigonometricae iacturam putarem Differentias omittens Finitas, quae tanta facilitate, ac veluti sponte sua ex antea positis effluunt. Revocatis itaque ad Exponentialium formam Trigonometricis Lineis ultro obtinentur

$$D \text{ Sin. H. } x = \frac{DC^x - DC^{-x}}{2} = \left(\frac{C^x}{2} \right) \left(\frac{(Dx)}{1} + \frac{(Dx)^2}{1.2} + \frac{(Dx)^3}{1.2.3} + \frac{(Dx)^4}{1.2.3.4} + \frac{(Dx)^5}{1.2.3.4.5} + \&c: \right)$$

$$\text{ex num°. 342.} \quad + \left(\frac{C^{-x}}{2} \right) \left(\frac{(Dx)}{1} - \frac{(Dx)^2}{1.2} + \frac{(Dx)^3}{1.2.3} - \frac{(Dx)^4}{1.2.3.4} + \frac{(Dx)^5}{1.2.3.4.5} - \&c: \right)$$

, ac

$$D \text{ Cofin. H. } x = \frac{DC^x + DC^{-x}}{2} = \left(\frac{C^x}{2} \right) \left(\frac{(Dx)}{1} + \frac{(Dx)^2}{1.2} + \frac{(Dx)^3}{1.2.3} + \frac{(Dx)^4}{1.2.3.4} + \frac{(Dx)^5}{1.2.3.4.5} + \&c: \right)$$

$$- \left(\frac{C^{-x}}{2} \right) \left(\frac{(Dx)}{1} - \frac{(Dx)^2}{1.2} + \frac{(Dx)^3}{1.2.3} - \frac{(Dx)^4}{1.2.3.4} + \frac{(Dx)^5}{1.2.3.4.5} - \&c: \right),$$

quae Elementa praecognita Differentiis etiam Finitis Tangentium, Cotangentium &c: in Sinus, Cofinusque semper resolubilium determinandis inferviunt. Ii ergo, quibus curae sit integras perscrutari Infinitesimas Differentias, ita se gerent in hoc Problemate definiendo.

$$d(\text{Sin. H. } x) = \left(\frac{\text{Cofin. H. } x + \text{Sin. H. } x}{2} \right) \left(dx + \frac{dx^2}{2} + \frac{dx^3}{2.3} + \frac{dx^4}{2.3.4} + \frac{dx^5}{2.3.4.5} + \frac{dx^6}{2.3.4.5.6} + \frac{dx^7}{2.3.4.5.6.7} + \&c: \right)$$

$$+ \left(\frac{\text{Cofin. H. } x - \text{Sin. H. } x}{2} \right) \left(dx - \frac{dx^2}{2} + \frac{dx^3}{2.3} - \frac{dx^4}{2.3.4} + \frac{dx^5}{2.3.4.5} - \frac{dx^6}{2.3.4.5.6} + \frac{dx^7}{2.3.4.5.6.7} + \&c: \right),$$

Q q q q q

d (Cofin.

$$d(\text{Cofin. H. x}) = \left(\frac{\text{Cofin. H. x} + \text{Sin. H. x}}{2} \right) \left(dx + \frac{dx^2}{2} + \frac{dx^3}{2.3} + \frac{dx^4}{2.3.4} + \frac{dx^5}{2.3.4.5} + \frac{dx^6}{2.3.4.5.6} + \frac{dx^7}{2.3.4.5.6.7} \&c: \right) \\ - \left(\frac{\text{Cofin. H. x} - \text{Sin. H. x}}{2} \right) \left(dx - \frac{dx^2}{2} + \frac{dx^3}{2.3} - \frac{dx^4}{2.3.4} + \frac{dx^5}{2.3.4.5} - \frac{dx^6}{2.3.4.5.6} + \frac{dx^7}{2.3.4.5.6.7} \&c: \right).$$

Observandum paullisper est, quod neglectis, uti Leibnitianis mos fuerat, superiorum Ordinum Differentialibus redeat prima Fluxio superius reperta, nempe tam $d(\text{Sin. H. x}) = (\text{Cofin. H. x}) dx$, quam $d(\text{Cofin. H. x}) = (\text{Sin. H. x}) dx$: nec difficilis inventu patet altera forma oecumenicae ipsius Differentiationis

$$d(\text{Sin. H. x}) = (\text{Cofin. H. x}) \left(dx + \frac{dx^3}{2.3} + \frac{dx^5}{2.3.4.5} + \frac{dx^7}{2.3.4.5.6.7} \&c: \right) + \\ (\text{Sin. H. x}) \left(\frac{dx^2}{2} + \frac{dx^4}{2.3.4} + \frac{dx^6}{2.3.4.5.6} \&c: \right),$$

$$d(\text{Cofin. H. x}) = (\text{Sin. H. x}) \left(dx + \frac{dx^3}{2.3} + \frac{dx^5}{2.3.4.5} + \frac{dx^7}{2.3.4.5.6.7} \&c: \right) + \\ (\text{Cofin. H. x}) \left(\frac{dx^2}{2} + \frac{dx^4}{2.3.4} + \frac{dx^6}{2.3.4.5.6} \&c: \right), \text{ eademque ratione}$$

$$D(\text{Sin. H. x}) = (\text{Cofin. H. x}) \left(Dx + \frac{Dx^3}{2.3} + \frac{Dx^5}{2.3.4.5} + \frac{Dx^7}{2.3.4.5.6.7} \&c: \right) + \\ (\text{Sin. H. x}) \left(\frac{Dx^2}{2} + \frac{Dx^4}{2.3.4} + \frac{Dx^6}{2.3.4.5.6} \&c: \right),$$

$$D(\text{Cofin. H. x}) = (\text{Sin. H. x}) \left(Dx + \frac{Dx^3}{2.3} + \frac{Dx^5}{2.3.4.5} + \frac{Dx^7}{2.3.4.5.6.7} \&c: \right) + (\text{Cofin. H. x}) \left(\frac{Dx^2}{2} + \frac{Dx^4}{2.3.4} + \frac{Dx^6}{2.3.4.5.6} \&c: \right),$$

vel potius

$$D(\text{Sin. H. x}) = \left(\frac{\text{Cofin. H. x} + \text{Sin. H. x}}{2} \right) \left(Dx + \frac{Dx^2}{2} + \frac{Dx^3}{2.3} + \frac{Dx^4}{2.3.4} + \frac{Dx^5}{2.3.4.5} + \frac{Dx^6}{2.3.4.5.6} + \frac{Dx^7}{2.3.4.5.6.7} \&c: \right) \\ + \left(\frac{\text{Cofin. H. x} - \text{Sin. H. x}}{2} \right) \left(Dx - \frac{Dx^2}{2} + \frac{Dx^3}{2.3} - \frac{Dx^4}{2.3.4} + \frac{Dx^5}{2.3.4.5} - \frac{Dx^6}{2.3.4.5.6} + \frac{Dx^7}{2.3.4.5.6.7} \&c: \right),$$

$$D(\text{Cofin. H. x}) = \left(\frac{\text{Cofin. H. x} + \text{Sin. H. x}}{2} \right) \left(Dx + \frac{Dx^2}{2} + \frac{Dx^3}{2.3} + \frac{Dx^4}{2.3.4} + \frac{Dx^5}{2.3.4.5} + \frac{Dx^6}{2.3.4.5.6} + \frac{Dx^7}{2.3.4.5.6.7} \&c: \right) \\ - \left(\frac{\text{Cofin. H. x} - \text{Sin. H. x}}{2} \right) \left(Dx - \frac{Dx^2}{2} + \frac{Dx^3}{2.3} - \frac{Dx^4}{2.3.4} + \frac{Dx^5}{2.3.4.5} - \frac{Dx^6}{2.3.4.5.6} + \frac{Dx^7}{2.3.4.5.6.7} \&c: \right).$$

Prior

Prior autem Formula magis idonea videtur elegantissimo comprobando Theoremati, cuius exstat mentio in num^{is}. 338. ac 343.; hypothefi enim Constantis $\tau \delta dx$, videlicet parvulae Lineae Arcui Infinitesimo Circulari analogae, renovata, nanciscemur, post

	$d(\text{Sin. H. } x) = (\text{Cofin. H. } x) dx$,	et	$d(\text{Cofin. H. } x) = (\text{Sin. H. } x) dx$, subiecta
Differentio- differentialia	$d^2(\text{Sin. H. } x) = (\text{Sin. H. } x) dx^2$		$d^2(\text{Cofin. H. } x) = (\text{Cofin. H. } x) dx^2$
	$d^3(\text{Sin. H. } x) = (\text{Cofin. H. } x) dx^3$		$d^3(\text{Cofin. H. } x) = (\text{Sin. H. } x) dx^3$
	$d^4(\text{Sin. H. } x) = (\text{Sin. H. } x) dx^4$		$d^4(\text{Cofin. H. } x) = (\text{Cofin. H. } x) dx^4$
	$d^5(\text{Sin. H. } x) = (\text{Cofin. H. } x) dx^5$		$d^5(\text{Cofin. H. } x) = (\text{Sin. H. } x) dx^5$
	-----		-----
	$d^{2n-1}(\text{Sin. H. } x) = (\text{Cofin. H. } x) dx^{2n-1}$		$d^{2n-1}(\text{Cofin. H. } x) = (\text{Sin. H. } x) dx^{2n-1}$
	$d^{2n}(\text{Sin. H. } x) = (\text{Sin. H. } x) dx^{2n}$		$d^{2n}(\text{Cofin. H. } x) = (\text{Cofin. H. } x) dx^{2n}$
	-----		-----

nusquam ob Transcendentem Functionem abrupta Fluxionum Serie. Exinde, suffectis aequipollentibus, fit

$$D(\text{Sin. H. } x) = \frac{d(\text{Sin. H. } x) Dx}{dx} + \frac{d^2(\text{Sin. H. } x) Dx^2}{2dx^2} + \frac{d^3(\text{Sin. H. } x) Dx^3}{2 \cdot 3 \cdot dx^3} + \frac{d^4(\text{Sin. H. } x) Dx^4}{2 \cdot 3 \cdot 4 \cdot dx^4} + \&c.;$$

$$D(\text{Cofin. H. } x) = \frac{d(\text{Cofin. H. } x) Dx}{dx} + \frac{d^2(\text{Cofin. H. } x) Dx^2}{2 \cdot dx^2} + \frac{d^3(\text{Cofin. H. } x) Dx^3}{2 \cdot 3 \cdot dx^3} + \frac{d^4(\text{Cofin. H. } x) Dx^4}{2 \cdot 3 \cdot 4 \cdot dx^4} + \&c.;$$

ad praefati nimirum Theorematis canonem accommodata alterutra Aequatio.

358. Quinimmo Series illae numero terminorum Infinitae, quarum adiumento primi Gradus Fluxiones, nulloque labore, adquisivimus, parient cuiuslibet Ordinis Differentialia Hyperbolicorum Cofinum, ac Sinuum. Et re quidem vera, cum ex num^o. 355. innotescant

$$d(\text{Cofin. H. } x) = d\left(1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 3 \cdot 4} + \frac{x^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{x^8}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \&c.\right) = dx\left(x + \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{x^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \&c.\right),$$

$$d(\text{Sin. H. } x) = d\left(x + \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{x^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{x^9}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \&c.\right) = dx\left(1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 3 \cdot 4} + \frac{x^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{x^8}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \&c.\right),$$

erunt

erunt etiam

$$d^2(\text{Cofin. H. } x) = dx^2 \left(1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 3 \cdot 4} + \frac{x^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \&c: \right) = dx^2 (\text{Cofin. H. } x)$$

$$d^3(\text{Cofin. H. } x) = dx^3 \left(x + \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} \&c: \right) = dx^3 (\text{Sin. H. } x)$$

$$d^4(\text{Cofin. H. } x) = dx^4 \left(1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 3 \cdot 4} \&c: \right) = dx^4 (\text{Cofin. H. } x)$$

&c:

&c:

&c:

$$d^2(\text{Sin. H. } x) = dx^2 \left(x + \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{x^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \&c: \right) = dx^2 (\text{Sin. H. } x)$$

$$d^3(\text{Sin. H. } x) = dx^3 \left(1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 3 \cdot 4} + \frac{x^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \&c: \right) = dx^3 (\text{Cofin. H. } x)$$

$$d^4(\text{Sin. H. } x) = dx^4 \left(x + \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} \&c: \right) = dx^4 (\text{Sin. H. } x)$$

&c:

&c:

&c:

;

quibuscum universis Fluxionum - Fluxionibus superiores congruunt ad unguem.

359. Pari gressu procedunt quae Circulo competunt; ac, tametsi venustas defuit, atque elegantia, quum analogae Formulae eodem ordine servato repeti debeant, nihilo tamen minus, ut magis constet de ipsarum similitudine et harmonia, fere omnia iisdem verbis, ac methodo nunc recensebo.

TRIGONOMETRIA CIRCULARIS.

$$\frac{C^{x\sqrt{-1}} + C^{-x\sqrt{-1}}}{2} = \text{Cofin. C. } x \text{ dat } d(\text{Cofin. C. } x) = \frac{(C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}) dx\sqrt{-1}}{2} =$$

$$- \left(\frac{C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}}{2\sqrt{-1}} \right) dx = - (\text{Sin. C. } x) dx;$$

$$\frac{C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}}{2\sqrt{-1}}$$

$$\frac{C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}}{2\sqrt{-1}} = \text{Sin. C. x ducit ad } d(\text{Sin. C. x}) = \left(\frac{C^{x\sqrt{-1}} + C^{-x\sqrt{-1}}}{2\sqrt{-1}} \right) dx \sqrt{-1} = (\text{Cofin. C. x}) dx;$$

$$\frac{C^{2x\sqrt{-1}} - 1}{\sqrt{-1}(C^{2x\sqrt{-1}} + 1)} = \text{Tang. C. x affert } d(\text{Tang. C. x}) = \frac{-4(C^{2x\sqrt{-1}}) dx}{-(C^{2x\sqrt{-1}} + 1)^2} = \frac{dx}{\left(\frac{C^{x\sqrt{-1}} + C^{-x\sqrt{-1}}}{2} \right)^2} = \frac{dx}{(\text{Cofin. C. x})^2};$$

$$\frac{2C^{x\sqrt{-1}}}{C^{2x\sqrt{-1}} + 1} = \text{Sec. C. x elargitur } d(\text{Sec. C. x}) = - \frac{2(C^{3x\sqrt{-1}} - C^{x\sqrt{-1}}) dx \sqrt{-1}}{(C^{2x\sqrt{-1}} + 1)^2} =$$

$$\frac{\left(\frac{C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}}{2\sqrt{-1}} \right) dx}{\left(\frac{C^{x\sqrt{-1}} + C^{-x\sqrt{-1}}}{2} \right)^2} = \frac{(\text{Sin. C. x}) dx}{(\text{Cofin. C. x})^2};$$

five diverso tramite

$$d(\text{Tang. C. x}) = d\left(\frac{\text{Sin. C. x}}{\text{Cofin. C. x}} \right) = \frac{\text{Cofin. C. x. } d(\text{Sin. C. x}) - \text{Sin. C. x. } d(\text{Cofin. C. x})}{(\text{Cofin. C. x})^2} =$$

$$\frac{((\text{Cofin. C. x})^2 + (\text{Sin. C. x})^2) dx}{(\text{Cofin. C. x})^2} = \frac{dx}{(\text{Cofin. C. x})^2};$$

$$d(\text{Sec. C. x}) = d\left(\frac{1}{\text{Cofin. C. x}} \right) = \frac{(\text{Sin. C. x}) dx}{(\text{Cofin. C. x})^2} = \frac{(\text{Tang. C. x}) dx}{\text{Cofin. C. x}} = (\text{Tang. C. x})(\text{Sec. C. x}) dx;$$

$$d(\text{Cotang. C. x}) = d\left(\frac{1}{\text{Tang. C. x}} \right) = \frac{-dx}{(\text{Cofin. C. x})^2 (\text{Tang. C. x})^2} = \frac{-dx}{(\text{Sin. C. x})^2} = \frac{-(\text{Cotang. C. x})^2 dx}{(\text{Cofin. C. x})^2};$$

$$d(\text{Cofec. C. x}) = d\left(\frac{1}{\text{Sin. C. x}} \right) = \frac{-(\text{Cofin. C. x}) dx}{(\text{Sin. C. x})^2} = \frac{-(\text{Cotang. C. x}) dx}{\text{Sin. C. x}} = -(\text{Cot. C. x})(\text{Cofec. C. x}) dx;$$

R r r r r

d (Sin.

$$d(\text{Sin. Vers. C. } x) = d(1 - \text{Cofin. C. } x) = (\text{Sin. C. } x) dx;$$

$$d(\text{Chor. C. } x) = d\left(2 \text{Sin. C. } \frac{x}{2}\right) = 2 \left(\text{Cofin. C. } \frac{x}{2}\right) \frac{dx}{2} = (\text{Cofin. C. } \frac{x}{2}) dx;$$

$$d(\text{Chor. Compl. C. } x) = d\left(2 \text{Cofin. C. } \frac{x}{2}\right) = -2 \left(\text{Sin. C. } \frac{x}{2}\right) \frac{dx}{2} = -(\text{Sin. C. } \frac{x}{2}) dx.$$

Haec omnia variis modis, semperque unifona, confirmantur: est enim

$$d(\text{Sin. C. } x) = d\left(x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.5.6.7} + \&c:\right) = dx \times$$

$$\left(1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \&c:\right) = dx (\text{Cofin. C. } x);$$

$$d(\text{Cofin. C. } x) = d\left(1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \&c:\right) = -dx \times$$

$$\left(x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.5.6.7} + \&c:\right) = -dx (\text{Sin. C. } x);$$

$$d(\text{Tang. C. } x) = d\left(\frac{x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.5.6.7} + \&c:}{1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \&c:}\right) =$$

$$\frac{dx}{\left(1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \&c:\right)^2} = \frac{dx}{(\text{Cofin. C. } x)^2};$$

$$d(\text{Sec. C. } x) = d\left(\frac{1}{1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \&c:}\right) =$$

$$\frac{-dx \left(-x + \frac{x^3}{2.3} - \frac{x^5}{2.3.4.5} + \frac{x^7}{2.3.4.5.6.7} - \&c:\right)}{\left(1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \&c:\right)^2} = \frac{dx (\text{Sin. C. } x)}{(\text{Cofin. C. } x)^2};$$

vel denuo, si mavis,

$$d(\text{Sin. C. } x) = d\left[\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^\infty - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^\infty}{2\sqrt{-1}}\right] = \frac{dx\sqrt{-1}}{\infty} \cdot \infty \times$$

$$\left[\frac{(1 + \frac{x\sqrt{-1}}{\infty})^\infty}{2\sqrt{-1}} - \frac{(1 - \frac{x\sqrt{-1}}{\infty})^\infty}{2\sqrt{-1}}\right]$$

$$\left[\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1} + \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1}}{2\sqrt{-1}} \right] = dx (\text{Cofin. C. } x);$$

$$d(\text{Cofin. C. } x) = d \left[\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} + \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{2} \right] = + \frac{dx\sqrt{-1}}{\infty} \cdot \infty \times$$

$$\left[\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1} - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1}}{2} \right] = - dx (\text{Sin. C. } x),$$

$$\text{quia } + dx\sqrt{-1} \left[\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1} - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1}}{2} \right] = -$$

$$dx \left[\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{2\sqrt{-1}} \right], \text{ prouti Aequatio notissima } \sqrt{-1}$$

$$= \frac{-1}{\sqrt{-1}} \text{ meridiana luce clarius ostendit.}$$

360. Primis Differentiolis repertis Differentio-Differentialia Sinuum, Cofinumque statim oculis subiiciam; hicque magis natus ordo fortasse videbitur. Hac spe fretus geminam sequor numerorum 357. ac 358. methodum ad incoeptam Theoricen prosequendam.

$$d^2(\text{Sin. C. } x) = d \left(1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \&c. \right) dx = - \left(x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.5.6.7} + \&c. \right) dx^2 = - (\text{Sin. C. } x) dx^2,$$

$$d^3(\text{Sin. C. } x) = d \left(-x + \frac{x^3}{2.3} - \frac{x^5}{2.3.4.5} + \frac{x^7}{2.3.4.5.6.7} \&c. \right) dx^2 = - \left(1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} \&c. \right) dx^3 = - (\text{Cofin. C. } x) dx^3,$$

$$d^4(\text{Sin. C. } x) = d \left(-1 + \frac{x^2}{2} - \frac{x^4}{2.3.4} + \frac{x^6}{2.3.4.5.6} \&c. \right) dx^3 = \left(x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \&c. \right) dx^4 = (\text{Sin. C. } x) dx^4,$$

&c: fine Limite;

$$d^2(\text{Cofin. C. } x) = d \left(-x + \frac{x^3}{2.3} - \frac{x^5}{2.3.4.5} + \frac{x^7}{2.3.4.5.6.7} \&c. \right) dx = \left(-1 + \frac{x^2}{2} - \frac{x^4}{2.3.4} + \frac{x^6}{2.3.4.5.6} \&c. \right) dx^2 = - (\text{Cofin. C. } x) dx^2,$$

$d^3(\text{Cofin. C. } x) =$

$$d^3(\text{Cofin. C. x}) = d\left(-1 + \frac{x^2}{2} - \frac{x^4}{2 \cdot 3 \cdot 4} + \frac{x^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \&c:\right) dx^2 = \left(x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} \&c:\right) dx^3 = (\text{Sin. C. x}) dx^3$$

$$d^4(\text{Cofin. C. x}) = d\left(x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} \&c:\right) dx^3 = \left(1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 3 \cdot 4} \&c:\right) dx^4 = (\text{Cofin. C. x}) dx^4,$$

$$d^5(\text{Cofin. C. x}) = d\left(1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 3 \cdot 4} \&c:\right) dx^4 = -\left(x - \frac{x^3}{2 \cdot 3} + \&c:\right) dx^5 = -(\text{Sin. C. x}) dx^5,$$

&c: in Infinitum, propter Functionis Transcendentis naturam. Sed et absque Serierum auxilio provenient

$$d^2(\text{Sin. C. x}) = d((\text{Cofin. C. x}) dx) = -(\text{Sin. C. x}) dx^2,$$

$$d^3(\text{Sin. C. x}) = d(-(\text{Sin. C. x}) dx^2) = -(\text{Cofin. C. x}) dx^3,$$

$$d^4(\text{Sin. C. x}) = d(-(\text{Cofin. C. x}) dx^3) = (\text{Sin. C. x}) dx^4,$$

$$d^5(\text{Sin. C. x}) = d((\text{Sin. C. x}) dx^4) = (\text{Cofin. C. x}) dx^5,$$

$$d^{2n}(\text{Sin. C. x}) = \pm (\text{Sin. C. x}) dx^{2n}, \text{ ubi Signum valet superius, si } n \text{ Par, inferius dum Impar,}$$

$$d^{2n+1}(\text{Sin. C. x}) = \pm (\text{Cofin. C. x}) dx^{2n+1}, \text{ eadem, quae supra est, Signorum dispositione,}$$

Praeterea

$$d^2(\text{Cofin. C. x}) = d(-(\text{Sin. C. x}) dx) = -(\text{Cofin. C. x}) dx^2,$$

$$d^3(\text{Cofin. C. x}) = d(-(\text{Cofin. C. x}) dx^2) = (\text{Sin. C. x}) dx^3,$$

$$d^4(\text{Cofin. C. x}) = d((\text{Sin. C. x}) dx^3) = (\text{Cofin. C. x}) dx^4,$$

$$d^5(\text{Cofin. C. x}) = d((\text{Cofin. C. x}) dx^4) = (-\text{Sin. C. x}) dx^5,$$

$$d^{2n}(\text{Cofin. C. x}) = \pm (\text{Cofin. C. x}) dx^{2n}, \text{ selecto superiore Signo dum } 2n \text{ per Quaternarium dividatur; alias inferiore,}$$

$$d^{2n+1}$$

$d^{2n+1}(\text{Cofin. C. } x) = \pm (\text{Sin. C. } x) dx^{2n+1}$ ex adverso superiore Signo posito
quando $2n$ divisionem per Quaternarium non admittat, atque vicissim.

Non tam Signorum, quam Cosinuum alternantium, et Sinuum perpetuo remeans Periodus Fluxionum-Fluxiones harumce Magnitudinum Transcendentium afficiens perinfignem attentionem meretur.

361. Simplex hinc transitus patet ad Differentias Finitas determinandas ipsarummet Linearum, quae in Trigonometria Circulari maximo usui sunt, nomenque habent antonomasticum Cosinuum, et Sinuum. Nec valde adlaborandum foret in aliis Lineis Trigonometricis, eoquod istae ex primis existant, et idcirco peculiare non postulent artificium. Revera, subrogato vice x in num°. 357. ac sequentibus $x\sqrt{-1}$, vel Arcu Circuli Imaginario, emerget

$$D(\text{Sin. C. } x) = D\left(\frac{C^{x\sqrt{-1}} - C^{-x\sqrt{-1}}}{2\sqrt{-1}}\right)$$

$$= \left(\frac{C^{x\sqrt{-1}}}{2\sqrt{-1}}\right) \left(\frac{(Dx)\sqrt{-1}}{1} - \frac{(Dx)^2}{1.2} - \frac{(Dx)^3\sqrt{-1}}{1.2.3} + \frac{(Dx)^4}{1.2.3.4} + \frac{(Dx)^5\sqrt{-1}}{1.2.3.4.5} - \&c.\right)$$

$$+ \left(\frac{C^{-x\sqrt{-1}}}{2\sqrt{-1}}\right) \left(\frac{(Dx)\sqrt{-1}}{1} + \frac{(Dx)^2}{1.2} - \frac{(Dx)^3\sqrt{-1}}{1.2.3} - \frac{(Dx)^4}{1.2.3.4} + \frac{(Dx)^5\sqrt{-1}}{1.2.3.4.5} + \&c.\right),$$

quae expressio in eam alterius formae facile vertitur

$$D(\text{Sin. C. } x) = \left(\frac{\text{Cofin. C. } x + (\text{Sin. C. } x)\sqrt{-1}}{2}\right) \times$$

$$\left((Dx) + \frac{(Dx)^2\sqrt{-1}}{2} - \frac{(Dx)^3}{2.3} - \frac{(Dx)^4\sqrt{-1}}{2.3.4} + \frac{(Dx)^5}{2.3.4.5} + \frac{(Dx)^6\sqrt{-1}}{2.3.4.5.6} - \&c.\right)$$

$$+ \left(\frac{\text{Cofin. C. } x}{2}\right)$$

S s s s s

$$+ \left(\frac{\text{Cofin. C. } x - (\text{Sin. C. } x) \sqrt{-1}}{2} \right) \times$$

$$\left((Dx) - \frac{(Dx)^2 \sqrt{-1}}{2} - \frac{(Dx)^3}{2.3} + \frac{(Dx)^4 \sqrt{-1}}{2.3.4} + \frac{(Dx)^5}{2.3.4.5} - \frac{(Dx)^6 \sqrt{-1}}{2.3.4.5.6} - \&c. \right),$$

aut, multiplicatione confecta, Imaginariisque omnibus eliminatis,

$$D(\text{Sin. C. } x) = (\text{Cofin. C. } x) \left(Dx - \frac{Dx^3}{2.3} + \frac{Dx^5}{2.3.4.5} - \frac{Dx^7}{2.3.4.5.6.7} + \&c. \right) - (\text{Sin. C. } x) \left(\frac{Dx^2}{2} - \frac{Dx^4}{2.3.4} + \frac{Dx^6}{2.3.4.5.6} - \frac{Dx^8}{2.3.4.5.6.7.8} + \&c. \right),$$

nimirum

$$D(\text{Sin. C. } x) = \frac{d(\text{Sin. C. } x)}{dx} Dx + \frac{d^2(\text{Sin. C. } x)}{2 \cdot dx^2} Dx^2 + \frac{d^3(\text{Sin. C. } x)}{2.3 \cdot dx^3} Dx^3 + \frac{d^4(\text{Sin. C. } x)}{2.3.4 \cdot dx^4} Dx^4 + \frac{d^5(\text{Sin. C. } x)}{2.3.4.5 \cdot dx^5} Dx^5 + \&c.$$

ad normam Theorematis in re Infinitesimali, et in sublimiori Finitorum Analyfi perquam amplissimae utilitatis. Universa, quae de Sinubus praedicantur, aequae valent ex numero 359. de Chordis, Arcuumve Subtensis; adeo, ut inutile sit distinctam adire Circularium Chordarum mentionem. Eadem dicas de Chordis Arcuum-Complementi, quum Cofinum Semi-Arcuum duplis coincidant. Interea fructum uberrimum colligemus si Differentiae Infinitesimae Finitis substituantur: quod enim mutilum haecenus fuerat, plenissime et omnibus numeris absolvetur. Profecto

$$d(\text{Sin. C. } x) = d(\text{Sin. C. } x) + \frac{d^2(\text{Sin. C. } x)}{2} + \frac{d^3(\text{Sin. C. } x)}{2.3} + \frac{d^4(\text{Sin. C. } x)}{2.3.4} + \frac{d^5(\text{Sin. C. } x)}{2.3.4.5} + \frac{d^6(\text{Sin. C. } x)}{2.3.4.5.6} + \&c.,$$

sive, quemadmodum in Hyperbolicorum Sinuum tractatione,

$$d(\text{Sin. C. } x) = (\text{Cofin. C. } x) \left(dx - \frac{dx^3}{2.3} + \frac{dx^5}{2.3.4.5} - \frac{dx^7}{2.3.4.5.6.7} + \&c. \right) - (\text{Sin. C. } x) \left(\frac{dx^2}{2} - \frac{dx^4}{2.3.4} + \frac{dx^6}{2.3.4.5.6} - \frac{dx^8}{2.3.4.5.6.7.8} + \&c. \right);$$

unde statim elucet, primo inventum Differentiale $d(\text{Sin. C. } x) = dx (\text{Cofin. C. } x)$ nonnisi partem esse seu natu priorem ex Infinitis numero terminis Fluxionis integrae generalioris, quae Differentiis omnium Ordinum cumulatis in Infinitum progreditur. Nihil etiam vetat isthac potius Formula aequipollente

$$d(\text{Sin. C. } x)$$

$$d(\text{Sin. C. } x) = \frac{(\text{Cofin. C. } x + (\text{Sin. C. } x)\sqrt{-1})}{2} \left(dx + \frac{dx^2\sqrt{-1}}{2} - \frac{dx^3}{2.3} - \frac{dx^4\sqrt{-1}}{2.3.4} + \frac{dx^5}{2.3.4.5} + \frac{dx^6\sqrt{-1}}{2.3.4.5.6} - \&c. \right) \\ + \frac{(\text{Cofin. C. } x - (\text{Sin. C. } x)\sqrt{-1})}{2} \left(dx - \frac{dx^2\sqrt{-1}}{2} - \frac{dx^3}{2.3} + \frac{dx^4\sqrt{-1}}{2.3.4} + \frac{dx^5}{2.3.4.5} - \frac{dx^6\sqrt{-1}}{2.3.4.5.6} - \&c. \right)$$

Differentiale ipsum absolutissimum concinnare, non substantia, sed forma tantummodo Imaginarium, ideoque praeterito minus eleganter explicitum, quia non satis, ut decet et potest, ad limam cotemve ab Imaginariorum rubigine perpolitum.

362. Consimili pacto obtinentur Aequationes Finito - Differentiales

$$D(\text{Cofin. C. } x) = D \left(\frac{C^{x\sqrt{-1}} + C^{-x\sqrt{-1}}}{2} \right) = \left(\frac{C^{x\sqrt{-1}}}{2} \right) \left(\frac{(Dx)\sqrt{-1}}{1} - \frac{(Dx)^2}{1.2} - \frac{(Dx)^3\sqrt{-1}}{1.2.3} + \frac{(Dx)^4}{1.2.3.4} + \frac{(Dx)^5\sqrt{-1}}{1.2.3.4.5} - \&c. \right) \\ + \left(\frac{C^{-x\sqrt{-1}}}{2} \right) \left(-\frac{(Dx)\sqrt{-1}}{1} - \frac{(Dx)^2}{1.2} + \frac{(Dx)^3\sqrt{-1}}{1.2.3} + \frac{(Dx)^4}{1.2.3.4} - \frac{(Dx)^5\sqrt{-1}}{1.2.3.4.5} - \&c. \right),$$

scilicet

$$D(\text{Cofin. C. } x) = \left(\frac{\text{Cofin. C. } x + (\text{Sin. C. } x)\sqrt{-1}}{2} \right) \left((Dx)\sqrt{-1} - \frac{(Dx)^2}{2} - \frac{(Dx)^3\sqrt{-1}}{2.3} + \frac{(Dx)^4}{2.3.4} + \frac{(Dx)^5\sqrt{-1}}{2.3.4.5} - \frac{(Dx)^6}{2.3.4.5.6} \&c. \right) \\ + \left(\frac{-\text{Cofin. C. } x + (\text{Sin. C. } x)\sqrt{-1}}{2} \right) \left((Dx)\sqrt{-1} + \frac{(Dx)^2}{2} - \frac{(Dx)^3\sqrt{-1}}{2.3} - \frac{(Dx)^4}{2.3.4} + \frac{(Dx)^5\sqrt{-1}}{2.3.4.5} + \frac{(Dx)^6}{2.3.4.5.6} \&c. \right),$$

aut clarius expunctis Imaginariis

$$D(\text{Cofin. C. } x) = -(\text{Sin. C. } x) \left(Dx - \frac{Dx^3}{2.3} + \frac{Dx^5}{2.3.4.5} - \frac{Dx^7}{2.3.4.5.6.7} \&c. \right) - (\text{Cofin. C. } x) \left(\frac{Dx^2}{2} - \frac{Dx^4}{2.3.4} + \frac{Dx^6}{2.3.4.5.6} - \frac{Dx^8}{2.3.4.5.6.7.8} \&c. \right) \\ = \frac{d(\text{Cofin. C. } x)}{dx} Dx + \frac{d^2(\text{Cofin. C. } x)}{2 \cdot dx^2} Dx^2 + \frac{d^3(\text{Cofin. C. } x)}{2.3 \cdot dx^3} Dx^3 + \frac{d^4(\text{Cofin. C. } x)}{2.3.4 \cdot dx^4} Dx^4 + \frac{d^5(\text{Cofin. C. } x)}{2.3.4.5 \cdot dx^5} Dx^5 \&c.$$

absque ullo Limite. Exinde fit

$$d(\text{Cofin. C. } x) = d(\text{Cofin. C. } x) + \frac{d^2(\text{Cofin. C. } x)}{2} + \frac{d^3(\text{Cofin. C. } x)}{2.3} + \frac{d^4(\text{Cofin. C. } x)}{2.3.4} + \frac{d^5(\text{Cofin. C. } x)}{2.3.4.5} + \&c.,$$

vel

$d(\text{Cofin.}$

$$d(\text{Cofin. C. } x) = \left(\frac{\text{Cofin. C. } x + (\text{Sin. C. } x)\sqrt{-1}}{2} \right) \left(dx\sqrt{-1} - \frac{dx^2}{2} - \frac{dx^3\sqrt{-1}}{2 \cdot 3} + \frac{dx^4}{2 \cdot 3 \cdot 4} + \frac{dx^5\sqrt{-1}}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{dx^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \&c. \right) \\ + \left(\frac{-\text{Cofin. C. } x + (\text{Sin. C. } x)\sqrt{-1}}{2} \right) \left(dx\sqrt{-1} + \frac{dx^2}{2} - \frac{dx^3\sqrt{-1}}{2 \cdot 3} - \frac{dx^4}{2 \cdot 3 \cdot 4} + \frac{dx^5\sqrt{-1}}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{dx^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \&c. \right),$$

ac denique, spuria Imaginarietate sublata,

$$d(\text{Cofin. C. } x) = -(\text{Sin. C. } x) \left(dx - \frac{dx^3}{2 \cdot 3} + \frac{dx^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{dx^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \&c. \right) - (\text{Cofin. C. } x) \left(\frac{dx^2}{2} - \frac{dx^4}{2 \cdot 3 \cdot 4} + \frac{dx^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \frac{dx^8}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \&c. \right),$$

in quibus omnibus prima fronte Leibnitianum Differentiale $d(\text{Cofin. C. } x)$
 $= -dx(\text{Sin. C. } x)$, prae ceteris familiarius, adparet.

363. Nec minus evidenter de additarum Aequationum veritate constabit, nempe

$$dL(\text{Sin. C. } x) = \frac{(\text{Cofin. C. } x)}{\text{Sin. C. } x} dx = (\text{Cot. C. } x) dx = \left(\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \frac{x^7}{4725} - \frac{2x^9}{93555} - \&c. \right) dx;$$

$$dL(\text{Cofin. C. } x) = \frac{-(\text{Sin. C. } x)}{\text{Cofin. C. } x} dx = -(\text{Tan. C. } x) dx = \left(-x - \frac{x^3}{3} - \frac{2x^5}{15} - \frac{17x^7}{315} - \frac{62x^9}{2835} - \&c. \right) dx;$$

$$dL(\text{Tan. C. } x) = \frac{dx}{(\text{Cofin. C. } x)^2 (\text{Tan. C. } x)} = \frac{(\text{Cofin. C. } x) dx}{(\text{Cofin. C. } x)^2 (\text{Sin. C. } x)} = \frac{dx}{(\text{Cofin. C. } x) (\text{Sin. C. } x)};$$

$$dL(\text{Cot. C. } x) = \frac{-dx}{(\text{Sin. C. } x)^2 (\text{Cot. C. } x)} = \frac{-(\text{Sin. C. } x) dx}{(\text{Sin. C. } x)^2 (\text{Cofin. C. } x)} = \frac{-dx}{(\text{Sin. C. } x) (\text{Cofin. C. } x)};$$

$$dL(\text{Chor. C. } x) = dL\left(2 \text{Sin. C. } \frac{x}{2}\right) =$$

$$dL\left(x - \frac{x^3}{4^1 \cdot 2 \cdot 3} + \frac{x^5}{4^2 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{4^3 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{x^9}{4^4 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} - \&c. \right) \\ = \left(1 - \frac{x^2}{4^1 \cdot 2} + \frac{x^4}{4^2 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{4^3 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{x^8}{4^4 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} - \&c. \right) \left(\text{Cofin. C. } \frac{x}{2} \right) \\ \frac{x - \frac{x^3}{4^1 \cdot 2 \cdot 3} + \frac{x^5}{4^2 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{4^3 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{x^9}{4^4 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} - \&c.}{2 \left(\text{Sin. C. } \frac{x}{2} \right)} dx =$$

$$dL(\text{Chor. Comp. C. } x) = dL\left(2 \text{Cofin. C. } \frac{x}{2}\right) = \frac{-2\left(\text{Sin. C. } \frac{x}{2}\right) \frac{dx}{2}}{2 \left(\text{Cofin. C. } \frac{x}{2} \right)} = \frac{-(\text{Sin. C. } \frac{x}{2})}{2 \left(\text{Cofin. C. } \frac{x}{2} \right)} dx;$$

$dL(\text{Sec.}$

$$dL(\text{Sec. C. } x) = dL\left(\frac{1}{\text{Cofin. C. } x}\right) = \frac{(\text{Sin. C. } x)}{(\text{Cofin. C. } x)} dx = (\text{Tan. C. } x) dx = -(-\text{Tan. C. } x) dx = -dL(\text{Cofin. C. } x);$$

$$dL(\text{Cofec. C. } x) = dL\left(\frac{1}{\text{Sin. C. } x}\right) = \frac{-(\text{Cofin. C. } x)}{(\text{Sin. C. } x)} dx = -(\text{Cot. C. } x) dx = -dL(\text{Sin. C. } x);$$

$$d(\text{Cofin. C. } x)^n = n(\text{Cofin. C. } x)^{n-1} \times -(\text{Sin. C. } x) dx = -n(\text{Sin. C. } x)(\text{Cofin. C. } x)^{n-1} dx;$$

$$d(\text{Sin. C. } x)^z = (\text{Sin. C. } x)^z \cdot L(\text{Sin. C. } x) dz + (\text{Sin. C. } x)^{z-1} z (\text{Cofin. C. } x) dx, \text{ et idcirco}$$

$$d(\text{Sin. C. } x)^m = m(\text{Sin. C. } x)^{m-1} (\text{Cofin. C. } x) dx;$$

$$d(\text{Tan. C. } x)^{(\text{Sin. C. } x)} = (\text{Tan. C. } x)^{(\text{Sin. C. } x)} \cdot L(\text{Tan. C. } x)(\text{Cofin. C. } x) dx \\ + \frac{(\text{Sin. C. } x)(\text{Tan. C. } x)^{(\text{Sin. C. } x)-1}}{(\text{Cofin. C. } x)^2} dx, \text{ five potius}$$

$$d(\text{Tan. C. } x)^{(\text{Sin. C. } x)} = (\text{Tan. C. } x)^{(\text{Sin. C. } x)} \left(L(\text{Tan. C. } x)(\text{Cofin. C. } x) + \frac{1}{\text{Cofin. C. } x} \right) dx;$$

$$d(\text{Sin. C. } x)^{(\text{Cofin. C. } x)} = (\text{Cofin. C. } x)(\text{Sin. C. } x)^{(\text{Cofin. C. } x)-1} (\text{Cofin. C. } x) dx \\ - (\text{Sin. C. } x)^{(\text{Cofin. C. } x)} \cdot L(\text{Sin. C. } x)(\text{Sin. C. } x) dx, \text{ aut ad rem aptius}$$

$$d(\text{Sin. C. } x)^{(\text{Cofin. C. } x)} = (\text{Sin. C. } x)^{(\text{Cofin. C. } x)-1} ((\text{Cofin. C. } x)^2 - (\text{Sin. C. } x)^2 L(\text{Sin. C. } x)) dx:$$

fed non vacat ultra progredi, quum iis fundamentis cetera omnia sustineantur. Igitur ex praeostensis

$$dL(\text{Sin. C. } x) \times dL(\text{Cofin. C. } x) = -(\text{Cot. C. } x)(\text{Tan. C. } x) dx^2 = -1^2 \cdot dx^2 = -dx^2;$$

$$dL(\text{Tan. C. } x) = -dL(\text{Cot. C. } x), \text{ ac vicissim; nec non alia forma } dL(\text{Tan. C. } x) + dL(\text{Cot. C. } x) = 0;$$

$$dL(\text{Sec. C. } x) = -\frac{dx^2}{dL(\text{Cofec. C. } x)}; \text{ et versa vice } dL(\text{Cofec. C. } x) = -\frac{dx^2}{dL(\text{Sec. C. } x)}; \text{ quapropter}$$

$$dL(\text{Sec. C. } x)(dL(\text{Cofec. C. } x)) = -dx^2 = dL(\text{Sin. C. } x) \times dL(\text{Cofin. C. } x),$$

prouti adlatae Formulae docent;

T t t t

S $\frac{dx}{dx}$

$$\int \frac{dx}{(\text{Sin. C. } x)} = L(\text{Tan. C. } \frac{x}{2}), \text{ quia } dL(\text{Tan. C. } \frac{x}{2}) = \frac{dx}{2(\text{Cofin. C. } \frac{x}{2})(\text{Sin. C. } \frac{x}{2})} =$$

$$\frac{dx}{\text{Sin. C. } (\frac{x}{2} + \frac{x}{2})} = \frac{dx}{\text{Sin. C. } x};$$

$$\int \frac{(\text{Cofin. C. } x)}{\text{Sin. C. } x} dx = \int dL(\text{Sin. C. } x) = L(\text{Sin. C. } x);$$

$$-\int \frac{(\text{Sin. C. } x)}{\text{Cofin. C. } x} dx = \int dL(\text{Cofin. C. } x) = L(\text{Cofin. C. } x);$$

$$dL(\text{Sin. C. } x) \times dL(\text{Tan. C. } x) = \frac{dx^2}{(\text{Sin. C. } x)^2} = (\text{Cofec. C. } x)^2 dx^2;$$

$$dL(\text{Cofin. C. } x) \times dL(\text{Tan. C. } x) = \frac{-dx^2}{(\text{Cofin. C. } x)^2} = -dL(\text{Cofin. C. } x)(dL(\text{Cot. C. } x)) = -(\text{Sec. C. } x)^2 dx^2;$$

$$dL(\text{Sin. C. } x) \times dL(\text{Cofin. C. } x) + dL(\text{Sin. C. } x) \times dL(\text{Tan. C. } x) = (\text{Cot. C. } x)^2 dx^2;$$

$$dL(\text{Sin. C. } x) \times dL(\text{Cofin. C. } x) + dL(\text{Cofin. C. } x) \times dL(\text{Cot. C. } x) = (\text{Tan. C. } x)^2 dx^2;$$

et aliae sexcentae deduci possent Aequalitates, quae Lectorum patientiam certe minuerent.

364. Percelebres exstant in *Philosophicis Transactionibus* anni MDCLXXXV. Series a Vallisio evulgatae, dum occasione Mapparum in Nauticos usus conditarum Epistolam scriberet ad Richardum Norrisium. Eae sunt

$$\begin{aligned} \text{Summa Secantium Circ.}^{\text{rium}} &= \frac{x^1}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \frac{x^{13}}{13} + \&c: \\ \text{Summa Tangentium Circ.}^{\text{rium}} &= \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \frac{x^8}{8} + \frac{x^{10}}{10} + \frac{x^{12}}{12} + \frac{x^{14}}{14} + \&c: \end{aligned} \left. \begin{array}{l} \text{posito } x \text{ Recto-Sinu Arcus} \\ \text{Circularis, ad quem referatur} \\ \text{Secantium vel Tangentium} \\ \text{Collectio;} \end{array} \right\}$$

quibus expressionibus obtinendis Methodus Divisionis par fuit a Mercatore prae omnibus adinventum. Reapse, statim ac x Circularem Arcum de more designet, aut duplum Areae Sectoris,

$$\text{Summa Secantium} = \int (\text{Sec. C. } x) dx = \int \frac{(\text{Sec. C. } x)d(\text{Sin. C. } x)}{\text{Cofin. C. } x} = \frac{d(\text{Sin. C. } x)}{(\text{Cofin. C. } x)^2} = \frac{d(\text{Sin. C. } x)}{1 - (\text{Sin. C. } x)^2} = (d$$

$$= (d(\text{Sin. C. } x)) \left(1 + (\text{Sin. C. } x)^2 + (\text{Sin. C. } x)^4 + (\text{Sin. C. } x)^6 + (\text{Sin. C. } x)^8 + (\text{Sin. C. } x)^{10} + \&c. \right)$$

$$= \frac{(\text{Sin. C. } x)^1}{1} + \frac{(\text{Sin. C. } x)^3}{3} + \frac{(\text{Sin. C. } x)^5}{5} + \frac{(\text{Sin. C. } x)^7}{7} + \frac{(\text{Sin. C. } x)^9}{9} + \frac{(\text{Sin. C. } x)^{11}}{11} + \&c.;$$

$$\text{atque Summa Tangentium} = \int (\text{Tan. C. } x) dx = \int \frac{(\text{Tan. C. } x) d(\text{Sin. C. } x)}{\text{Cofin. C. } x} = \int \frac{(\text{Sin. C. } x) \times d(\text{Sin. C. } x)}{1 - (\text{Sin. C. } x)^2}$$

$$= d(\text{Sin. C. } x) \left((\text{Sin. C. } x) + (\text{Sin. C. } x)^3 + (\text{Sin. C. } x)^5 + (\text{Sin. C. } x)^7 + (\text{Sin. C. } x)^9 + (\text{Sin. C. } x)^{11} + \&c. \right)$$

$$= \frac{(\text{Sin. C. } x)^2}{2} + \frac{(\text{Sin. C. } x)^4}{4} + \frac{(\text{Sin. C. } x)^6}{6} + \frac{(\text{Sin. C. } x)^8}{8} + \frac{(\text{Sin. C. } x)^{10}}{10} + \frac{(\text{Sin. C. } x)^{12}}{12} + \frac{(\text{Sin. C. } x)^{14}}{14} + \&c.;$$

quae gemina Series in Loxodromia definienda, five Latitudinibus, ut vocant, *crescentibus* supputandis maximam commoditatem praefert. Accidit tamen in collatione istarum Serierum hoc periucundum, tametsi invisum Vallisio, quod simul cumulatis Summis Secantium, atque Tangentium habeatur Series numero terminorum Infinita, ubi nulla perfectarum desit $\tau\theta$ Sinus x Potestatum, scilicet

$$\sum \text{Sec.}^{\text{ium}} + \sum \text{Tan.}^{\text{ium}} = \frac{x^1}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} + \frac{x^{10}}{10} + \frac{x^{11}}{11} + \frac{x^{12}}{12} + \frac{x^{13}}{13} + \frac{x^{14}}{14} + \&c.;$$

Huius elegantissimae Progressionis intima ratio e longinquo petenda non erit. Summa etenim, circa quam nunc versamur, abit ex praedictis in Formu-

$$\text{lam } \int \frac{(1 + \text{Sin. C. } x) d(\text{Sin. C. } x)}{1 - (\text{Sin. C. } x)^2} = \int \frac{d(\text{Sin. C. } x)}{1 - \text{Sin. C. } x} =$$

$$-L(1 - \text{Sin. C. } x) = - \left(\frac{(\text{Sin. C. } x)^1}{1} + \frac{(\text{Sin. C. } x)^2}{2} + \frac{(\text{Sin. C. } x)^3}{3} + \frac{(\text{Sin. C. } x)^4}{4} + \frac{(\text{Sin. C. } x)^5}{5} + \frac{(\text{Sin. C. } x)^6}{6} + \&c. \right)$$

$$= \frac{(\text{Sin. C. } x)^1}{1} + \frac{(\text{Sin. C. } x)^2}{2} + \frac{(\text{Sin. C. } x)^3}{3} + \frac{(\text{Sin. C. } x)^4}{4} + \frac{(\text{Sin. C. } x)^5}{5} + \frac{(\text{Sin. C. } x)^6}{6} + \frac{(\text{Sin. C. } x)^7}{7} + \frac{(\text{Sin. C. } x)^8}{8} + \&c.;$$

vi numeri 88.; adeo, ut hoc prodeat Theorema „ Summas Tangentium Secantiumque eidem Arcui Circuli additarum simul collectas Logarithmum Hyperbolicum (negative sumptum Numeri $1 - \text{Sin. C. } x$, aut Sinus-Versi Complementi Arcus x peraequare; quod eodem redit ac dicere Logarithmum Numeri $\frac{1}{1 - \text{Sin. C. } x}$, five $\frac{1}{\text{Cofin. Vers. C. } x}$ „ Accedit, Series illas

fingu-

singulariter etiam consideratas exprimi Logometrice posse: namque a numeris 102. et 105. dimanant protinus Aequalitates

$$\text{Summa Secantium} = \frac{1}{2} \left(L \left(\frac{1+x}{1-x} \right) \right), \text{ nec non Summa Tangentium} = -\frac{1}{2} \left(L \left(\frac{1+x}{1-x} \right) \right);$$

$$\text{nimirum Summa Secantium} = \frac{1}{2} L \left(\frac{1 + \text{Sin. C. } x}{1 - \text{Sin. C. } x} \right), \text{ ac Summa Tangentium}$$

$$= -\frac{1}{2} L \left((1 + \text{Sin. C. } x)(1 - \text{Sin. C. } x) \right): \text{quamobrem universa Proble-$$

mata Loxodromica Logarithmorum adiumento resolvere postremae Formulae docent.

365. Eadem aggressus Problemata in *Transactionibus* anni MDCXCV. VI. Edmundus Halleius alio se vertit inquirens Summas Secantium, Tangentiumve non ex Lineis Trigonometricis, sed ope Arcuum Circularium, ad quos illae pertineant. Sic, Arcu Circulari nuncupato v ,

$$S_{\text{Sec.}}^{\text{ium}} = S \left(1 + \frac{v^2}{2} + \frac{5v^4}{24} + \frac{61v^6}{720} + \frac{277v^8}{8064} + \frac{50521v^{10}}{3628800} + \&c. \right) dv \text{ ex num}^\circ. 215.$$

$$= v + \frac{v^3}{6} + \frac{v^5}{24} + \frac{61v^7}{5040} + \frac{277v^9}{72576} + \frac{50521v^{11}}{39916800} + \&c. = L \left(\text{Tan. C.} \left(45^\circ + \frac{v}{2} \right) \right) \text{ ex num}^\circ. 248.;$$

$$S_{\text{Tan.}}^{\text{ium}} = S \left(v + \frac{v^3}{3} + \frac{2v^5}{15} + \frac{17v^7}{315} + \frac{62v^9}{2835} + \frac{1382v^{11}}{155925} + \frac{21844v^{13}}{6081075} + \&c. \right) dv \text{ ex num}^\circ. 208.$$

$$= \frac{v^2}{2} + \frac{v^4}{12} + \frac{v^6}{45} + \frac{17v^8}{2520} + \frac{31v^{10}}{14175} + \frac{691v^{12}}{935550} + \frac{10922v^{14}}{42567525} + \&c. = L \left(\text{Sec. C. } v \right) \text{ ex num}^\circ. 247:$$

haeque Halleianae expressiones consimiliter ad Logometriam referuntur. Mirificum equidem est, quam perfecte consentiant Series postremo loco recensitae cum inventis Cotesii: eius quippe Problemata I^{um}. ac II^{um}., quae tertiam *Harmoniae Mensurarum* Partem aperiunt, Quadraturas Figurae Tangentium, ac Figurae Secantium complectuntur iisdem pene verbis, quibus supra usi sumus de Tangentium, atque Secantium Summis agentes. Profecto, quum nihil aliud sonent Summae antea repertae praeter Curvarum, quae Tangentium, ac Secantium Lineae adpellantur, Tetragonismum, non poterat quin plenissimus ille consensus accideret, cui penitus intelligendo sufficiet ani-

animadversio Tan. C. $(45^\circ + \frac{v}{2})$, et Tangentem Semissis Complementi Cir-

cularis Arcus v , aut Tangentem Arcus $(45^\circ - \frac{v}{2}) = \frac{Q-v}{2}$ eodem gaudere

Logarithmo, dummodo excipias Signum in priore positivum, at in altera

negativum, ob universas Gradus Imparis Potestates in Serie $v + \frac{v^3}{6} +$

$\frac{v^5}{24} + \frac{61v^7}{5040} + \frac{277v^9}{9(8064)} + \frac{50521v^{11}}{39916800}$ &c., ac insuper Summam Tangentis

atque Secantis Arcus eiusdem v Tangentem Arcus $(45^\circ + \frac{v}{2})$ exaequare.

Quod ut ostendam, praemitto esse

$$\text{Tan. C. } v, \text{ five Tan. C. } \left(\frac{v}{2} + \frac{v}{2}\right) = \frac{\text{Tan. C. } \frac{v}{2} + \text{Tan. C. } \frac{v}{2}}{1 - (\text{Tan. C. } \frac{v}{2})(\text{Tan. C. } \frac{v}{2})} =$$

$$\frac{2 \text{ Tan. C. } \frac{v}{2}}{1 - (\text{Tan. C. } \frac{v}{2})^2}, \text{ ideoque Tan. C. } v + \text{Sec. C. } v = \text{Tan. C. } v +$$

$$\left(1 + (\text{Tan. C. } v)^2\right)^{\frac{1}{2}} = \frac{2 \text{ Tan. C. } \frac{v}{2}}{1 - (\text{Tan. C. } \frac{v}{2})^2} + \left(1 + \frac{(2 \text{ Tan. C. } \frac{v}{2})^2}{(1 - (\text{Tan. C. } \frac{v}{2})^2)^2}\right)^{\frac{1}{2}},$$

nimirum, redactis in communem Denominatorem Fractionibus, =

$$\frac{2 \text{ Tan. C. } \frac{v}{2}}{1 - (\text{Tan. C. } \frac{v}{2})^2} + \left(\frac{(1 + \text{Tan. C. } \frac{v}{2})^2}{(1 - (\text{Tan. C. } \frac{v}{2})^2)^2}\right)^{\frac{1}{2}} = \frac{(1 + \text{Tan. C. } \frac{v}{2})^2}{1 - (\text{Tan. C. } \frac{v}{2})^2} =$$

$$\frac{1 + \text{Tan. C. } \frac{v}{2}}{1 - \text{Tan. C. } \frac{v}{2}} = \text{Tan. C. } (45^\circ + \frac{v}{2}) \text{ ex praecitato num}^\circ. 248. \text{ Ista non}$$

intempestiva inlustratio aliquantulum commentatur ea, quae fuit Cotesius, nec ipsius Operum Editor Smithius in brevibus ad calcem *Notis* exposuit.

366. Multa de Curvarum Quadraturis a Lineis Trigonometricis origi-

V v v v v nem

nem propriam ducentium nunc enarrare fas esset; sed partim Doctrinae huiusce, eiusque Confectariorum facilitas, partim provincia a tot antea Scriptoribus luculenter usurpata me remonent. Verumtamen argumenti vel titulus ipse mihi etiam invito risum excitat, dum memoria repeto, gravissimum Virum persaepe in Etruria hac ipsa aetate labente, perinde ac si sublimē fuisset et novum, proposuisse ad instar *Experimenti Crucis* resolvendum Problema determinationis Curvae illius non in eodem plano iacentis, quam in vacuo describat punctum Ponderis funiculae appensi, atque, uti solet, adpositi ianuarum Valvis circa suos cardines, quum aperiantur, de more rotantibus. Praesumpti siquidem Aenigmatis Auctor certe non vidit, istam puncti Ponderis semitam esse Lineam Chordarum in Superficie Recti Cylindri depictam, cuius Basis sit Circulus Semi-Diametrum habens Latitudini Valvae aequalem, et sane Lineam ipsissimam, quam facillimo Circini ductu, tam in regione positivorum, tam in adversa negativorum, quisque in Superficie Cylindrica, uti Circulum in Plano, describere possit ad Robervallii modum, cui iampridem, quidquid contra sentiat Montuclas, innotuit sub nomine Gemellae aut Sociae Trochoidis (*). Quinimmo ea Aenigmatistae Linea est duplicis Curvaturae Limes, qui Oculos Fenestrasve terebratae a Vivianio suo Sphaerae, aut geminae Veliformis Testudinis Florentinae circumscribit, et Aequatione, si velles, distinguitur Infinitesimali

$$dx = \frac{2R dy}{(4R^2 - y^2)^{\frac{1}{2}}},$$

positis x Circuli Baseos Arcubus veluti Abscissis, $2R$ Diametro, et y Ordinatis super Baseos Planum normaliter insistentibus, sive alia Aequatione,

$$\text{pariter Finitorum Algebrae vires transcendente, } y = x - \frac{x^3}{4^1 \cdot 2 \cdot 3} + \frac{x^5}{4^2 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{4^3 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{x^9}{4^4 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} - \frac{x^{11}}{4^5 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} + \&c:$$

$$\text{dum Radius ob maiorem simplicitatem Unitas fuerit, vel denique } y = 2 \text{ Sin. C.}$$

(*) Auctor est singulari vir ingenio et doctrina Pascalius in *Historia Trochoidis* Cl. Robervallium eo pervenisse, illamque tenuisse Curvam ante annum MDCXXXV.

$$2 \text{ Sin. C. } \frac{x}{2} = \frac{\left(1 + \frac{x\sqrt{-1}}{2 \cdot \infty}\right)^{\infty} - \left(1 - \frac{x\sqrt{-1}}{2 \cdot \infty}\right)^{\infty}}{\sqrt{-1}} = \text{Sin. C. } x \text{ in Circu-}$$

lo, cui 2 pro Radio conveniat. Postrema haec expressio decernit, Lineam eandem nonnisi forma differre a notissima Sinuum-Rectorum Linea simplicis Curvaturae, vel ab Ungulae Cylindricae Semi-orthogonalis terminatrice, quam usque ab incunabilis Indivisibilium Geometriae Mathematicarum rerum historia docet exacto certatim Tetragonismo, aliorumque symptomatum elegantia exornatam (*). Ea est demum Curva perinignis Sphaero-Cylindrica, quae una cum aliis primariis ac secundariis, primi et alterius Nominis, Cylindri ac Sphaerae dupliciter Curvis Sectionibus multo ante Vivianaeum Aenigma, anno MDCXCII.^{mo} propositum, Galliae Geometris innotuerat, atque praesertim Matheseos non fucatae, sed profundioris adprime gnaro Tolosanae Curiae Senatori Fermatio, et Iesuitae Laloverae, qui in *Veterum Geometria promota* Editionis anni MDCLX.^{mi} valde plura toto Libro II.^o de Sectionibus Cyclo-Cylindricis in lucem edidit prae illis postmodum evulgatis a Grandio, *Vivianaeorum Problematum* sub anni MDCXCIX.^{mi} lapsum *Demonstrationem Geometricam* conscribente.

367. Novitatem fortassis aliquam praesetulisset loco obsoleti Sphaero-Cylindricae Curvae ortus elegantissima eius affectio haecenus in obscuro posita, suae nimirum indoles Ichnographiae, sive, quemadmodum aiunt, Projectionis Orthographicae, praeter Circulum suapte natura ichnographice subiectum, utpote Basin Cylindri, cuius in Superficie Linea iacet circino descriptibilis, atque Parabolen Apollonianam a Grandio repertam (**). Tertiam enim Ichnographiam Orthogonalem repraesentat Bifolium Aequatione gaudens

(*) Stephanus de Angelis Iesuata inter alios *Tractatum Geometricum de Superficie Ungulae* inscriptum anno MDCLXI., tunc Patavii Matheseos docens, Venetis Typographeis vulgavit. At Pascalius haec omnia de Ungulis possidebat anno MDCLVIII., quum sub istud tempus Problematum *de Cycloide* celeberrimorum in promptu habuerit enodationem ab Ungulis illis petitam.

(**) Exstat in Propositione 1.^a memoratae Latinae Versionis *Vivianaeorum Problematum*. Ioannes Lamius eruditissimus in Vivianii Vita Opus ipsi tribuit ineditum *De Terebratione Solidorum*. Quid hoc sit, ubinam sit, equidem nescio; ideoque Parabolae-Ichnographicae inventum Grandio adscribens neutiquam videbor iniurius.

gaudens quarti Gradus $x^4 - a^2x^2 + a^2y^2 = 0$, atque in Plano duobus simul aliis Normali descriptum. Triplicem hanc ostendit Projectionem Schema

XXXI^{um}, earumque mutuum situm in Hemisphaerii Quadrante $\Phi Y \omega \Psi T \Theta \Phi D$, ubi $\Phi LZPT$ Sectionem designat Sphaero-Cylindricam, vel Semissem Lineae Subtenfarum positivarum Semi-Circulo ΦVSD Orthogonaliter insistentem, $TH\Phi D$ Trilineum Parabolae Ichnographicae, et TQD Semi-Petalon, aut quartam Bifolii partem, quod integrum demonstratur ab Schemate XXXII^o. in Nodi formam $BQTS DPTLB$, Axe $DB = 2a$, scilicet Diametro Globi aut Circuli Maximi $ABCD$, in quo inscribitur, praeditum. Posset itaque Archetypum Fornicis Veliformis in Sphaera data aequae Quadrabilis delineari ope terebrationis a Cylindraceo confectae Basin habente aequalem ipsi Bifolio, seu Lemniscatoidi, cuius meminit etiam Gabriel Cramerus (*), Axemque HT Circulo Sphaerae Maximo perpendicularem. Notandum autem occurrit, hoc Bifolium ternas inter Projectiones ambitus Testudinis Vivianaeae summopere excellere, eoquod sit Curva vere unica, dum ceterae vel bis repetito Circulo coalescunt, vel duplici eiusdem Parabolae Segmento converse posito, et ideo Systema, ut aiunt, Curvarum conficiunt, ac dumtaxat spuriam imitantur Quarti Ordinis Lineam. Projectionis istius singularis, nedum a Viviano, Grandioque, verum etiam a Pascaliq, Lalovera, Fermatio, omnibusque, quod sciam, inobservatae, contemplationem inivi, quum Pisis Iurisprudenciae causa commorarer; atque tunc fervens, et animo exultans, nesciusque haec nugae veram non promovere Mathesin, statim Amicis mei studiosissimis inventum communicavi una cum praecipuis Bifolii illius proprietatibus, quas, servato penitus antiquo ordine, subdo.

1. Curvae

(*) In *Introductionis ad Curvarum Analysin* pag^a. 470. exponitur Curva Aequationis $a^2x^2 - a^2y^2 + y^4 = 0$, remeatque in pag^a. 495., ubi quaestio fit de Problemate quodam *Maximorum*. Aequatio enim Curvae est ibidem ita concinnata, $a^2y^2 - r^2x^2 + x^4 = 0$, versoque r in a nostrum obtinetur Bifolium. Confundi autem hoc nequit cum Lemniscata Bernoullii Aequationem habente $y^4 + (2x^2 + b^2)y^2 + x^4 - b^2x^2 = 0$, nimirum cum Ellipfi Cassiniana, cuius Focorum Distantia sit $= b\sqrt{2}$, Rectangulum constans $= \frac{b^2}{2}$; quam congruentiam utriusque Lemniscatae, Cassinii, ac Bernoullii, nemo, si recte memini, haecenus retulit.

1. Curvae Sub-Tangens $= \frac{(a^2 - x^2)x}{-2x^2 + a^2}$.

3. Abscissae Ordinatis
Maximis responden-
tes

$$= \pm a \sqrt{\frac{1}{2}}$$

5. Concurfus Perimetri
utriusque Semissis Bi-
folii ac Circuli de-
scripti super Semi-A-
xem accidunt tum,

$$\text{quum Abscissae sint } 0, +a, \pm \left(\frac{a}{2} \sqrt{5} - \frac{a}{2} \right),$$

qui postremus valor
TV=TN reperitur
subsidio divisionis Se-
mi-Axis TB aut TD
in Proportionem divi-
na.

2. Ordinatarum Maxima EQ, FP, FS, EL
 $= \pm \frac{a}{2} = \text{Folii unius Semi-Axi.}$

4. Tangentes in Puncto Duplici T Decus-
sationis Ramorum, sed *tertia* Speciei ob
geminum Nodi Flexum-contrarium, ef-
ficiunt Angulum Rectum, quemadmo-
dum in celebri Lemniscata Bernoullia-
na, aut Cassiniana.

6. Area totius Bifolii est Geometrici Te-
tragonismi capax, quia peraequat $\frac{4}{3} a^2$,

sive $\frac{2}{3}$ ABCD inscripti Quadrati. Ergo uni-

cum Petalon Area gaudebit aequali Trien-
ti $\frac{2}{3}$ ABCD.

7. Inde consequitur, Spatium Bifolii Ichno-
graphici Marginum Testudinis Florenti-
nae medietatem esse Areae Ichnographiae
Parabolicae, ac Fornicis Veliformis Su-
perficiem se habere ad Aream subiecti
Bifolii in tripla Ratione, dum ad illam
Ichnographiae Parabolicae est in Ratione
Sesquialtera, quam Rationem Grandius
neglexit, tametsi iucunditate nulli se-
cundam.

368. Summam equidem simplicitatem in definiendo Tetragonismo Areae
totius ab integra Circini revolutione super Rectum Cylindrum circumse-
ptae, eiusve Areae partium, dum extrema ipsius Curvae Convexa se mu-
tuo contingant, nuperrimae Trigonometricae Formulae pollicentur. Profe-
cto, quum ex hypothese assumpta totum Linea Sphaero-Cylindrica com-
plectatur Cylindrum, erit Areae Quadrans, Semissi Circuli insistens, ae-
qualis $\int y dx$, factis post Integrationem Ordinatum-Adplicata $y = \text{Diametro}$
Baseos eiusdem Cylindri, et Arcu $x = \text{Semi-Peripheriae}$. Sed $\int y dx =$

$\int dx$ (

$$S dx \left(x - \frac{x^3}{4^1 \cdot 2 \cdot 3} + \frac{x^5}{4^2 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{4^3 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \&c: \right) = \frac{x^2}{2} - \frac{x^4}{4^1 \cdot 2 \cdot 3 \cdot 4} +$$

$$\frac{x^6}{4^2 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \frac{x^8}{4^3 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \frac{x^{10}}{4^4 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} - \&c: \right) = 4 \cdot \text{Sin. Verf. C. } \frac{x}{2} =$$

$2 \left(2 \text{ Sin. V. C. } \frac{x}{2} \right)$ ex numero 165. Igitur Areae illius Quadrans $= 4 \cdot 1 = 4.$,

vel Quadrato Baseos Diametri, scilicet Spatium totum ab in se remeante Sphaero-Cylindrica Curva clausum par fiet quadruplicato Diametri praedictae Quadrato. Quin etiam universalius, ac aequè evidenter, Trilineum quodlibet arcu Circuli, Ordinata, atque Arcu Lineae Sphaero-Cylindricae circumscriptum indefinitam Quadraturam admittet; quippe quum $2 \left(2 \text{ Sin.}$

$\text{Verf. C. } \frac{x}{2} \right)$ praebeat Aream Parallelogrammatis Rectangularis, cuius Basis constans sit, nimirum toties citata Diameter, vel Cuspidum Circini describentis aut generatoris distantia, et Altitudo sit dupla Versi-Sinus pertinentis ad Semissem Arcus-Abscissae. Admodum difficilior manet Areae Mensura a Secundariis Lineis omnibus Sphaero-Cylindricis comprehensae, cuius Syntheticam investigationem absolutissimam, quam Lalovera non prodidit, ut habet Montucla, sed deducendam indigitavit a Theoremate Pascali in Epistola *de Cycloide* ad Hugenum Detonvillii sub nomine ficto latentis (*), non est huius loci a mea *de Cochlearibus Solidis* Disquisitione huc revocare.

369. Insolens potius perscrutari iuvat Curvarum Sinuum, Cosinumque, tam Circularium, quam Hyperbolicorum, ingenium, quod iubet Fluxionum-Fluxiones Ordinarum, nusquam abruptas, tali periodo procedere, ut alternatim ipsaemet Curvae recurrant, et vacua unius intermedia converse ab altera compleantur; hac tamen Legis differentia, quod in Circularibus Curvarum situs a Positivarum parte abeat regulariter in adversam, non ita in Curvis Hyperbolicae additis Trigonometriae. Ista reproductio Curvarum, quamvis minus elegans affini Logarithmicae regeneratione, semper sibi Qualitate aut Signo consimilis, adserta in numero 347., summam, et sine numeris 357. 358. 360. statui sic potest.

$$y = \text{Sin. C.}$$

(*) Epistolae Gallicae Titulus exstat huiusmodi in Volumine V°. praecitatae Pascali Operum Collectionis *Dimensio Perimetri quarumcunque Cycloidum*.

$$y = \text{Sin. C. } x = \frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{2\sqrt{-1}},$$

$$\frac{dy}{dx} = z = \infty \cdot \frac{\sqrt{-1}}{\infty} \left(\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1} + \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1}}{2\sqrt{-1}} \right) =$$

$$\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} + \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{2}, \text{ prouti in numero 359.},$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} = v = \infty \cdot \frac{\sqrt{-1}}{\infty} \left(\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1} - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1}}{2} \right) =$$

$$- \left(\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{2\sqrt{-1}} \right),$$

$$\frac{d^3y}{dx^3} = \frac{d^2z}{dx^2} = \frac{dv}{dx} = t = \infty \cdot \frac{-\sqrt{-1}}{\infty} \left(\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1} + \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1}}{2\sqrt{-1}} \right) =$$

$$- \left(\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} + \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{2} \right),$$

$$\frac{d^4y}{dx^4} = \frac{d^3z}{dx^3} = \frac{d^2v}{dx^2} = \frac{dt}{dx} = \Phi = \infty \cdot \frac{-\sqrt{-1}}{\infty} \left(\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1} - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1}}{2} \right) =$$

$$\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{2\sqrt{-1}},$$

$$y' = \text{Sin.}$$

$$y' = \text{Sin. H. } x = \frac{\left(1 + \frac{x}{\infty}\right)^{\infty} - \left(1 - \frac{x}{\infty}\right)^{\infty}}{2},$$

$$\frac{dy'}{dx} = z' = \frac{\infty}{\infty} \left[\frac{\left(1 + \frac{x}{\infty}\right)^{\infty-1} - \left(1 - \frac{x}{\infty}\right)^{\infty-1}}{2} \right] = \frac{\left(1 + \frac{x}{\infty}\right)^{\infty} + \left(1 - \frac{x}{\infty}\right)^{\infty}}{2},$$

quemadmodum in num^o. 355.,

$$\frac{d^2 y'}{dx^2} = \frac{dz'}{dx} = v' = \frac{\infty}{\infty} \left[\frac{\left(1 + \frac{x}{\infty}\right)^{\infty-1} - \left(1 - \frac{x}{\infty}\right)^{\infty-1}}{2} \right] = \frac{\left(1 + \frac{x}{\infty}\right)^{\infty} - \left(1 - \frac{x}{\infty}\right)^{\infty}}{2},$$

$$\frac{d^3 y'}{dx^3} = \frac{d^2 z'}{dx^2} = \frac{dv'}{dx} = t' = \frac{\infty}{\infty} \left[\frac{\left(1 + \frac{x}{\infty}\right)^{\infty-1} + \left(1 - \frac{x}{\infty}\right)^{\infty-1}}{2} \right] = \frac{\left(1 + \frac{x}{\infty}\right)^{\infty} + \left(1 - \frac{x}{\infty}\right)^{\infty}}{2},$$

$$\frac{d^4 y'}{dx^4} = \frac{d^3 z'}{dx^3} = \frac{d^2 v'}{dx^2} = \frac{dt'}{dx} = \phi' = \frac{\infty}{\infty} \left[\frac{\left(1 + \frac{x}{\infty}\right)^{\infty-1} - \left(1 - \frac{x}{\infty}\right)^{\infty-1}}{2} \right] = \frac{\left(1 + \frac{x}{\infty}\right)^{\infty} - \left(1 - \frac{x}{\infty}\right)^{\infty}}{2},$$

-----;

$$y = \text{Cofin. C. } x = \frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} + \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{2},$$

$$\begin{aligned} \frac{dy}{dx} = z = \frac{\infty}{\infty} \cdot \frac{\sqrt{-1}}{\infty} \left(\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1} - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1}}{2} \right) = \\ = \left(\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{2\sqrt{-1}} \right), \text{ ad instar numeri 359.}, \end{aligned}$$

$$\frac{d^2 y}{dx^2} = \frac{dz}{dx} = v = -\frac{\infty}{\infty} \cdot \frac{\sqrt{-1}}{\infty} \left(\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1} + \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1}}{2\sqrt{-1}} \right) =$$

$$\left(\frac{1 +}{\infty} \right)$$

$$= \left(\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} + \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{2} \right),$$

$$\frac{d^3 y}{dx^3} = \frac{d^2 z}{dx^2} = \frac{dv}{dx} = t = -\infty \cdot \frac{\sqrt{-1}}{\infty} \left(\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1} - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1}}{2} \right) =$$

$$\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} - \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{2\sqrt{-1}},$$

$$\frac{d^4 y}{dx^4} = \frac{d^3 z}{dx^3} = \frac{d^2 v}{dx^2} = \frac{dt}{dx} = \Phi = \infty \cdot \frac{\sqrt{-1}}{\infty} \left(\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1} + \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty-1}}{2\sqrt{-1}} \right) =$$

$$\frac{\left(1 + \frac{x\sqrt{-1}}{\infty}\right)^{\infty} + \left(1 - \frac{x\sqrt{-1}}{\infty}\right)^{\infty}}{2},$$

$$y' = \text{Cofin. H. } x = \frac{\left(1 + \frac{x}{\infty}\right)^{\infty} + \left(1 - \frac{x}{\infty}\right)^{\infty}}{2},$$

$$\frac{dy'}{dx} = z' = \frac{\infty}{\infty} \left[\frac{\left(1 + \frac{x}{\infty}\right)^{\infty-1} - \left(1 - \frac{x}{\infty}\right)^{\infty-1}}{2} \right] = \frac{\left(1 + \frac{x}{\infty}\right)^{\infty} - \left(1 - \frac{x}{\infty}\right)^{\infty}}{2},$$

ut in num°. 355.,

$$\frac{d^2 y'}{dx^2} = \frac{dz'}{dx} = v' = \frac{\infty}{\infty} \left[\frac{\left(1 + \frac{x}{\infty}\right)^{\infty-1} + \left(1 - \frac{x}{\infty}\right)^{\infty-1}}{2} \right] = \frac{\left(1 + \frac{x}{\infty}\right)^{\infty} + \left(1 - \frac{x}{\infty}\right)^{\infty}}{2},$$

$$\frac{d^3 y'}{dx^3} = \frac{d^2 z'}{dx^2} = \frac{dv'}{dx} = t' = \frac{\infty}{\infty} \left[\frac{\left(1 + \frac{x}{\infty}\right)^{\infty-1} - \left(1 - \frac{x}{\infty}\right)^{\infty-1}}{2} \right] = \frac{\left(1 + \frac{x}{\infty}\right)^{\infty} - \left(1 - \frac{x}{\infty}\right)^{\infty}}{2},$$

$$\frac{d^4 y'}{dx^4} = \frac{d^3 z'}{dx^3} = \frac{d^2 v'}{dx^2} = \frac{dt'}{dx} = \Phi' = \frac{\infty}{\infty} \left[\frac{\left(1 + \frac{x}{\infty}\right)^{\infty-1} + \left(1 - \frac{x}{\infty}\right)^{\infty-1}}{2} \right] = \frac{\left(1 + \frac{x}{\infty}\right)^{\infty} + \left(1 - \frac{x}{\infty}\right)^{\infty}}{2},$$

Yyyyy

Ita meo

Ita meo sensu rectius intimiusque perspicitur ratio non ingratae affectionis, ob quam, nullo Limite prohibente, Neutoniana Differentio-Differentialia, ad quemlibet Ordinem pertinentia, Sinuum aut Cosinum alterutrius Trigonometriae se mutuo repetant, alternis vicibus Fluents ipsos regenerent, et iisdemmet Curvis-Lineis veluti renascentibus novam originem praebeant ultra humani conceptus nimis angusta confinia.

370. Discrimen autem non leve Curvas Sinuum, ac Cosinum Circularium ab illis separat Hyperbolicorum, quas *analogas* nuncupare non videtur inopportunum. Porro, si in Axe Abscissarum Circulares Sinus y , aut, quod idem est, Cosinus $\sqrt{1-y^2}$ adnumerentur, fiet Arcus vel Ordinatum-Applicata

x Infinite diversorum valorum capax: namque $x = y + \frac{1 \cdot y^3}{2 \cdot 3} + \frac{1 \cdot 3 \cdot y^5}{2 \cdot 4 \cdot 5} +$

$$\frac{1 \cdot 3 \cdot 5 \cdot y^7}{2 \cdot 4 \cdot 6 \cdot 7} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot y^9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot y^{11}}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 11} + \&c: = \frac{1}{\sqrt{-1}} L (\sqrt{1-y^2} + y \sqrt{-1})$$

$$= \frac{\infty}{\sqrt{-1}} ((\sqrt{1-y^2} + y \sqrt{-1})^{\frac{1}{\infty}} - 1) = v \pm n\pi, \text{ dum } v \text{ Arcuum mi-}$$

nimus, iuxta numerorum 295. et 296. praecepta, subintelligatur. Curva sane Cosinum eandem adamussim Figuram induit ac ea Sinuum-Rectorum; adeo, ut, quidquid de postrema differant Geometrae, alteram quoque complecti neminem lateat. Hanc Curvarum Sinuum et Cosinum aequipollentiam nedum evincit sermo nuper institutus, atque natura Trigonometriae Circularis, verum etiam Calculus ipse confirmat. Si etenim libeat Abscissas Curvae Cosinum indigitare, uti adsolet, sub unico grammate z , facillime constat Abscissarum z novum Axem, ac earum supputationis initium transferri debere in parte positivorum super Ordinatarum Axem per longitudinem Quadrantis Radio 1 descripti, et Aequationem inter Arcum x , Cosinumque

$$z \text{ tali modo exhiberi } x = \frac{\infty}{\sqrt{-1}} ((z + \sqrt{1-z^2} \cdot \sqrt{-1})^{\frac{1}{\infty}} - 1) =$$

$$\frac{1}{\sqrt{-1}} L (z + \sqrt{1-z^2} \cdot \sqrt{-1}) = \frac{1}{1-z^2} + \frac{1 \cdot 1-z^2}{2 \cdot 3} + \frac{1 \cdot 3 \cdot 1-z^2}{2 \cdot 4 \cdot 5} +$$

$$\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}.$$

$$\frac{1 \cdot 3 \cdot 5 \cdot 1 - z^2}{2 \cdot 4 \cdot 6 \cdot 7}^{\frac{7}{2}} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 1 - z^2}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9}^{\frac{9}{2}} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 1 - z^2}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 11}^{\frac{11}{2}} + \&c. \text{ Haec}$$

autem luculentissime decernit z nulescenti respondere Arcum-Ordinatam $1 +$

$$\frac{1}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 11} + \&c., \text{ scilicet } \frac{\Pi}{4}, \text{ quum}$$

e-contra Sinu y evanescente Arcus simul in Nihilum abeat. In Curva vero Sinuum Aequilaterae Hyperbolae Apolloniana fit Arcus-analogus x'

$$= y - \frac{1 \cdot y^3}{2 \cdot 3} + \frac{1 \cdot 3 \cdot y^5}{2 \cdot 4 \cdot 5} - \frac{1 \cdot 3 \cdot 5 \cdot y^7}{2 \cdot 4 \cdot 6 \cdot 7} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot y^9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 9} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot y^{11}}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 11} + \&c. =$$

$$L(\sqrt{y^2+1} + y) = \infty \left((\sqrt{y^2+1} + y)^{\frac{1}{\infty}} - 1 \right) = v' \pm n\Pi\sqrt{-1}; \text{ nimirum,}$$

Abscissae cuilibet y unica Realis convenit Ordinata v' , Lineam-Rectam peraequans, quae cum Semi-Latere Transverso 1 constituat Rectangulum aequalem Areae Hyperbolicae determinanti Naturalem Logarithmum Numeri dati $\sqrt{y^2+1} + y$, ceteris Ordinatis abeuntibus in Imaginarias. Mirabile autem occurrit symptoma, quod eadem expressio $\pm n\Pi$, quae innumeris Ordinatis, praeter primam v Realem, in flexuosa Sinuum Circularium Linea originem elargitur, oculis quoque subiiciat in Imaginariam versa $\pm n\Pi\sqrt{-1}$ Impossibilium sine numero Ordinarum Curvae Sinuum Hyperbolicorum expressionem, ac qualitatem; ita, ut initio Abscissarum $y = 0$ sint in prima Ordinatum-Applicatae Reales $\pm n\Pi$, et in altera $\pm n\Pi\sqrt{-1}$, cuius phoenomeni intima causa simul cum praecedentis ratiocinationis totius fundamentis ex Capite V^o. petenda. Demum Aequatio $x = L(\sqrt{y^2+1} + y)$, vel $C^x = y + \sqrt{1+y^2}$, aut demum $C^{2x} - 2yC^x - 1 = 0$ satis abundanter ostendit Sinuum Hyperbolicorum Lineam Logarithmicis Curvis latissimo sensu acceptis adscribi debere, quarum constructio ab Hyperbolae inter Asymptotos Quadratura, et a fufius expositis in num^o. 353. deduci facile possit. Pari methodo Linea Hyperbolicorum Cofinum in Aequatione $x = L(z + \sqrt{z^2-1})$, vel $C^x = z + \sqrt{z^2-1}$, five $C^{2x} - 2zC^x + 1 = 0$ refertur ad ipsam Logometriam, cuius doctrinae argumentum aliis Trigonometricis Functionibus ope Curvarum repraesentandis amplificare non difficile foret.

371. Quemadmodum Hyperbolae contigit absque Trianguli-characteristici, aut aliorum conficiendi Schematis elementorum consideratione, ita etiam simili pacto Imaginaria Pseudo-Aequatio Circuli Logometrica praestabit, quae in numero 356., commodiora Theoremata. Sed ratum antea firmumque sit, Aequationem $x\sqrt{-1} = L(z + \sqrt{1-z^2} \cdot \sqrt{-1}) = L(z + \sqrt{z^2-1})$, Imaginario Circulo addictam, non Circulo, sed Hyperbolae Reali Aequilaterae referri, et Differentiatione facta non, uti vulgo putatur, perfectum Realem Circulum tribuere $dx\sqrt{-1} = dL(z + \sqrt{z^2-1})$

$$= dz + \frac{zdz}{\sqrt{z^2-1}} = \frac{dz}{\sqrt{z^2-1}}$$
, scilicet $dx = \frac{dz}{\sqrt{1-z^2}}$, quia vera Circuli Aequatio est $dx = \frac{-dz}{\sqrt{1-z^2}} = \frac{-d\text{Cofin. C. } x}{\text{Sin. C. } x}$ ad normam numeri 359.; quae

dicta obiter sint, ne Theoriae in Capite V°. tot argumentis firmatae Differentialia nunc recensita, et recensenda opponantur. Ad hoc, ut melius tutiusque de re gravissima constet, examini subiiciatur altera Imaginaria Logometricae Aequationis forma $x\sqrt{-1} = L(y\sqrt{-1} + \sqrt{1-y^2})$, vel $dx\sqrt{-1}$

$$= dy\sqrt{-1} - \frac{ydy}{\sqrt{1-y^2}} = \frac{dy\sqrt{-1}(\sqrt{y^2-1} - y)}{(y\sqrt{-1} + \sqrt{1-y^2})(\sqrt{y^2-1})}, \text{ et ideo } dx = \frac{dy}{\sqrt{1-y^2}} \text{ verum}$$

Circulum exhibens. Istum autem Fluxionem Logarithmicam determinandi modum nulla Lex Calculi finit, eo quod, ut reapse stricto Logometrico Algorithmo foret locus, existere semper deberet Proportio Geometrica $dN:dL::$

N : ad *Constantem* Magnitudinem, nimirum nostra in hypothesi $\frac{(dy\sqrt{-1})}{\sqrt{1-y^2}} \times$

$(\sqrt{y^2-1} - y): dx\sqrt{-1}:: y\sqrt{-1} + \sqrt{1-y^2}$: ad *Quantitatem Constantem* Imaginariam, quum ex adverso inita Differentiatio pro quarto Analogiae termino Realem Unitatem tacito sensu supponat. Est itaque absurda Logometriae rite intellectae applicatio ad Circuli naturam, et symptomata definienda; ac solummodo, dum Calculi Fitione Legem eandem Differentiandi in Imaginariis Logarithmis Hyperbolicis Imaginariorum Numerorum teneamus perinde ac si haberent Modulum aut Subtangentem Realem 1, quam certe, nisi universo Logometriae Algorithmo a fundamentis concusso, non habent,

habent, hac arte in maiorem contrahitur elegantiam Circulares adfectiones scribendi forma, compendiosior evadit Cyclometria, methodusque obtinetur; etsi mendoso subnixa principio, quae nec intime, nec substantialiter, sed quodam permissi erroris artificio vel Pseudo-Differentiationis Logometricae ope, et adparenter duntaxat, Circulum Hyperbolae unica Formula Analytica nectat. Qui haec omnia cum iampridem adfertis in num°. 172. ac nonnullis sequentibus comparaverit, nusquam se ab Imaginaria ista Logometria illaqueari patietur, sed fictione in Calculis peragendis affatim comoda libenter utens ab intima rei veritate artificium Analystarum secernet, aeternumque res diffitas, nec eadem Formula Analytice *Continua* $x = L(v + \sqrt{v^2 - 1})$, vel $x \sqrt{-1} = L(\pm v \mp (\sqrt{1 - v^2}) \sqrt{-1})$ invicem coniunctas sibi suadebit Circulum, atque Hyperbolen. Recessit a primum suppositae Continuitatis, equidem peringeniosa, sententia Foncenexius vel ipse in *Taurinensium Philosophiae ac Matheseos Miscellaneorum* posteriore Volumine, Societatis illius Acta annorum MDCCLX. ac LXI. complectente; semperque, uti solet, feliciori mentis acumine in eam abiuravit perspicacissimus Dalembertius (*). Quumque consimili facilitate, qua nos Finita tractantes summo rigore demonstravimus in num°. 178. completam Formulae, cuius nunc mentio, expressionem non esse $C^x = v + \sqrt{v^2 - 1}$, veruntamen $\pm C^x = v + \sqrt{v^2 - 1} = \text{Cofin. H. } x + \text{Sin. H. } x$, liquido constiterit mutilam mancamque futuram analogae etiam Formulae derivationem $C^{x\sqrt{-1}} = v + \sqrt{v^2 - 1} = v + (\sqrt{1 - v^2}) \sqrt{-1} = \text{Cofin. C. } x + (\text{Sin. C. } x) \sqrt{-1}$, nisi plenius ita $\pm C^{x\sqrt{-1}} = \text{Cofin. C. } x + (\text{Sin. C. } x) \sqrt{-1}$ exponatur, hac nova arte nanciscimur $\pm dx = \frac{dv}{\sqrt{1 - v^2}}$, sive potius $dx = \frac{\mp dv}{\sqrt{1 - v^2}}$ omnibus numeris absolutam Aequationem inter Arcum Circularem x , eiusque Cofinum v , cui supra adhaesos naevos aliquos, nec non deficientem perfectionem quamdam summo iure adnotavimus.

Z z z z z

372. Incoepum

(*) Consulantur praesertim *Additiones* post sextum praecitatum Opusculum Mathematicum in ipsorum Collectionis Volumine I°. ubi de pristina differit opinione Foncenexii, et LaGrangii.

372. Incoeptum prosequens argumentum demonstratas Aequalitates in usitatiorem redigo formam :

$$S \frac{dz}{\sqrt{-1}(\sqrt{1-z^2})} = L(z + \sqrt{z^2-1}) = L(z + (\sqrt{1-z^2})\sqrt{-1}); \text{ quae sunt ad Circulum translatae,}$$

$$S \frac{dv \sqrt{-1}}{\sqrt{1-v^2}} = L(v \sqrt{-1} + \sqrt{1-v^2}) = L(v \sqrt{-1} + (\sqrt{v^2-1})\sqrt{-1}) \text{ ubi } z \text{ Cofinus} < 1, \text{ et } v \text{ Sinus} < 1.$$

Prior Formula adamussim coincidit cum priore in num°. 356. prolata, nec posterior ab illa intime differt, tametsi specialius Reali Circulo contemplando videatur idonea. Et re quidem vera, suffecto $v \sqrt{-1}$ loco z in Aequationum prima, emergit altera

$$S \frac{dv}{\sqrt{1+v^2}} = L(v \sqrt{-1} + \sqrt{1+v^2} \cdot \sqrt{-1}), \text{ ex vul-$$

gariter assumptis rei Logometricae Canonibus, analogae superius dictae

$$S \frac{dv \sqrt{-1}}{\sqrt{1-v^2}} = L(v \sqrt{-1} + \sqrt{1-v^2}). \text{ Poterit etiam ita prior Aequalitas con-$$

cinnari:

$$S \frac{dz \sqrt{-1}}{\sqrt{z^2-1}} = \sqrt{-1} \cdot L(z + \sqrt{z^2-1}) = L((z + \sqrt{z^2-1})^{\sqrt{-1}}) =$$

$$L((\text{Cofin. H. } x + (\text{Sin. H. } x))^{\sqrt{-1}}) = L(\text{Cofin. H. } (x \sqrt{-1}) + \text{Sin. H. } (x \sqrt{-1})).$$

Haec Formularum celebriorum conlatio iterum monet, quam parum in Circulo primae expressioni Analyticae fidendum sit, quae revera Hyperbolam directe magis refert Aequilateram, et quanta facilitate diducantur

$$\text{Arcus Analogus Hyp. } x = S \frac{dz}{\sqrt{-1}(\sqrt{1-z^2})}, \text{ subindeque } S \frac{dz}{\sqrt{1-z^2}} = x \sqrt{-1} \text{ Pseudo-Arcui, quem placuit Circularem nuncupare,}$$

$$= S \frac{dv \sqrt{-1}}{\sqrt{v^2-1}}, \text{ quod idem est ac } S \frac{dv \sqrt{-1}}{\sqrt{1-v^2}} = x \sqrt{-1}, \text{ cui Imaginario nomen Arcus Circuli impositum.}$$

Igitur Sinu-Recto Arcus x existente v Confectaria praecipua Fictionis expositae sunt ea, quae consequuntur, nempe

$$dx =$$

$dx = \frac{dv}{\sqrt{1-v^2}}$, quae Aequalitas, posito Cofinu $\sqrt{1-v^2} = z$, evadet dx

$$= \frac{-dz}{v} = \frac{-dz}{\sqrt{1-z^2}}; \text{ et praeterea factis } y \text{ Verso-Sinu, } V \text{ Tangente, } \Theta$$

Co-Tangente, Φ Secante, atque Ψ Cofecante provenient

$$1-y=z; -dz=dy; 1-z^2=2y-y^2; dx = \frac{dy}{\sqrt{2y-y^2}}; \text{ videlicet}$$

$$V = \frac{\sqrt{1-z^2}}{z}; \frac{-dz}{\sqrt{1-z^2}} = dV + \frac{Vdz}{z} = dV - \frac{V^2 dV}{1+V^2} = \frac{dV}{1+V^2}, \text{ scilicet } dx = \frac{dV}{1+V^2};$$

$$\Theta = \text{Tan. C. } \frac{\Pi}{4} - x, \text{ ideoque } -dx = \frac{d\Theta}{1+\Theta^2}, \text{ vel dicas } dx = \frac{-d\Theta}{1+\Theta^2};$$

$$\Phi = \frac{1}{z}, -dz = \frac{d\Phi}{\Phi^2}, \frac{-dz}{\sqrt{1-z^2}} = \frac{d\Phi}{\Phi^2} \left(\frac{\Phi}{\sqrt{\Phi^2-1}} \right) = \frac{d\Phi}{\Phi\sqrt{\Phi^2-1}} = dx;$$

$$\Psi = \text{Sec. C. } (Q-x), \text{ quamobrem } -dx = \frac{d\Psi}{\Psi\sqrt{\Psi^2-1}}, \text{ nimirum } dx = \frac{-d\Psi}{\Psi\sqrt{\Psi^2-1}}.$$

Nihil addendum censeo de Arcuum x Chordis T, vel Subtensis Complementaribus Γ : namque ex praemissis $T = 2 \text{ Sin. C. } \frac{x}{2}$, aut $\text{Sin. C. } \frac{x}{2} = \frac{T}{2}$,

$$\text{unde oritur } dx = \frac{dT}{\sqrt{1-\left(\frac{T}{2}\right)^2}}; \text{ nec non, propter } \Gamma = \text{Chord. C. } \frac{\Pi}{2} - x$$

$$= 2 \text{ Sin. C. } \left(\frac{\Pi}{4} - \frac{x}{2} \right) = 2 \text{ Sin. C. } \left(Q - \frac{x}{2} \right), \text{ erit } dx = \frac{-d\Gamma}{\sqrt{1-\left(\frac{\Gamma}{2}\right)^2}},$$

mutato tantum quo ad alteram Formulam Signo, uti in Cofinibus Sinuum respectu usuenit.

373. Ii, quibus Circuli Radium seu Transversum Hyperbolae Semi-Axem explicitum in animo sit contemplari, in retroactis omnibus Formulis vota nullo labore adimplebunt subnexum intuentes Elenchum.

Circulus

Circulus.

Rad. ^{ius} seu S. ^{emi} Axis.

Hyperbole.

Sinus v

$$dx = \frac{R dv}{(R^2 - v^2)^{\frac{1}{2}}}$$

$$dx = \frac{R dv}{(v^2 + R^2)^{\frac{1}{2}}}$$

Cofinus z

$$dx = \frac{-R dz}{(R^2 - z^2)^{\frac{1}{2}}}$$

$$dx = \frac{R dz}{(z^2 - R^2)^{\frac{1}{2}}}$$

Tangens V

$$dx = \frac{R^2 dV}{R^2 + V^2}$$

$$dx = \frac{R^2 dV}{R^2 - V^2}$$

Cotangens Θ

$$dx = \frac{-R^2 d\Theta}{R^2 + \Theta^2}$$

$$dx = \frac{-R^2 d\Theta}{\Theta^2 - R^2}$$

Secans Φ

$$dx = \frac{R^2 d\Phi}{\Phi(\Phi^2 - R^2)^{\frac{1}{2}}}$$

Cofecans Ψ

$$dx = \frac{-R^2 d\Psi}{\Psi(\Psi^2 - R^2)^{\frac{1}{2}}}$$

Sin. Ver. y

$$dx = \frac{R dy}{(2Ry - y^2)^{\frac{1}{2}}}$$

$$dx = \frac{R dy}{(2Ry + y^2)^{\frac{1}{2}}}$$

Chord. T.

$$dx = \frac{R dT}{(R^2 - \frac{T^2}{4})^{\frac{1}{2}}}$$

Chor. Com. Γ

$$dx = \frac{-R d\Gamma}{(R^2 - \frac{\Gamma^2}{4})^{\frac{1}{2}}}$$

Cofin. Ver. Y

$$dx = \frac{-R dY}{(2RY - Y^2)^{\frac{1}{2}}}$$

Univerfae

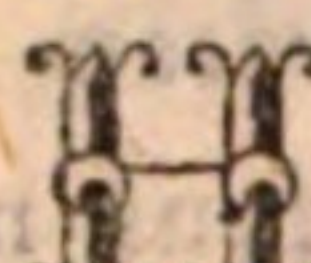
Universae expressiones hoc modo plenius concinnatae Coronidum instar emergunt a superius inventis, dummodo Lex Homogeneitatis, semper subintellecta, servetur.

374. Trigonometricam Doctrinam perinsignium Theorematum foecundissimam penitus evacuare impossibilitatis notam praesefert. Haftenus dicta sufficiant, ut de ea iuxta promissam simplicitatem etiam in Calculo Fluxionum Fluentiumque tractanda nulla maneat suspicio. Quibus principiis praecognitis omnes ferme difficultates solventur, quae non paucos torsere Geometras. Exemplo sit longior, quam par fuisset, dubitatio circa veram Lineae Sinuum, aut Curvae Sphaero-Cylindricae Formam. Viderant ocyus nonnulli, Sinuum-Rectorum Lineam in Plano genitam innumeris constare Ramis, ac sese porrigere in Infinitum. Istam Ramorum Infinities multiplicium repetitionem confirmabat et ipsa Trochois, innumeris aequae praedita Ramis, nec non eius Comes, quae Sinuum-Versorum Curva vocatur. Ex adverso alii noverant, Sinuum Lineam, atque Sphaero-Cylindricam super Cylindrum depictam unica in se redeunte Perimetro circumscribi. Quidni, aiebant, nativa genesis Curvae in Superficie Recti Cylindri cum Trigonometrica veritate consentit? Erroris species in eo sita erat, quod postremi Lineam Sinuum aut Chordarum in Cylindro non concipiebant ex infinitis gyris existentem, quamvis ita perfecte superplicatam, ut unicam circumvolutionem mentiretur. His enim plicis expansis, simul cum exfoliata pariter in Infinitum Superficie Cylindrica, ac Baseos Peripheria, Curva Sinuum acquirit in Plano innumeros illos Ramos Inflexionum Punctis distinctos, quos sub Pseudo-Perimetro unius Ellipseos, vel quasi-Ellipseos, super Cylindrum iacens celabat. Haec autem palmaria est, atque oecumenica Linearum omnium in se redeuntium, uti et iis insistentium omnium Superficierum proprietas.



CAPUT IX.

ILLUSTRATIO CELEBRIS CONTROVERSIAE
CIRCA QUANTITATUM NEGATIVARUM LOGARITHMOS.

375. istoria certaminis, si non acerrimi, saltem diuturni, quod Saeculo hoc ipso labente plerosque Analystarum divisit, Negativorum Logarithmos investigantes, adeo percrebuit, ut, praeter nonnullos Schediasmatum Titulos celeris calami ductu e regione adnotatos, ab illa censeam nunc abstinendum (*). Sit satis superque quaestionis, fortasse nimium per ora hominum adhuc

(*) *Commercium Philosophicum ac Mathematicum* primam sistit Epocham quaestionis huiusce Analyticae, inter Amicos Leibnitium, et Ioannem Bernoullium anno MDCCXII^{mo}. longius actae, nimirum a Martio ad Iulium fere totum, variis in Epistolis ab illius Operis II^o. Volumine comprehensis. Lis itidem Epistolaris de ipsomet argumento Leonardum Eulerum Leibnitii partes tenentem, ac Dalembertium Bernoullii, vertentibus annis MDCCXLVI^{mo}. VII^{mo}., sive XLVII^{mo}. VIII^{mo}., detinuit. Sententiam in Epistolis latam protulit in lucem Eulerus anno MDCCLI^{mo}. inter *Memorabilia Scientiarum Academiae Berolinensis* ad annum MDCCXLIX^{um}. relata; sed quaedam ante postremi istius anni lapsum attigerat ad rem facientia in Cap. 21^{mo}. Voluminis IIⁱ. *Introductionis in Analysin Infinitorum*. Valmesleyus in praecitatis *Memorabilibus*, labente anno MDCCLV^{mo}. editis, Theoremata omnia Euleriana confirmavit ope Calculi Infinitesimalis. Propriam mox evulgavit opinionem Dalembertius, tam anno MDCCLXI^{mo}. in sexto *Opusculorum Mathematicorum* Voluminis Iⁱ., tam in alia obiter Disquisitione Voluminis Vⁱ. anno MDCCLXVIII^{mo}. Horumce cogitatorum excerptum Auctor ipse in *Encyclopediae Parisiensis* Articulo *τὴ Logarithmi* sub anni finem MDCCLXVI^{mi}. impertierat. Eodem hoc anno in Scarellae, nescio cuius, *Commentariis*

adhuc volitantis, epitomen nosse in eo sitam, quod Leibnitius negavit Reales Negativarum Magnitudinum Logarithmos, ex adverso eos adfirmante Ioanne Bernoullio. Subtilia equidem hinc inde prolata argumenta novis accessionibus ditantur Analysis: sed, uti assolet, quum peringeniosi Viri pro sua quilibet opinione, perinde ac si pro aris et focis, pugnantes Eruditorum Iudicium sibi felix faustumque optant, a rationum adsertarum pondere ac varietate difficilior, periculique et dubitationis plenissima quaestio ipsa adhuc nos manet. Dum spes adsit aliqua eam dirimendi, nonnisi purissimos Logometriae latices qui supra libaverit experiri unquam poterit. Saepius enim methodorum obscuritati vitioque dandum est parum in enodandis clare, et ad rem apte gravissimis difficultatibus felicitatis; quas ideo si melioribus fortuna vero auspiciis, non mihi pusillo Geometrae, sed Neutonianio Binomio novus cumulabitur honos.

376. Magnitudinum Negativarum Logarithmos in Systemate dumtaxat Hyperbolico, vel Modulo - Unitate contemplari idem profecto erit ac omnium Systematum, Modulorumve simul complecti universitatem. Nec, ut mihi gradum faciam ad argumenti tractationem, e-longinquo petendum opinor dubium Geometris Leibnitianis subortum de impossibili Rationum conceptu $\tau\frac{1}{2} + 1 : -1$, et generalius $+m : -n$, unde vel ipso in limine Iudicii autumant definitam Logarithmorum impossibilitatem, qui ad numeros Negativos pertineant

his animadversiones adparuere nonnullae; at plenius sublimiusque Imaginarias versantes Magnitudines controversiam inlustrare, et prope dirimere conati fuerant Ludovicus Lagrangius, atque Eques Foncenexius in *Miscellaneis Philosophico-Mathematicis Taurinensibus* annorum MDCCLIX^{mi}. LX^{mi}. ac LXI^{mi}. Fertur quoque Karstenium, Halae-Saxonum Matheseos Doctorem, apodictice in hac palaestra sese exercuisse, totamque istam Theoriam fusius in Volumine V°. *Commentariorum Bavaricae Electoralis Academiae* iuxta Leibnitii atque Euleri placita pertractasse. Aliqua demum habent Riccati, et Saladinii *Analyticae Institutiones* Editionis Bononiensis, nec non Opellae in num°.

173^{mo}. memoratae, quibus ansam dedit earum *Institutionum* Iudicium Mutinae publici iuris factum in *Diario Litteratorum Italico* anni MDCCLXXIII^{mi}, et adversus ipsum vulgata Riccati Epistola in nova Veneta Collectione *Opusculorum Scientificorum ac Philologicorum* sub annum MDCCLXXVI^{um}. Plura minoris pretii recensens Lectoris otium fallere non immerito existimarer.

tineant (*). Enimvero, quum Quantitatum Ratio nec stare nec concipi possit praeterquam in earum *Magnitudinis*, non situs, non qualitatis, non plurium existendi modorum, qui in Locis apparent Geometricis, comparatione, nescio quo iure Logometriae Negativorum illi labefactasse sibi suaserint Fundamenta, atque in imam Problematis evolvendi radicem immedicabile vulnus tulisse. Absit a sincero Veritatis cultu quisquis Magnitudinum Negativarum *entitatem* veluti quandam, genusve a Positivis distinctum effingere audeat, reiue luculentissimae statutum ordinem perturbare fallacibus ideis denuo excitatis, quae horrendum olim pepererunt Monstrum Nihilominorum, et Infinito-maiorum.

377. Seclusis igitur iis, quae, vel absurda, vel inutilia, et moram dicendis afferrent, et prope captata obscuritate evidentiam ipsam inficerent, libet potius quiddam de triplicis Magnitudinum *Continuitatis* discrimine praefari, ne singularis verborum delectus, et a ceteris Scriptoribus leviter attacta nonnullarum vocum significatio germano Veritatis sensui infidientur. Apte hoc nimirum, ac sine temporis iactura: tota etenim rerum summa gemino cardine sustinetur, Unitatis videlicet Lineae seu Loci ab

Aequatione expressi $\pm C^x = y$, Realisque valoris Formulae $\frac{\sqrt{x-1}}{1} - \frac{\sqrt{x-1}^2}{2} + \frac{\sqrt{x-1}^3}{3} - \frac{\sqrt{x-1}^4}{4} + \frac{\sqrt{x-1}^5}{5} - \frac{\sqrt{x-1}^6}{6} + \&c:$ etiam dum $+v$

transilit ad $-v$, subindeque in regione Negativorum analytice progreditur; quae duo maximam partem a Continuitate rite intellecta dependent. Quamvis insolens sit in Mathematicis Disciplinis dubitationi obnoxia vocabulorum potentia, nihilo tamen minus ea profecto est investigatio nunc ob oculos

(*) Guillelmi Leibnitii, etiam in re Mathematica grandiloquam quandoque, ac sibi potissimum charam Metaphysicen peramantis, speculatio huiusmodi exstat in *Actis Eruditorum Lipsiensibus* anni MDCCXII^{mi}. Non absimilis videtur in *Novis Geometriae Elementis* Antonii Arnaldi sententia fatentis, ultra ingenii sui vires repositam esse Proportionem hanc Continuum Geometricam $+1:-1:+1:-1:\dots$. Quae Vallisius, Grandius, Fontenellius &c.; ceteroquin eruditissimi, de *Plusquam-Infinitis* adinvenisse censuerunt, ferme omnia huic innituntur labili Negativorum uti Quantitatum sui generis considerationi.

los posita, quae studio ac cura praeter morem abundantiore institui debeat, ut cuncta sibi consentiant ex sparsis superius elementis in unum colligenda, atque maxima effulgeat perspicuitas, in illis praesertim a Finitorum penu, ex limpidissimo scilicet fonte, desumptis.

378. *Arithmetica* Continuitas, vulgari saltem modo recepta, subsultim incedit, et Negativas excludit penitus Magnitudines, communi cuilibet Numerico Problemati resolvendo sane inidoneas. *Analytica* vero, et *Geometrica* Continuitas Negativos aequae ac Positivos valores censui subiiciunt, quippe quae nedum habent rationem Numeri, uti in se est, absoluti, sed etiam exprimere potis sunt ipsius modificationem, aut, in Continuo diceret non inepte, situs convenientiam, ob quam fit, ut nonnullae Magnitudines ab eodem computationis puncto in contraria veluti actae minuendi vi gaudeant, interea dum aliae ex adverso amplificandi; nonnullae dextrorsum, sursum; aliae contra sinistrorsum, deorsum existant, et discriminatim notentur. Haec autem diversitas interest, quod Formula Numerum repraesentans perfectam Continuitatem servet *Analyticam* dummodo sensim, per gradus, et per intervalla quamminima ex uno in alterum ferri possit valorem; sed ad Continuitatem *Geometricam* strictim diiudicandam accedat saepius oportet, praeter valorum sine ullo saltu progressum, Unitas Formulae illius, quae nempe adeo fit comparata, ut exstet Rationalis, at in plures Rationales Factores nec dispesci, nec dividi queat. Facile, neque immerito, ex dictis adparet artis periculum si in Curvis Algebraicis aestimare velis tutissime, an sint unum Continuum Geometricum, an potius plurium Continuatorum Systema; quod de Transcendentibus Curvis iudicium non dispar ab Algebraicarum solummodo Continuitate Geometrica, ex quarum Tetragonismo dimanant, caute erit atque prudenter hauriendum (*). Cave a Seriebus, etiamsi Rationalibus, numero terminorum Infinitis in hoc Iudicio sistendo, nisi aliunde constet eas germanam puramque noscere originem, integras esse, nec falsas aut sine iure neglectis Residuis admodum perturbatas: et haec, si fauste contigerint omnia, albo erunt signanda lapillo.

379. Quae diversicoloris Continuitatis confinia posita, ut ea, qua decet, sedulitate regantur, neve praeceptis nimium ingenii conceptus suum

B b b b b b

cuique

(*) Relege calcem numeri 178^{mi}.

cuique Ius quandoque non tribuat in re summae lubricitatis atque periculi, exemplum unum aut alterum clariore in luce locabit. Sunt aliquae interea de Continuitate ipsa minus obviae, nec invenustae advertendae Leges, quibus, si otium suppeteret, plures etiam dicare pagellas, non sine usu perquammaximo, censuissem. Terminus Generalis, ut aiunt, Seriei Numerorum Triangularium est $\frac{x^2+x}{2}$, Pyramidalium $\tau\delta \frac{x^3+3x^2+2x}{6}$; parique ritu ac facilitate

C

&c:	45	36	28	21	15	10	6	3	1	0	0
&c:	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	0
&c:	1	1	1	1	1	1	1	1	1	1	1

Differentiae

secun-

D

&c:	-120	-84	-56	-35	-20	-10	-4	-1	0	0	0
&c:	36	28	21	15	10	6	3	1	0	0	0
&c:	-8	-7	-6	-5	-4	-3	-2	-1	0	1	1
&c:	1	1	1	1	1	1	1	1	1	1	1

Differentiae

ter-

Licet insuper, haudquaquam praetergresso strictae Arithmetices Limite, intervalla Numerorum minuere aliis medio

G

&c:	15	$\frac{99}{8}$	10	$\frac{63}{8}$	6	$\frac{35}{8}$	3	$\frac{15}{8}$	1	$\frac{3}{8}$	0	$-\frac{1}{8}$	0
&c:	$-\frac{21}{8}$	$-\frac{19}{8}$	$-\frac{17}{8}$	$-\frac{15}{8}$	$-\frac{13}{8}$	$-\frac{11}{8}$	$-\frac{9}{8}$	$-\frac{7}{8}$	$-\frac{5}{8}$	$-\frac{3}{8}$	$-\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
&c:	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Differentiae

secun-

H

&c:	$-\frac{1287}{48}$	-20	$-\frac{693}{48}$	-10	$-\frac{315}{48}$	-4	$-\frac{105}{48}$	-1	$-\frac{15}{48}$	0	$\frac{3}{48}$	0	$-\frac{3}{48}$	0
&c:	$\frac{327}{48}$	$\frac{267}{48}$	$\frac{213}{48}$	$\frac{165}{48}$	$\frac{123}{48}$	$\frac{87}{48}$	$\frac{57}{48}$	$\frac{33}{48}$	$\frac{15}{48}$	$\frac{3}{48}$	$-\frac{3}{48}$	$-\frac{3}{48}$	$\frac{3}{48}$	$\frac{3}{48}$
&c:	$-\frac{10}{8}$	$-\frac{9}{8}$	$-\frac{8}{8}$	$-\frac{7}{8}$	$-\frac{6}{8}$	$-\frac{5}{8}$	$-\frac{4}{8}$	$-\frac{3}{8}$	$-\frac{2}{8}$	$-\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{2}{8}$
&c:	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Differentiae

ter-

facilitate se quaerenti praeberet in Triangulo-Triangularibus, Triangulo-Pyramidalibus, et ceteris Numeris Figuratis, quin etiam et Polygonis absque ullo Limite. Dum communioris Arithmeticae angustos intra cancellos manere libeat, emergunt primum, commodi ergo variis seiunctae Litteris A, B, Progressiones, quales in Trigono suo Pascalius, et partim ante ipsum Maurolycus Messanenensis consideravit,

A

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 28 \quad 36 \quad 45 \quad \&c:$$

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad \&c:$$

$$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \&c:$$

dae aequales,

B

$$1 \quad 4 \quad 10 \quad 20 \quad 35 \quad 56 \quad 84 \quad 120 \quad \&c:$$

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 28 \quad 36 \quad \&c:$$

$$2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad \&c:$$

$$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \&c:$$

tiae aequales.

infertis, quales intercalavit Vallisus, Litterisque E, F clarius ostendit hoc Paradigma,

E

$$\frac{3}{8} \quad 1 \quad \frac{15}{8} \quad 3 \quad \frac{35}{8} \quad 6 \quad \frac{63}{8} \quad 10 \quad \frac{99}{8} \quad 15 \quad \&c:$$

$$\frac{3}{8} \quad \frac{5}{8} \quad \frac{7}{8} \quad \frac{9}{8} \quad \frac{11}{8} \quad \frac{13}{8} \quad \frac{15}{8} \quad \frac{17}{8} \quad \frac{19}{8} \quad \frac{21}{8} \quad \&c:$$

$$\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \&c:$$

dae aequales,

F

$$\frac{15}{48} \quad 1 \quad \frac{105}{48} \quad 4 \quad \frac{315}{48} \quad 10 \quad \frac{693}{48} \quad 20 \quad \frac{1287}{48} \quad \&c:$$

$$\frac{15}{48} \quad \frac{33}{48} \quad \frac{57}{48} \quad \frac{87}{48} \quad \frac{123}{48} \quad \frac{165}{48} \quad \frac{213}{48} \quad \frac{267}{48} \quad \frac{327}{48} \quad \&c:$$

$$\frac{3}{8} \quad \frac{4}{8} \quad \frac{5}{8} \quad \frac{6}{8} \quad \frac{7}{8} \quad \frac{8}{8} \quad \frac{9}{8} \quad \frac{10}{8} \quad \&c:$$

$$\frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{8} \quad \&c:$$

tiae aequales.

380. Nihil obstat Numerorum vacuis ex novorum accessione interpolandis, seu, quod eodem recidit, Indicibus x et Differentiis Aequalibus, secundi, tertii, quarti, quinti &c. Ordinis in Infinitum, semper semperque minoribus efficiendis, dummodo in dextrorsum a Limite positis Seriebus A, E, B, F contingant haec omnia, ne vulgatae Arithmetices pomperia unquam violentur. Quinimmo illa Indicum, Differentiarum, et Intervallorum attenuatio eo usque progredi poterit, quo, Vacuis omnino completis, atque Incommensurabilium etiam ac Transcendentium Indicum x ratione habita, Series tandem vertantur in Analyticas Aequalitates $x^2 + x - 2y = 0$, $x^3 + 3x^2 + 2x - 6y = 0$ &c., quas in Continuo imitari perpulchrum est, nec scitu inamoenum. Primus forsitan Geometrarum Series Algebraicas, et earum Reciprocas cuiusvis Ordinis in Parabolicis et Hyperbolicis Curvis, omni mensura ac numero definitis, singulatim depinxi: sed, exceptis Schematibus XXXIII°. ac XXXIV°. , improbum hunc adolescentis aetatis laborem ceteras inter quisquilias obscura semper in pulvere delitescere nauci puto aestimandum (*). Locus itaque Serierum A, E est Ramus KLM Parabolae Apolloniana, cuius Parameter-Linearis $NP = 2$, Punctumque K, vel Abscissarum aut Indicum $+x$ initium, distat ab Axe NQ per Semi-Ordinatam

$$KR = \frac{1}{2}, \text{ et a Vertice quota est Sagitta } RN = \frac{1}{4}. KR = \frac{1}{8}. \text{ Crus autem}$$

OST Parabolae Cubicae, seu Linearum tertii Ordinis postremae a Neutono, si non fideliter, acutissime tamen enumeratarum, Parametrum-Planam P

$$\text{habentis} = \frac{6^3}{37} = \frac{216}{37} = 5 \frac{31}{37}, \text{ Punctumque } \Theta, \text{ a quo Indices incipiant}$$

$+x$, ita situm, ut Tangente ducta ex Puncto Inflexionis Λ sit ei Norma-

$$\text{lis } \Theta\Delta = \frac{1}{\sqrt{37}}, \text{ et Tangentis ipsius Segmentum } \Lambda\Delta = 6. \Theta\Delta = \frac{6}{\sqrt{37}}, \text{ Lo-}$$

cum statuit Serierum B, F quomodocunque interpolatarum. Ii vero Loci, quorum Puncta *discreta* aequalibus super Abscissarum Axem Intervallis ad

Normam

(*) Nonnulla huiusce Theorices lineamenta duxit Carraeus occasione Problematis Trevolitensis, in *Actis Academiae Regiae Scientiarum*, quae Lutetiae-Parisiiorum est, pro anno MDCCI^{mo}.

Normam respondent, et Continuitatem graphice repraesentant Arithmeti-
cam Ordinatarum in Serie A, E Triangulare, vel B, F Pyramidalium, eva-
dunt Analytice Continui statim atque illa Puncta in Continuum abeant Pe-
rimetrum. Quo vixdum facto, Analytica ipsamet exigit Continuitas, ut Ne-

gativi quilibet Numeri accedant ad Terminos Generales $\frac{x^2 + x}{2}$,

$\frac{x^3 + 3x^2 + 2x}{6}$, aut sinistrorsum a Limite Series etiam C, G et D, H super-

veniant, Extremisque Ordinatarum in Curvilinea prope coeuntibus alter
Parabolarum Ramus KNYEV, ΘΛΑΥΦΣΖ cum primo simul existat.

381. Hac in Metamorphosi Arithmeticae Continuitatis, dum scilicet iu-
ra recipit, induitque regulas Analyticae, non pauca obiter sunt adnotanda.
Ac primum qui Series C, G vel D, H secum ipse volutaverit, illico sentiet,
quod, ut farta testis Differentiarum aequalium Lex maneat, nempe Serierum
ipsarum basis et unica causa, Subtractio Numerorum in dextra regione sae-
pissime Additio fiat sinistrorsum, ac vicissim, Nullitatesque repetitae Finitis
Numeris, uti Progressionum partes, intermisceantur necesse sit; quae omnia
peregrina sunt, et prope monstri similia in Arithmetice Disciplinis. Ad
haec nullus dubito, quin liquido constet transitum a dextris Numeris ad fini-
stros, seu Positivos seu Negativos, in illis Seriebus Nihilorum interpositorum
ope contingere, vel ex adverso Infinitorum in Triangulare, et Pyrami-
dali Reciproci, quarum Termini Generales fuerint $\frac{2}{x^2 + x}$, $\frac{6}{x^3 + 3x^2 + 2x}$,

istaeque Nihila, aut Infinita eo plura fore interiacentia, quotus ad unguem est
Aequalium Ordo Differentiarum. In Parabolicis Lineis ea Nihila *Nodi* sunt,
quos curvi separant *Ventres* alternatim positi, et Infinita sunt *Asymptoti* pa-
rallaelae, quas inter mediae stant *Hyperbolici Generis Curvae* alternis in pla-

gis iacentes. Ita, dum Parabola Schematis XXXIIIⁱⁱ. duos *Nodos* K, Ψ, uni-
cumque *Ventrem* habet KNY, Curva e-contra Reciprocorum Triangulare,
Aequatione praedita $x^2y + xy - 2 = 0$, eadem est atque Linea Hyper-
bolica tertii Ordinis, penes Neutonum sexagesima, nimirum ille *Hyperbolae-
Hyperbolismus*, in cuius generali Aequatione $x^2y + ex = cy + d$ sint $e = 0$,

$c = \frac{1}{4}$, $d = 2$, aut ipsemet, de quo mentio facta in num^o. 270^{mo}. uti Apol-

C c c c c c

lonianae

Ionianae Hyperbolae Quadratrice-Verforia. Nec difficilior videtur Linea e quarti Ordinis sine Diametro imago, Aequatione $x^3y + 3x^2y + 2xy - 6 = 0$ distinctae Reciprocorum Pyramidalium, quippe quae tribus gaudet Asymptotis invicem parallelis loco trium Nodorum Θ, Λ, Φ Parabolae Cubicae Schematis XXXIVⁱ, quartamque habet Asymptoton aliis normalem, et iconem parcius recedentem ab ea diligentissime meo quodam in Schemate expressa. Istud insuper moneo, Continuitatem rite callentibus non tanta visum iri circumseptum sublimitate quod olim miraculo proximum sensit in *Traſtatu de Triangulo Arithmetico* Pascalius, videlicet affectiones *Continui* ab illis plerumque Magnitudinis *Discretae*, toto coelo distitae, derivari (*).

382. Extra rem non puto contemplationem, quae propius admodum Logometriae pandit, etiam in Continuo, secreta. In elevationibus Parabolis et Hyperbolis spectantibus ad Terminos Generales Serierum Algebraicarum earumque Reciprocis, scilicet $x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + \dots + kx^3 + lx^2 + mx + n = r. y^{\pm 1}$, *Ventres* ac *Nodi*, visibiles aut invisibiles, totidem sunt, quot Numerus exprimit $n - 1$ atque n respective, totidemque ex adverso Hyperbolae inscriptae, quot pariter Unitates habet $n - 1$ vel Index maximae Potestatis Unitate minutus, et Asymptoti illarum, invicem parallelae, quot Index n Unitates, dummodo Aequationis propositae $x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + \dots + kx^3 + lx^2 + mx + n = 0$ Radices omnes fuerint Reales. Quum igitur, respectu habito ad unum saltem Logarithmicae Ramum, Aequatio Curvae $C^x = y$ repraesentari ita possit $1 + x^1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots + \frac{x^\infty}{1.2.3.4 \dots \infty} = \left(1 + \frac{x}{\infty}\right)^\infty = y$, eiusque Reciproca, sive *Anti-Logistica* $C^x.y = 1$ alterius ope Aequalitatis $y + xy + \frac{x^2y}{2} + \frac{x^3y}{6} + \frac{x^4y}{24} + \frac{x^5y}{120} + \dots + \frac{x^\infty y}{1.2.3.4 \dots \infty} = \left(1 + \frac{x}{\infty}\right)^\infty . y = 1$, erit in ipsa Logistica

(*) Non absurda equidem foret Continui definitio „*Quantitas infinite Discreta est Continua* „.

gistica Abscissarum x initium aequipollens innumeris *Nodis* ac *Ventribus* in unicum Contactus Infiniti Ordinis Punctum coeuntibus, nec non in *Anti-Logistica* initium idem Abscissarum erit Punctum adeo *Singulare*, ut unica Asymptotos innumeris Asymptotis parallelis, et inscriptis Hyperbolis aequivaleat. Quae Asymptotorum Crurumque Hyperbolae, *Nodorum* ac *Ventrium* fine numero in *Singularium* praestantissimo *Anti-Logisticae* et *Logisticae* Puncto superpositio cum praestabilita Theorice Parabolarum Limitis (*) omnino consentit.

383. Ne vero censeas Logarithmicam Curvam ope nunc memoratae

$$\text{Aequationis } 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} \dots + \frac{x^\infty}{2.3.4.5 \dots \infty}$$

— $y = 0$ Paraboloeidon, unica Ordinata Ramoque praedictarum, Familiae tuto et absque novi examinis instituendi periculo adscribi posse. Indubium equidem est (**) Seriem illam, cui $\tau\delta$ y exaequatur, fidelem sese convergentemque praebere. Origo eiusdem Seriei non poterat certe optari simplicior, ac prope oculis aestimata, quum a Binomio Neutoni $\left(1 + \frac{x}{\infty}\right)^\infty$

nec hilum recedat. Veruntamen non integra est; nam ex toties praefatis

$$\text{ita componi deberet Aequalitas } 1^x \left(1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5} \dots + \frac{x^\infty}{2.3.4.5 \dots \infty} \right) - y = 0, \text{ seu brevius } 1^x \left(1 + \frac{x}{\infty} \right)^\infty - y$$

$= 0$ ad totum simul complectendum Transcendentem Logisticae Locum. Additus iste Coefficientens 1^x , cuius Index x nedum Irrationalis, quin etiam saepenumero Transcendens fit, atque Coefficientis ipsius valorem reddit Infinitiformem Cofin. C. $nx\Pi \pm (\text{Sin. C. } nx\Pi) \sqrt{-1}$ (***), quaedam requirit de Continuitate subtilius, Anatomicorum ad instar, introspicienda.

384. Nimis autem molestus essem, si novis exemplis condendis operam darem: qua de re iisdem insistens Parabolis Hyperbolisque Schematum iam

$$\text{depictorum facile illis assentior, qui Aequationum Formulas } \left(\frac{x^2 + x}{2} \right)^2 - y^2 = 0,$$

(*) Vide Numerum 347^{um}.

(**) Consule Numerum 56^{um}.

(***) Sic statuit Numerus 299^{us}.

$$= 0, \left(\frac{x^3 + 3x^2 + 2x}{6} \right)^2 - y^2 = 0, \left(\frac{x^2 + x}{2} \right)^2 y^2 - 1 = 0,$$

$$\left(\frac{x^3 + 3x^2 + 2x}{6} \right)^2 y^2 - 1 = 0, \text{ five, evolutis Quadratis, } x^4 + 2x^3 + x^2$$

$$- 4y^2 = 0, x^6 + 6x^5 + 13x^4 + 12x^3 + 4x^2 - 36y^2 = 0, \text{ nec non } x^4y^2 + 2x^3y^2 + x^2y^2 - 4 = 0, x^6y^2 + 6x^5y^2 + 13x^4y^2 + 12x^3y^2 + 4x^2y^2 - 36 = 0,$$

fore autumant Analytice Continuas, sed non *Geometricae*. Profecto, quum Loci Lineares Parabolici, idemque dicas de aliis Hyperbolici Generis, in illis latentes Aequalitatibus sic exprimi possint per Ra-

$$\text{dicum extractionem } \pm \left(\frac{x^2 + x}{2} \right) - y = 0, \text{ et } \pm \left(\frac{x^3 + 3x^2 + 2x}{6} \right) - y =$$

0, Unitas Curvae laeditur, evaditque ipsiusmet Loci adversis in plagis repetiti Systema, uti Perimetri punctulis consulto adumbratae demonstrant. Hoc universalissime contingit Parabolis omnibus et Hyperbo-

lis, quarum Aequatio fuerit $a^2 x^{2m} = y^{\pm 2n}$, utpote resolubilis in $\pm ax^m - y^n = 0$, scilicet in Factorum $(ax^m - y^{\pm n})(-ax^m - y^{\pm n})$ utro-

rumque Rationalium Productum $= 0$, vel tandem in Systema duorum, aequalium, similium, et situ tantummodo discrepantium Locorum $ax^m - y^{\pm n} = 0$, $ax^m + y^{\pm n} = 0$. Idcirco Hyperbola Neutoniana IIⁱ. Schema-

tis, quamvis sub Aequationis $y^4 = \frac{1}{x^2}$ aut $y^4 - x^{-2} = 0$ velo pro-

posita etiam in regione Ξ Negativorum videretur existere, uti punctis ostenditur, Analyticae reapse, non vero *Geometricae*, obtemperaret Continuitati. Sed in sententiam illorum pari facilitate non ibo, qui, dum Factorum Irrationalitas Variabiles tantum afficiat, nunquam autem dum Constantes

sive Parametros, Loci Linearis Ramos ad unicum et *incomplexum*, ut aiunt, spectare Locum, salva *Geometrica* Continuitate, arbitrantur. Concedam equidem primum, quum meridiana luce clarius fit, Factores $(y + \sqrt{ax})(y - \sqrt{ax}) = 0$, seu $(y - 2 - \sqrt{4x} - \sqrt{4+2x})(y - 2 + \sqrt{4x} + \sqrt{4+2x})(y - 2 - \sqrt{4x} + \sqrt{4+2x})(y - 2 + \sqrt{4x} - \sqrt{4+2x}) = 0$

duo Crura eiusdemmet Parabolae Apolloniana, vel quatuor Ramos pariter in Infinitum excurrentes ipsiusmet Curvae Ordinis tertii $y^4 - 8y^3 + 16y^2 - 12y^2x + 48yx - 64x + 4x^2 = 0$ significare. Veruntamen non ab-

mili

mili iure, quo Curvas Aequationibus expressas $y = \sqrt{x} + \sqrt[4]{x^3}$, ac $y = \frac{1}{2}(\sqrt{6x-x^2} + \sqrt{6x+x^2} + \sqrt{36-x^2})$ nonnisi unum ex duobus Ramis in prima, cui Cuspis adest *secundae Speciei* (*), unumque in altera ex quatuor Cruribus, scriptorii veluti calami ductu invicem plexis ac in se redeuntibus, indicare ultro fateor (**), mihi quoque ii concedant oportet Aequalitates $y^2 = \sqrt{a^2} \cdot x^{\pm 1}$, $y^4 = \sqrt[6]{a^6} \cdot x^{\pm 3}$, $y^{2m} = \sqrt[2n]{A} \cdot x^{\pm \lambda}$, $y = x^{\pm 1} \cdot \sqrt{a}$, aliasque sexcentas ad Linearum Geometriam translatas, et facillime concinnandas, ut tota earum vis exhauriatur, conficere, propter Parametri Irrationalis aequivocam significationem, unicum Locum *Analytice* simul ac *Geometrice* Continuum, Aequationibus respectivis $y^4 - a^2 x^{\pm 2} = 0$, $y^8 - a^2 x^{\pm 6} = 0$, $y^{4m} - \sqrt[n]{A} \cdot x^{\pm 2\lambda} = 0$, $y^2 - ax^{\pm 2} = 0$ locupletatum. Quanto maiore id adferendum ratione si, prout in Logistica, Parameter a *Transcendentalitate* afficiatur, nonnisi Quadraturis exsuperanda? Meditanti Geometrae haec breviter dicta sufficient (***) .

D d d d d

385. Post

(*) Praeli vitio Aequatio Rationalis huiuscemodi Lineae Cuspidatae legitur in Num^o. 333^o. secundi Voluminis *Introductionis in Analysin Infinitorum* $y^4 - 2y^2x - 4yxx - x^3 = 0$, quum revera sit $y^4 - 2xy^2 - 4x^2y - x^3 + x^2 = 0$.

(**) Quilibet in Num^o. 284^o. eiusdem praefati Voluminis *Introductionis &c.* octuplicem ab Eulero adsertum istius Curvae Ramum perlegerit, erroris me arguet. Eliminatis tamen Irrationalibus nunquam non invenio Lineam Ordinis quarti ab Aequatione exhibitam $16y^4 + (8x^2 - 96x - 288)y^2 + (16x^3 - 576x)y + 5x^4 + 24x^3 - 72x^2 - 864x + 1296 = 0$. Locus, quem Ordinis octavi, seu melius octuplicis Ordinatae capax, affirmavit Auctor egregius, non est idcirco Linea *Geometrice* Continua, sed ex duobus Continuis composita, nimirum $(16y^4 + (8x^2 - 96x - 288)y^2 + (16x^3 - 576x)y + 5x^4 + 24x^3 - 72x^2 - 864x + 1296)(16y^4 + (8x^2 - 96x - 288)y^2 - (16x^3 - 576x)y + 5x^4 + 24x^3 - 72x^2 - 864x + 1296) = 0$. Quanta in Continuitatis regno perambulando formido mentem occupabit Geometrarum, si fortassis semel aut bis dormitaverit incomparabilis, ac prope divinus Eulerus!

(***) Seclusa hypothesi Asymmetriae Parametri in Aequatione genitrice, repugnant Continuitati *Geometricae* Parabola quatuor Ramis distincta, atque Hyperbole Cruribus suis quaternos Asymptotorum Angulos implens. Eas tamen generatim sub Aequatione $ax^{\pm k} = y^{\lambda}$, dum a sit Numerus positivus, et k, λ Numeri pares, consideravit Cramer^{us}

385. Post detectam nuperrime Continui *Geometrici* proprietatem plane intelligitur ratio antea latens in num^o. 383^{io}., ob quam Aequatio ad Logisticam $y = 1^x \left(1 + \frac{x}{\infty}\right)^\infty = 1^x \left(1 + \frac{x^1}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \dots + \frac{x^\infty}{1.2.3.4.5 \dots \infty}\right)$ geminam quoque habere possit Ordinatum - Applicatam $\pm y$ eidem Abscissae x respondentem. Si qua igitur proferret methodus certior nobis pingendi Logarithmicæ Curvæ Figuram, integram, genuinamque, sine Quadraturarum praesidio, id equidem nullo alio modo continget nisi elevatione Aequationis illius ad Quadratum; ita, ut, facta testâ Continuitate tam Analytica, quam *Geometrica*, ea profecto

$$\text{Logistica sit, qualem praebeat Aequalitas } y^2 = 1^{2x} \left(1 + \frac{x}{\infty}\right)^{2 \cdot \infty} = 1^{2x} \left(1 + \frac{(2x)^1}{1} + \frac{(2x)^2}{1.2} + \frac{(2x)^3}{1.2.3} + \frac{(2x)^4}{1.2.3.4} + \frac{(2x)^5}{1.2.3.4.5} + \dots + \frac{(2x)^\infty}{1.2.3.4.5 \dots \infty}\right)$$

$= C^{2x}$, scilicet $y = \pm C^x$. Tunc enim, Aequationum ad instar, et potissimum penultima, in calce numeri proxime elapsi animadversarum, pertinebit Logistica ad Paraboloeidon Genus illud Diametro orthogonali, et idcirco duobus Ramis similibus ac aequalibus in Infinitum excurrentibus praeditarum, absque eo quod disrumpatur *Geometrica* Continuitas in transitu ab $y = +C^x$ ad fociam $y = -C^x$, ac vicissim, uti solertissimus Imaginariorum interpres Foncenexius obiecit. Neque ista Aequatio Logisticae $y = \pm C^x$, five $y^2 = \left(1 + \frac{x}{\infty}\right)^{2 \cdot \infty}$ adversaretur alteri in num^o. 347^{mo}. super coniecturam τ_2

Indicis

rus in *Introductione ad Curvarum Analysin*. Opinor, Auctorem gravissimum Schematibus exornasse hasce Curvas *Geometricæ* Discontinuas in usum inventionis tantum Numeri Ramorum Parabolicorum, et Hyperbolicorum ad singulas Lineas spectantium, quemadmodum Eulerus Operis praecitati Volumine II^o. pag. 91. 98. 107. 113. 143. &c: iam pridem effecerat. Observandum manet $\tau_0 \sqrt[n]{A}$, ob $\sqrt[n]{A} = \sqrt[n]{\sqrt[n]{A}}$, quae est Magnitudo *Semi-Imaginaria*, unico positivo Signo gaudere, nimirum hoc in casu congeminationem valoris exuere.

Indicis $\infty = L\infty$ aedificatae $y^{2.\infty} = \frac{\left(\frac{2.\infty'-1}{2.\infty} + x\right)^{2.\infty'-1}}{\left(\frac{2.\infty'-1}{2.\infty}\right)^{2.\infty'-1}}$, in qua $\frac{2.\infty'-1}{2.\infty}$

$= L\infty$: Parabola enim hac postrema Aequatione gaudens, aequae ac alia

sic exhibita $y^{2(2.\infty)} = \frac{\left(\frac{2.\infty'-1}{2.\infty} + x\right)^{2(2.\infty'-1)}}{\left(\frac{2.\infty'-1}{2.\infty}\right)^{2(2.\infty'-1)}}$ respondens $y^2 = C^{2x}$, vel

$y = \pm C^x$, ex duobus existeret Ramis in Spatio finito sine Limite progredientibus; aegrotique somnio par esset ultra Infinitum, aut in Plusquam-Infiniti sinu nova Crura peruestigare.

386. Dum in prosequenda sumus Logarithmicæ Curvæ Aequatione

$y = \pm \left(1 + \frac{x}{\infty}\right)^{\infty}$, quæ initium statuit Abscissarum x Infinite distans ab evanescente Ordinata y , obviam veniunt nonnulla prope paradoxa de Continuitatis ingenio, usui in reliquis, nec parum, futura. Translato inprimis Puncto initiali Abscissarum Logisticae ubi Axis vel Asymptotos Curvam tangit, non ideo laeditur Geometrica Continuitas, quamvis, posito $x = v - L\infty$,

Aequatio illa Localis formam induat $y = \pm C^{v-L\infty} = \pm$

$\left(1 + \frac{v-L\infty}{r.L\infty}\right)^{rL\infty}$; unde ad hoc, ut Ordinatum - Applicata y ex Nihilo

Aliquid fiat, quomodocunque etiam parvulum, necesse fit Abscissam v ex Nihilo fieri Infinitam Harmonici Ordinis. Consimile accidit symptoma absque saltus ullius aut anomaliae suspicione in Hyperbola ipsa Apolloniæ ad Asymptotas relata, cuius vulgaris satis superque Aequatio $xy = a^2$ ita potius composita sit, ut formam habeat $\infty y - vy = a^2$ seu brevius $(\infty - v) y = a^2$, in qua scilicet ab Asymptoti Pseudo-extremitate Centrum versus supputentur Abscissæ. Quinimmo idem prædicari posset universaliter de omnibus Curvarum quarumlibet Ramis Hyperbolicis; notandumque est Ordinem Infiniti $\tau\theta$ Abscissæ ab extremo Asymptoti Puncto numeratae, cui Applicata respondere incipiat finiti valoris, eundem semper fore cum Ordine Asymptoti ipsius Infinitæ. Sed, quod palmarium reor in

hoc

hoc argumento tractando, Aequatio etiam illa $M(Lx \rightarrow \infty \text{ Harm.}) = y$,

cuius mentio in num.^o 352^o. et saepius infra in postremo Capite recurret, Continuitate plenissima gaudet, etiamsi Finitis Abscissis $M.Lx$ Ordinatum-Applicatae y semper Infinitae convenient, ac signanter Harmonici Ordinis, et ab $y=0$ dum $x=0$, nullo, prope dixeris, intermedio gradu facto, ad $y=\infty$ dum x vel pusillum habeat valorem profiliatur. Conscripta equi-

dem Aequalitas consentit cum altera $Lx = \frac{y}{M} - L\infty = z - L\infty$, sive

$x = C^{z-L\infty}$, uti supra. Cuncta igitur pendent a translatione Initii numerationis Abscissarum, quod, cum, ubivis situm fuerit, haudquaquam immutet cuiuslibetque Linearis Loci progressum aut *Continuitatem*, id nequidem poterit ab Infinito veluti absorptum.

387. Haftenus Aequationis ad Logisticam *Exponentialitas*, si liceat nomen istud insolens usurpare, nimirum Irrationalitas indefinita, ac quemcunque Gradum exsuperans, $\approx C^x$ vel 1^x impedimento fuit, quominus omnibus numeris absolutus de plenaria illius Curvae Figura sermo haberetur. Ut recta tutaque de Perimetro, Ramis, Asymptotis, Diametris &c: Linearum Transcendentium, quas inter et Logistica enumeratur, pronuntiari sententia possit, genesis ipsa aut *Transcendentalitatis* origo prima ac nativa dispicienda. Facillime prae ceteris Scriptoribus voti compotes erimus in desiderata conformatione absolutissima Logarithmicae Curvae. Iamdudum etenim statutum fuit (*) Localem Aequationem Logisticam, cum Quadratura Hyperbolae Apolloniana inter Asymptotos nedum amice, sed necessario coniunctam, ne mutila mancaque sit, tali modo exhiberi ac componi

debere $\left(\frac{C^x+C^{-x}}{2}\right)^2 - \left(\frac{C^x-C^{-x}}{2}\right)^2 = 1^2 = 1 = v^2 - z^2$, scilicet $\pm$

$\left(\frac{C^x+C^{-x}}{2}\right) = v$, nec non $\pm \left(\frac{C^x-C^{-x}}{2}\right) = z$, aut demum $\pm C^{\pm x} = v \pm$

$z = y$, et brevius $\pm C^x = y$; uti minuendis, si non penitus oblitterandis, Lineae illius Paradoxis navans operam, exemplo Analogiae ducto ab Aequatione communis Parabolae $\pm \sqrt{ax} = y$ olim coniecit ingeniosissimus

Dalembertius.

(*) Potissimum in numero 178^o.

Dalembertius. Istam ex quadruplici Hyperbolae Ramo (de quibus Ramis, an sint vere ac *geometricae* inter se simul omnes Continui, dubitatio suboriens in Elementa Matheseos peccaret) depromptam Logisticae formam, gemino circum Asymptoton Axemve suum aequali Crure perinfignem, et Aequatione conclusam Exponentiali $\pm C^x = y$ seu $C^{2x} = y^2$ Continuitatis *Geometricae* Legibus consona, alijs methodis eodem ducentibus praepositam velim, quia ab intima profluit rei natura, nullo eget Calculo Infinitesimali, rectissimoque itinere metam attingit. Nec excipiam demonstrationem a Foncenexio latam, non feliciter satis ex Evolutae Figura ad Logisticam spectantis prodeuntem, quam si persequi, meo sensu nimium obliquam, utile existimarem, ceteris Parabolarum Limitis affectionibus in numero 345°. traditis adiungere potis essem. Revera in Parabolis, quarum Aequatio canonica $(m+x)^m = v^m = y$, habetur ex *Institutionibus* tam

$$y = \frac{y}{\frac{1}{m} - 1} \left(\frac{1+m^2 v^{2m-2}}{m^2 v^{2m-2}} \right) = z \text{ Ordinatum - Applicatae ad Evolutam perti-}$$

$$\text{nentis, tam } x = \frac{v^{2m-1}}{\frac{1}{m} - 1} \left(\frac{1+m^2 v^{2m-2}}{m v^{2m-2}} \right) = u \text{ Curvae ipsiusmet Abscissae;}$$

videlicet, dum Index $m = \infty$, pro Logistica emergunt eae Co-Ordinatae

$$z = -y + y \left(\frac{1+y^2}{y^2} \right) = \frac{2y^2+1}{y}, \text{ atque } u = x - \left(\frac{\infty+x}{\infty} \right) = \frac{\infty^2 \cdot y^2}{\infty(\infty+x)}$$

$$= x - 1 - y^2, \text{ ideoque } z = \frac{2C^{2x}+1}{C^x} \text{ et } u = x - C^{2x} - 1, \text{ ac ve-}$$

$$\text{ra Aequatio Evolutae } L \left(\frac{z}{4} \pm \sqrt{\frac{z^2}{16} - \frac{1}{2}} \right) - \left(\frac{z}{4} \pm \sqrt{\frac{z^2}{16} - \frac{1}{2}} \right)^2 - 1 = u,$$

cuius constructio ab Hyperbolae Quadratura, seu ex data Logarithmica pendet. Plura de huiusce Evolutae conformatione, et affectionibus praecipuis ab Aequalitate nunc primum plenissime eruta, ac *Constructioni* facillimae accommodata differere possem, sed unum monebo. Aequatio, quae legitur a Foncenexio concinnata (*), nempe $z^2 = \frac{4u-4u^2-1}{1-u}$, eadem est cum

$$\text{a Foncenexio concinnata (*), nempe } z^2 = \frac{4u-4u^2-1}{1-u}, \text{ eadem est cum}$$

Eeeee

partiali

(*) In Volumine II°. *Miscellaneorum Societatis Taurinensis &c.* Dissertatio Cl. Auctoris

partiali $\left(\frac{z}{4} \pm \sqrt{\frac{z^2}{16} - \frac{1}{2}}\right)^2 + 1 = u$, habetque vagam, aut nullibi sistens

Abscissarum Initium, nimirum non est vera Aequatio *Localis* ad Evolutam Logisticae, meoque iudicio aliquantulum idcirco claudicat argumentum duplicis Rami exinde ab Auctore deductum. Quisquis autem vellet hoc idem moliri

ope Aequationis repertae, et omnibus numeris absolutae $L\left(\frac{z}{4} \pm \sqrt{\frac{z^2}{16} - \frac{1}{2}}\right)$

$-\left(\frac{z}{4} \pm \sqrt{\frac{z^2}{16} - \frac{1}{2}}\right)^2 - 1 = u$, confestim experiretur eam esse tanto di-

scrimini imparem, quin prius abunde constiterit de Aequalitate $L\left(\frac{z}{4} +$

$\sqrt{\frac{z^2}{16} - \frac{1}{2}}\right) = L\left(-\frac{z}{4} - \sqrt{\frac{z^2}{16} - \frac{1}{2}}\right)$, scilicet de quaestione ipsa Loga-

rithmorum Negativis Numeris respondentium. Tunc etenim foret $L\left(\frac{z}{4} +$

$\sqrt{\frac{z^2}{16} - \frac{1}{2}}\right) - \left(\frac{z}{4} + \sqrt{\frac{z^2}{16} - \frac{1}{2}}\right)^2 - 1 = u = L\left(-\frac{z}{4} -$

$\sqrt{\left(-\frac{z}{4}\right)^2 - \frac{1}{2}}\right) - \left(-\frac{z}{4} - \sqrt{\left(-\frac{z}{4}\right)^2 - \frac{1}{2}}\right)^2 - 1$, nec non

$L\left(\frac{z}{4} - \sqrt{\frac{z^2}{16} - \frac{1}{2}}\right) - \left(\frac{z}{4} - \sqrt{\frac{z^2}{16} - \frac{1}{2}}\right)^2 - 1 = u = L\left(-\frac{z}{4} +$

$\sqrt{\left(-\frac{z}{4}\right)^2 - \frac{1}{2}}\right) - \left(-\frac{z}{4} + \sqrt{\left(-\frac{z}{4}\right)^2 - \frac{1}{2}}\right)^2 - 1$, nimirum cuilibet

Abscissae u conveniret Ordinatum-Applicata duplex $\pm z$, et geminum idcirco Crus Logarithmicae adscriberetur.

388. Antequam Figuram Logisticae inventam alio argumentorum robore confirmem non abs re erit advertisse, huic determinandae conformationi inidoneam futuram nudam, aut a Figura *quadrandae* Curvae seiunctam

Expressionem Integrale $\int \frac{dx}{x} = y$, quippe quae, neglecta etiam Constante,

Auctoris ab Aequatione Exponentiali $y = mC^x$ gradum sibi facit ad Logometricam $x = L(my)$. Hanc ita emendandam cenfeo $x = L\left(\frac{y}{m}\right)$.

te, vaga indeterminataque sit quoad *Geometricam* Continuitatem, nec mo-

do praebeat $C^x = y$, at insuper $C^{2x} = y^2$, seu universalius $C^{mx} = y^m$, uti ceteris accidit Summationibus. Sin vero in Summatione illius Formulae ad Analyticam simul, et *Geometricam* Continuitatem obtinendam sese vertat Mathematicorum ingenium, prae oculis semper ponatur oportet Hyperbolae Apolloniana conformatio notissima, ut ex indeterminata Aequa-

tione Integrali Analytica Continua $C^{mx} = y^m$ eam unicam illi adipiscantur *Geometrice* etiam Continuum. Quod statim ac fuerit effectum, iam constat

ex num^o. 352^o. Magnitudinibus Negativis eosdem addictos esse Logarithmos, qui aequalibus Positivis, nimirum Logisticam duobus Ramis gaudere, et Aequatione Locali $\pm C^x = y$ vel $C^{2x} = y^2$, nec unquam aliter, esse exornatam. Ne egregium hoc argumentum, Ioanni, uti dictum, Bernoullio seniori charissimum, labefactari sinam, paucula de Continuitate ipsa *Geometrica* manent superaddenda tum, quum a Quadraturis universali modo conceptis ea pendeat. Sunt enim non pauci, qui ita ratiocinantur. Quoniam modo Quadraturis fidendum, dum saepissime mutilos truncatosque exhibent Locos, quos Matheseos *Elementa* ipsa integros statuunt, *Geometriceque* et indubitanter Continuos? Sphaericae Superficie AC DH Zonarum Tetragonismus Aequatione conclusus $\frac{R}{I} \cdot \Pi x = ay$ dat in Schemate XXXV^o.

pusillam tantum Rectae partem AB (cum non sese ultra porrigat genitrix Magnitudo) prae Infinita eius Loci Recti-Linearis longitudine NABL in regione Positivorum, atque Negativorum. Consimili pacto *Cubatrix* Conoeidos Parabolicae CAB Frustrorum a Parabolae Apollonianaeversione circa proprium Axem prodeuntium videtur Ramus solummodo AEF Apollonianaepa-

riter Parabolae ad suam Tangentem relatae, qualem Aequationis $S \frac{\Pi}{2} \cdot z^2$.

$dx = S \frac{\Pi}{2} \cdot ax \cdot dx = \frac{a\Pi}{4} \cdot x^2 = b^2 y$ medietas suppeditat, excluso nem-

pe altero Curvae ipsius Ramo AD, cum in Negativorum parte AG nulla Conois existat, nec existere in Schemate XXXVI^o. unquam possit, nisi susdeque habitis Continuitatis *Geometricae* Legibus. Ad haec, in re adamus-

im Logometrica, quam manibus attrectamus, Foncenexius obiecit, non tutam inire rationem eum, qui Numeris Negativis addictos investigare conetur Logarithmos Reales ope Aequationis $n \int \frac{dx}{x} = L'x$, vel Areae Hyperbolicae Apolloniana; quandoquidem ex numeris XIV^o. 354^{to}. ac 384^{to}.

eadem Aequalitas Frustra etiam Solidi-Acuti Infinite longi ab Hyperbola, cuius Aequatio est $y^2x = 1$, circa suam Asymptoton Axemve Abscissarum x rotante geniti repraesentat, absque eo quod ulla ibidem mentio τx Negativi fieri queat, nam Figura Genitrix in Negativorum regione non extat (*).

389. Nihilo tamen minus praecipua Foncenexii oppositio, ceteroquin ingeniosissima, nedum infirmabitur, sed tota corruet ad Continuitatis *Geometricae* veros canones revocata. Hoc unum praemitto luculentissimum, Lineas Negativas Areasve, dum circum Axem eadem directione cum Positivis in gyrum agantur, Superficies aut Solida Negativa producere. Exinde est, quod, in Infinitum aucta altitudine AH Trilinei Parabolici ALH praefatam gignentis Conoeiden, Ramoque *Cubatricis* Curvae pariter in Infinitum protenso, Area Genitrix ALH positiva Infinita ultra, si liceat sic fari, progrediatur, et ex LHM, perinde ac de Ellipsoide Infinite magna loqueremur, sibi gradatim totum iungat negativum Trilineum HMA. Iure igitur Continuitatis *Geometricae* servato, postquam perventum fuerit ad Solidum Infinitum a Trilineo ALH genitum, et ad Punctum extremum F Infinite distans Curvae *Cubatricis*, Solida consequuntur Rotunda ex Areis orta ALHMBK partim positivis, partimque negativis, Punctumque H uti pertinens ad Infinitam Circuli Peripheriam ex regione G versus A gyrum claudit, et alterum sibi socium tenet *Cubatricis* Ramum DA, cuius Ordinatum-Applicatae semper positivae Arearum quoque ALHMBK Producta, quod evidentissimum est, positiva fore demonstrant. Idem dicas, inspecto Schemate II^o., de Solidi illius Hyperbolici-Acuti Frustris genitis versatione cuiusvis Quadrilinei Positivi BYKI, BOLI, BYMI, &c: usque ad Infinite longum BGXI. Heic enim non desinit Solido-

(*) Ita ferunt *Illustrationes &c:* in *Miscellaneis* Augustae Taurinorum editis pro Annis MDCCLX^{mo}. LXI^{mo}.

Solidorum progressus; siquidem Area BGXI, Abscissaque IX in Infinito non abrumpuntur, quum prima ex adverso ob geminum Hyperbolae Crus sensim evadat BGXVΣM, BGXVΦL, BGXVZK, &c.; positiva partim ac negativa, et Asymptotos IX quasi in se rediens ex parte U per E, H denuo perveniat ad I. Praeter Ramum igitur *Cubatricis* ABMC, qui Solida respicit a Quadrilineis orta ex BI dextrorsum aut sinistrorsum computatis circa Asymptoton-Axem HIX, Ramus alter existet, similis et aequalis, $\mu\nu$ in Latere opposito. Huius, aequae ac prioris, munus erit ope Differentiae Ordinarum $y\epsilon$, ωE , $\nu\theta$ &c: Asymptoto OHΩ aequidistantium significare super continuationem pene Circulatoriam Rectae positivae HIXUH vel, quod eodem recidit, super Segmentis negativae Rectae HεEθU Proportionem Frustrorum a BGXVΔ seu o, BGXVZKY seu BYKI, BGXVΦLΘ seu BΘLI, &c: Quadrilineis omnibus positivis ortorum, videlicet significare pro Negativis Ordinatis $H\epsilon = -HI = -1$, Hε, Hθ &c: Logarithmos o, QP, QN &c: illis Positivarum $HI = 1$, HK, HL &c: perfecte aequales. Subtilior etiam aditus patet ad considerationem in Schemate XXXV°. Sphaericae Superficie. Porro Superficies integrae Sphaerae enascitur a versatione solius Semi-Peripheriae ACD positivae. Sed huiusmodi oppido habetur ea genitrix Curva, quae, positiva semper existens, a D versus H procedat, et in A gyrum compleat, quin potius et innumeras sine fine peragat revolutiones. Superficies itaque ab Arcubus positivis ACDH, ACDHACDH, ACDHACDHACDH, &c: absque ullo Limite genitae, quomodolibet partim aut omnino superponantur, ac sese contrahant celentque, nihilominus, si exfoliarentur, uti ACDH', ACDH'EFGH'', &c: Proportionem tenent Ordinarum ad Rectam indefinitam ABL; cuius Ramus alter AMN ad Superficies negativas ab Arcu quovis negativo AHDC, AHDCAHDC, &c: rotante prodeuntes in Infinitum repraesentandas apte conducit. Quisquis haec premat vestigia nunquam in *Geometricae* Continuitatis labyrintho exspes ac fido duce carens errabit, sibi que potius singulos evolvat casus, qui, cuncti nativo ordine expositi, peculiarem fortasse Tractatum desiderant.

390. Alacriore nunc animo ad *Quadraturas* revertor, unde fueram digressus. Transcendens utraque Formula, cui totum innititur Logometricae doctrinae aedificium, denuo oculis subiiciatur, nimirum,

$$x = L \left(1 + \frac{x^1}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} + \frac{x^6}{1.2.3.4.5.6} + \dots \right)$$

Fffffff

$$+ \frac{x^7}{1.2.3.4.5.6.7} + \frac{x^8}{1.2.3.4.5.6.7.8} + \frac{x^9}{1.2.3.4.5.6.7.8.9} + \&c:)$$

$$= L\left(1 + \frac{x}{\infty}\right)^{\infty} = L(C^x) = Ly, \text{ et}$$

$$x = L\left(1 + \frac{x}{\infty}\right)^{\infty} = L(C^x) = Ly = \frac{\overline{y-1}^1}{1} - \frac{\overline{y-1}^2}{2} + \frac{\overline{y-1}^3}{3} - \frac{\overline{y-1}^4}{4} + \frac{\overline{y-1}^5}{5} - \frac{\overline{y-1}^6}{6} + \frac{\overline{y-1}^7}{7} - \frac{\overline{y-1}^8}{8} + \frac{\overline{y-1}^9}{9} - \&c: (*).$$

Prior, Logarithmo dato, seu data Area Hyperbolica Apolloniana inter curvam Perimetrum Asymptotonque, vel data Abscissa super Asymptoton Logisticae, valde confert ad inveniendum Transcendentem Numeri, aut *Baseos* Areae Hyperbolicae, sive Ordinatum-Applicatae ad Logisticam valorem: secunda ex hoc postremo valore cognito ducit vicissim ad valorem pariter Transcendentem Logarithmi, aut Areae Hyperbolicae, vel Abscissae in Logistica. Dum in altera fiat $\tau\delta$ y negativum, puta $= -m$, equidem erit

$$L(-m) = -\frac{(m+1)^1}{1} - \frac{(m+1)^2}{2} - \frac{(m+1)^3}{3} - \frac{(m+1)^4}{4} - \frac{(m+1)^5}{5} - \frac{(m+1)^6}{6} - \frac{(m+1)^7}{7} - \frac{(m+1)^8}{8} - \frac{(m+1)^9}{9} - \&c:.$$

Haec Series Transcendens saepissime ob sui Divergentiam perquammaximam *Plusquam-Harmonicum* Infinitum designare videtur; quod nomen, castis Geometrarum auribus malesonans, parcius in posterum adhibebo. Veruntamen ob eam Divergentiam crassa foret hallucinatio, si absurdum confestim aut impossibilem diceremus $\tau\delta -m$ Logarithmum. Termini etenim cuiusque Seriei Divergentes, nisi alia accesserint severiora argumenta, nullum ius tribuunt Seriem

(*) Ii vere caecutiunt, qui sophismatis nescio quibus inficias iverint $\tau\delta \left(1 + \frac{x}{\infty}\right)^{\infty}$ peraequare posse Numerum Finitum utcunque magnum C^x . Sic fallitur, et fallere vellet Auctor Opellae, cui Titulus *Iosephi Calandrellii Specimen Analyticum &c: editum* Romae anno MDCCLXXVIII^o. Praesertim lege pag^{am} . 23^{am} . atque seq^{em} .

riem ipsam statuendi iudicio nimium praecipiti Imaginariam (*). Qui con-

tra senferit, Imaginarietatis etiam arguere debuisset Seriem $\frac{(m-1)^1}{1} - \frac{(m-1)^2}{2}$

$$+ \frac{(m-1)^3}{3} - \frac{(m-1)^4}{4} + \frac{(m-1)^5}{5} - \frac{(m-1)^6}{6} + \frac{(m-1)^7}{7} - \frac{(m-1)^8}{8} +$$

$$\frac{(m-1)^9}{9} - \&c: = Lm \text{ quum } m > 2 (**); \text{ qua iniuria lata ferme omnes Ma-}$$

gnitudinum Positivorum Logarithmi absurdum fuissent, et parvula haecenus, incerta, miserrimaque iaceret Logometria. Artis est vel Divergentiam, quoad liceat, emendare, vel nativam Serierum Divergentium originem Summamque perquirere, vel ex singulis casibus, in quibus Convergens fieri possit aliqua Series, an sit Realis sive Impossibilis, definire, ut omnia pacato animo, nullaue inaudita parte librentur. Interim pendente iudicio scitu dignum existimo, ex Serie illa derivari $L(-0) = -$

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \&c: \right), \text{ videlicet } = L(0):$$

recte quidem, atque potenter. Constat praeterea $L(-\infty) = -$

$$\frac{(\infty+1)^1}{1} - \frac{(\infty+1)^2}{2} + \frac{(\infty+1)^3}{3} - \frac{(\infty+1)^4}{4} + \frac{(\infty+1)^5}{5} - \frac{(\infty+1)^6}{6} -$$

$$\frac{(\infty+1)^7}{7} + \frac{(\infty+1)^8}{8} - \frac{(\infty+1)^9}{9} \&c: = - \frac{\infty^1}{1} - \frac{\infty^2}{2} - \frac{\infty^3}{3} - \frac{\infty^4}{4}$$

$$- \frac{\infty^5}{5} - \frac{\infty^6}{6} - \frac{\infty^7}{7} - \frac{\infty^8}{8} - \frac{\infty^9}{9} - \&c:, \text{ uti fert Series praemissa,}$$

levi opificio (***) in Convergentem redigi Seriem, ac exaequare $L\infty$. Statim ac igitur indocilis, falsaue Series, Negativorum continens Logarithmos, forte fortuna in Convergentium classe locatur, Logarithmi Negativorum illis aequalium Positivorum pares emergunt. Quidni et in ceteris casibus? Profecto in expressio-

ne

(*) Attentissime ediscantur exempla de Serierum quomodolibet Divergentium Ufu, Veritate, ac Summa, quae passim docet Eulerus in Parte II^a. *Institutionum Calculi Differentialis*.

(**) Vide praesertim numeros 54^{um}. 101^{um}. 107^{um}.

(***) Consule numeros 118^{um}. 119^{um}. , ac potissimum eorum postremum.

$$\text{ne } L(L(1+v)) = -\infty \text{ Harm}^o + \frac{\infty^1 v^1}{1^2} - \frac{\infty^2 v^2}{2^2} + \frac{\infty^3 v^3}{3^2} - \frac{\infty^4 v^4}{4^2} \\ + \frac{\infty^5 v^5}{5^2} - \frac{\infty^6 v^6}{6^2} + \frac{\infty^7 v^7}{7^2} - \frac{\infty^8 v^8}{8^2} + \frac{\infty^9 v^9}{9^2} - \&c: (*), \text{ quae univer-}$$

se complectitur Logarithmos adscriptos cunctis Numeris Positivis si v positi-
vus, cunctisque Negativis si v negativus Unitate minor, aequae occurrit
Divergentiae vitium pro gemina hypothese; adeo, ut nulla maior sit ratio
de unius Impossibilitate Logarithmi, quin et simul de alterius minus cautam
proferendi sententiam.

391. Verius autem instituta anatome Seriei universaliter Divergentis

$$- \left(\frac{z+1^1}{1} + \frac{z+1^2}{2} + \frac{z+1^3}{3} + \frac{z+1^4}{4} + \frac{z+1^5}{5} + \frac{z+1^6}{6} + \frac{z+1^7}{7} + \frac{z+1^8}{8} + \frac{z+1^9}{9} + \&c: \right) = L(-z) \text{ dummodo } z \text{ Ne-}$$

gativum fit, patebit luculentissime eam iuxta num^{um}. 335^{um}. sic exprimi

$$\text{posse } L(-z) = S(-d(z+1)) (1 + (z+1)^1 + (z+1)^2 + (z+1)^3 + (z+1)^4 + (z+1)^5 + (z+1)^6 + (z+1)^7 + (z+1)^8 \\ + \&c:) = S \frac{-d(z+1)}{1-(z+1)} = S \frac{-dz}{-z} + A = S \frac{+dz}{+z} + A = S \frac{dz}{z}$$

$$- \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c: \right) \text{ in Infini-}$$

$$\text{tum} = Lz, (**). \text{ Tam } S \frac{dz}{z}, \text{ quam } S \frac{-dz}{-z} \text{ Infinitatem itaque Harmo-}$$

$$\text{nicam sinu suo complectuntur; quandoquidem prior } S \frac{dz}{z} = L(z) + \infty$$

Harm:

(*) Ita est in calce numeri 124^{ti}.

(**) Confer cum ista Constantis additione quae fufius ostendi in numeris 352.

Harm: , atque eodem ritu posterior $S \frac{-dz}{-z} = L(-z) \rightarrow \infty$ Harm: $= L(z)$

$\rightarrow \infty$ Harm: . Transitus tamen ab $S \frac{dz}{z} = S(d(z-1))(1-(z-1)^1 + (z-1)^2 - (z-1)^3 + (z-1)^4 - (z-1)^5 + (z-1)^6 - (z-1)^7 + (z-1)^8 - \&c:)$ $\rightarrow \infty$ Harm: $= S(-d(-z+1))(1+(-z+1)^1 + (-z+1)^2 + (-z+1)^3 + (-z+1)^4 + (-z+1)^5 + (-z+1)^6 + (-z+1)^7 + (-z+1)^8 + \&c:)$ $\rightarrow \infty$ Harm. $=$

$S \frac{-d(-z+1)}{1-(-z+1)} \rightarrow \infty$ Harm: ad $S \frac{-dz}{-z} = S(-d(z+1))(1 + (z+1)^1 + (z+1)^2 + (z+1)^3 + (z+1)^4 + (z+1)^5 + (z+1)^6 + (z+1)^7 + (z+1)^8 + \&c:)$ $\rightarrow \infty$ Harm. $= S \frac{-d(z+1)}{1-(z+1)} \rightarrow \infty$ Harm: ,

nimirum transitus ab Area Hyperbolica in regione Positivorum ad aliam in partibus Negativorum , atque vicissim , quum et *Analytice* et *Geometrice* fit Continuum nihil obstante Infinito Harmonico (*), reddit quoque *Analytice* ac *Geometrice* Continuum transitum non dissimilem ab $L(z)$ ad

$L(-z)$, scilicet ab $S \frac{dz}{z} \rightarrow \infty$ Harm: ad $S \frac{-dz}{-z} \rightarrow \infty$ Harm: . Hoc

tantummodo discrimen intererit, quod ab $S \frac{dz}{z}$ ad $S \frac{-dz}{-z}$, five ab $Lz \rightarrow \infty$

Harm: ad $L(-z) \rightarrow \infty$ Harm: mediante Nihilo, nempe $L0 \rightarrow \infty$ Harm: , transitus ille contingat, dum ex adverso ab Lz ad $L(-z)$ mediante In-

finito Harmonico Negativo, videlicet $-(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c:)$. Ea igitur Linea hac Aequatione gaudens

G g g g g g

y =

(*) Mutatio simplicissima $z \rightarrow -z$ in $-z$, ac converse, in Formulis omnibus paullo supra invicem comparatis meridiana luce clarius hoc ipsum confirmat.

$$y = \frac{(x-1)^1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \frac{(x-1)^7}{7}$$

$$- \frac{(x-1)^8}{8} + \frac{(x-1)^9}{9} - \&c., \text{ nimirum Logistica, cuius Abscissarum } x \text{ Axis}$$

Asymptoto sit normalis, geminum habebit, aequalem, *Analytice* et *Geometrice* Continuum Ramum: namque, posito $x = n$, evadet $y = \frac{(n-1)^1}{1} -$

$$\frac{(n-1)^2}{2} + \frac{(n-1)^3}{3} - \frac{(n-1)^4}{4} + \frac{(n-1)^5}{5} - \frac{(n-1)^6}{6} + \frac{(n-1)^7}{7} - \frac{(n-1)^8}{8}$$

$$+ \frac{(n-1)^9}{9} - \&c., \text{ ac, facto } x = -n, \text{ profluet } y = \frac{(-n-1)^1}{1} -$$

$$\frac{(-n-1)^2}{2} + \frac{(-n-1)^3}{3} - \frac{(-n-1)^4}{4} + \frac{(-n-1)^5}{5} - \frac{(-n-1)^6}{6} + \frac{(-n-1)^7}{7}$$

$$- \frac{(-n-1)^8}{8} + \frac{(-n-1)^9}{9} - \&c. = - \left(\frac{(n+1)^1}{1} + \frac{(n+1)^2}{2} + \frac{(n+1)^3}{3} + \right.$$

$$\left. \frac{(n+1)^4}{4} + \frac{(n+1)^5}{5} + \frac{(n+1)^6}{6} + \frac{(n+1)^7}{7} + \frac{(n+1)^8}{8} + \frac{(n+1)^9}{9} + \&c. \right) =$$

$$\frac{(n-1)^1}{1} - \frac{(n-1)^2}{2} + \frac{(n-1)^3}{3} - \frac{(n-1)^4}{4} + \frac{(n-1)^5}{5} - \frac{(n-1)^6}{6} + \frac{(n-1)^7}{7} -$$

$$\frac{(n-1)^8}{8} + \frac{(n-1)^9}{9} - \&c. \text{ ex nuper ostensis. Summa idcirco vel Magnitu-}$$

tudo, cuius Symbolum est Series perpetuo divergens $\frac{(m+1)^1}{1} -$

$$\frac{(m+1)^2}{2} - \frac{(m+1)^3}{3} - \frac{(m+1)^4}{4} - \frac{(m+1)^5}{5} - \frac{(m+1)^6}{6} - \frac{(m+1)^7}{7} -$$

$$\frac{(m+1)^8}{8} - \frac{(m+1)^9}{9} - \&c., \text{ tametsi pene desperatam, ac fortassis Imagina-}$$

riam suspicaremur, manat ex Transcendente *Quadraturarum* penu Rea-
 lis, eademque cum $\frac{(m-1)^1}{1} - \frac{(m-1)^2}{2} + \frac{(m-1)^3}{3} - \frac{(m-1)^4}{4} +$

$$\frac{(m-1)^5}{5} - \frac{(m-1)^6}{6} + \frac{(m-1)^7}{7} - \frac{(m-1)^8}{8} + \frac{(m-1)^9}{9} - \&c. . \text{ Nec mi-}$$

rum,

rum, eoquod, Residuis in computum actis, Serierum illarum prior ita plenius exprimi debeat $S(-d(z+1))(1+(z+1)^1+(z+1)^2+(z+1)^3+(z+1)^4+(z+1)^5+(z+1)^6+(z+1)^7+(z+1)^8+(z+1)^9+\dots+\frac{(z+1)^\infty}{-z})$, facta, post absolutam Integrationem, z

$=m$. Hoc enim Residuum $S(-d(z+1))(\frac{(z+1)^\infty}{-z})$, seu potius

$S(-d(z+1))(\frac{(z+1)^\infty}{1-(z+1)})$, quod negligi haudquaquam potest utcumque

parvus fuerit Numerus z dummodo positivus, Divergentem ac prope rebel-

lem praefatam Seriem emendat, statuitque $L(-m) = -\frac{(m+1)^1}{1}$

$-\frac{(m+1)^2}{2} - \frac{(m+1)^3}{3} - \frac{(m+1)^4}{4} - \frac{(m+1)^5}{5} - \frac{(m+1)^6}{6} - \frac{(m+1)^7}{7} - \frac{(m+1)^8}{8} - \frac{(m+1)^9}{9} - \&c: + S(d(z+1))(\frac{(z+1)^\infty}{z})$ in hypothesi z

$=m$, nimirum $L(-m) = -\infty + \infty$, uti saepius $L(+m) = \frac{(m-1)^1}{1}$

$-\frac{(m-1)^2}{2} + \frac{(m-1)^3}{3} - \frac{(m-1)^4}{4} + \frac{(m-1)^5}{5} - \frac{(m-1)^6}{6} + \frac{(m-1)^7}{7}$

$-\frac{(m-1)^8}{8} + \frac{(m-1)^9}{9} - \&c: = \infty - \infty$, scilicet absque ulla compa-

rationis initae absurditate, ulliusve, fortassis praeconcepi, Imaginarij vitio latente. Iniuria ergo Leibnitius (*) Imaginarietatis arguere solebat $\tau\delta L(-1)$,

quia $= -\frac{2^1}{1} - \frac{2^2}{2} - \frac{2^3}{3} - \frac{2^4}{4} - \frac{2^5}{5} - \frac{2^6}{6} - \frac{2^7}{7} - \frac{2^8}{8} - \frac{2^9}{9} - \&c:$

nullus siquidem ipse scivit, hanc Seriem admodum Divergentem, ob Residuum Infinite excrefcens, commentitiam mancamque futuram, nisi Residuum illud

(*) Perlege Pag^{am}. 269^{am}. Voluminis IIⁱ. *Commercii Philosophici et Mathematici* ac seq^{es}. usque ad 316^{am}. in quibus saepissime agitur de Numeri Negativi cuiusque Logarithmo, nec invenies hoc argumentum, quod tamen Leibnitii fuisse affirmat Eulerus in *Actis Berolinensibus* anni MDCCXLIXⁱ.

illud accesserit (*), tumque fieri $L(-1) = -\frac{2^1}{1} - \frac{2^2}{2} - \frac{2^3}{3} - \frac{2^4}{4} - \frac{2^5}{5} - \frac{2^6}{6} - \frac{2^7}{7} - \frac{2^8}{8} - \frac{2^9}{9} - \&c: + S(d(z+1))\left(\frac{(z+1)^\infty}{z}\right) = -\frac{2^1}{1} - \frac{2^2}{2} - \frac{2^3}{3} - \frac{2^4}{4} - \frac{2^5}{5} - \frac{2^6}{6} - \frac{2^7}{7} - \frac{2^8}{8} - \frac{2^9}{9} - \&c: - \frac{2^{\infty-2}}{\infty-2} - \frac{2^{\infty-1}}{\infty-1} - \frac{2^\infty}{\infty} + \infty'$, aut, id quod miraculo proximum videbatur, non absurde $= -\frac{2^1}{1} - \frac{2^2}{2} - \frac{2^3}{3} - \frac{2^4}{4} - \frac{2^5}{5} - \frac{2^6}{6} - \frac{2^7}{7} - \frac{2^8}{8} - \frac{2^9}{9} - \&c: - \frac{2^{\infty-2}}{\infty-2} - \frac{2^{\infty-1}}{\infty-1} - \frac{2^\infty}{\infty} + \frac{2^\infty}{\infty} + \frac{2^{\infty-1}}{\infty-1} + \frac{2^{\infty-2}}{\infty-2} + \&c: + \frac{2^9}{9} + \frac{2^8}{8} + \frac{2^7}{7} + \frac{2^6}{6} + \frac{2^5}{5} + \frac{2^4}{4} + \frac{2^3}{3} + \frac{2^2}{2} + \frac{2^1}{1} = 0$ ad instar $\tau\tilde{z}$ $L(+1)$. Eodem Divergentiae laborant incommodo Series etiam $L(-m)$ $= 2 \left(\frac{\left(\frac{-m-1}{-m+1}\right)^1}{1} + \frac{\left(\frac{-m-1}{-m+1}\right)^3}{3} + \frac{\left(\frac{-m-1}{-m+1}\right)^5}{5} + \frac{\left(\frac{-m-1}{-m+1}\right)^7}{7} + \frac{\left(\frac{-m-1}{-m+1}\right)^9}{9} + \&c: \right)$, ac $L(-m) = \frac{\left(\frac{m+1}{m}\right)^1}{1} + \frac{\left(\frac{m+1}{m}\right)^2}{2} + \frac{\left(\frac{m+1}{m}\right)^3}{3} + \frac{\left(\frac{m+1}{m}\right)^4}{4} + \frac{\left(\frac{m+1}{m}\right)^5}{5} + \frac{\left(\frac{m+1}{m}\right)^6}{6} + \frac{\left(\frac{m+1}{m}\right)^7}{7} + \frac{\left(\frac{m+1}{m}\right)^8}{8} + \frac{\left(\frac{m+1}{m}\right)^9}{9} + \&c: (**)$, quae, dum $-m = -1$, praeberent $L(-1) = 2 \left(-\frac{2^1}{0} - \frac{1}{3} \cdot \frac{2^3}{0} - \frac{1}{5} \cdot \frac{2^5}{0} - \frac{1}{7} \cdot \frac{2^7}{0} - \frac{1}{9} \cdot \frac{2^9}{0} - \&c: \right) = \frac{2^1}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \frac{2^4}{4} + \frac{2^5}{5} + \frac{2^6}{6} + \frac{2^7}{7} + \frac{2^8}{8} + \frac{2^9}{9} + \&c:$; adeo, ut Residuorum, illic etiam Infiniti valoris, nempe

$$2S \frac{\left(d\left(\frac{-m-1}{-m+1}\right)\right)\left(\frac{-m-1}{-m+1}\right)^\infty}{1 - \left(\frac{1+m}{1-m}\right)^2}, \text{ ac } S - m \left(\frac{m+1}{m}\right)^\infty d\left(\frac{m+1}{m}\right), \text{ necessario ha-}$$

benda

(*) Volutentur prae ceteris num.ⁱ 35^{us} . 73^{ius} .

(**) Substitutione confecta tales evadunt Series recensitae in Num.^{is} 102^{do} , 108^{vo} .

benda sit ratio, nisi in condendis monstris tempus terere, ac denuo caespitare libuerit.

392. Sed ab altiore etiam origine doctrinam ipsam repetentes nativae integritati eam restituere conabimur. Series illa toties ob oculos posita —

$$\left(\frac{(z+1)^1}{1} + \frac{(z+1)^2}{2} + \frac{(z+1)^3}{3} + \frac{(z+1)^4}{4} + \frac{(z+1)^5}{5} + \frac{(z+1)^6}{6} + \frac{(z+1)^7}{7} + \frac{(z+1)^8}{8} + \frac{(z+1)^9}{9} + \&c: \right) = L(-z)$$

nascitur ab evolutione $\tau\tilde{z}$ Exponentialis $(-z)^x$ ad instar Potestatis Binomii $(1 - (z+1))^x$. Atqui captiosam fallacemque fore talem evolutae Potestatis Seriem multiplici argumentorum robore iam fuit plenissime constitutum (*). Est igitur monstri simile quidquid ex ea derivetur, nisi prudenti artificio emendatum. Ars autem Analytica vitium illud eliminat dum $(-z)^x$ in $(-1)^x z^x$, sive in $(-1)^x (1 - (z-1))^x$ traducatur, quo confecto statim emergit $-z =$

$$\begin{aligned} & - \left(1 + \frac{(Lz)^1}{1} + \frac{(Lz)^2}{1.2} + \frac{(Lz)^3}{1.2.3} + \frac{(Lz)^4}{1.2.3.4} + \frac{(Lz)^5}{1.2.3.4.5} + \frac{(Lz)^6}{1.2.3.4.5.6} + \frac{(Lz)^7}{1.2.3.4.5.6.7} + \frac{(Lz)^8}{1.2.3.4.5.6.7.8} + \frac{(Lz)^9}{1.2.3.4.5.6.7.8.9} + \&c: \right) = -C^{Lz} = - \\ & C \left(\frac{(z-1)^1}{1} + \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} + \frac{(z-1)^4}{4} + \frac{(z-1)^5}{5} + \frac{(z-1)^6}{6} + \frac{(z-1)^7}{7} + \frac{(z-1)^8}{8} + \frac{(z-1)^9}{9} + \&c: \right) \end{aligned}$$

Nec inficias ibimus $\tau\tilde{z} - z$, semper Negativum, tum Geometrice progredi si Exponens Infinitinomialis adscriptus inceperit Arithmetice, qui erit idcirco ex Logometriae unica Lege verus $\tau\tilde{z} - z$ Logarithmus; nimirum erit $L(\pm z) =$

$$\frac{(z-1)^1}{1} - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \frac{(z-1)^5}{5} - \frac{(z-1)^6}{6} + \frac{(z-1)^7}{7} - \frac{(z-1)^8}{8} + \frac{(z-1)^9}{9} - \&c: \text{rite evoluto Binomio. Porro, tametsi } -z \text{ gemino repraesentari queat modo, scilicet,}$$

H h h h h h

aut

(*) Iuvabit Lectorem pervolutatio celerrima numerorum 30^{mi} . 31^{mi} . 71^{mi} . 72^{di} . 73^{ii} . 87^{mi} . 88^{vi} .

$$\begin{aligned}
 \text{aut per } 1 &= \frac{A}{1} + \frac{A^2}{1.2} + \frac{A^3}{1.2.3} + \frac{A^4}{1.2.3.4} + \frac{A^5}{1.2.3.4.5} + \frac{A^6}{1.2.3.4.5.6} \\
 &+ \frac{A^7}{1.2.3.4.5.6.7} + \frac{A^8}{1.2.3.4.5.6.7.8} + \frac{A^9}{1.2.3.4.5.6.7.8.9} + \&c.; \text{ aut per } \left(1 + \frac{E}{1} \right. \\
 &+ \frac{E^2}{1.2} + \frac{E^3}{1.2.3} + \frac{E^4}{1.2.3.4} + \frac{E^5}{1.2.3.4.5} + \frac{E^6}{1.2.3.4.5.6} + \frac{E^7}{1.2.3.4.5.6.7} + \\
 &\left. \frac{E^8}{1.2.3.4.5.6.7.8} + \frac{E^9}{1.2.3.4.5.6.7.8.9} + \&c. \right), \text{ positis } A = \frac{(z+1)^1}{1} + \frac{(z+1)^2}{2} + \\
 &\frac{(z+1)^3}{3} + \frac{(z+1)^4}{4} + \frac{(z+1)^5}{5} + \frac{(z+1)^6}{6} + \frac{(z+1)^7}{7} + \frac{(z+1)^8}{8} + \frac{(z+1)^9}{9} \\
 &+ \&c.; \text{ atque } E = \frac{(z-1)^1}{1} + \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} + \frac{(z-1)^4}{4} + \frac{(z-1)^5}{5} + \\
 &\frac{(z-1)^6}{6} + \frac{(z-1)^7}{7} + \frac{(z-1)^8}{8} + \frac{(z-1)^9}{9} + \&c.; \text{ hoc nihil ad rem pro Nu-}
 \end{aligned}$$

merorum Negativorum stabiliendo Systemate Logarithmico. Sit, equidem concedam invita minerva, $-z = C^{-A}$ uti $= -C^E$; veruntamen, ne repetam Seriei-Indicis A immedicabilem Divergentiam, ac Pseudo-Infinitum et nullius ufus valorem prae Serie-Indice E, Aequalitas, quae fundamentum totius esset Negativorum Logometriae, $(-z)^x = C^{-xA}$ non omnes, prouti deberet, Negativos, quemadmodum potest altera Formula $-z^x = -C^{xE}$, sed innumeros Positivos, et quandoque Imaginarios loco Negativorum, qui defunt, includit. Sin autem aliquis mordicus teneat Aequationem, perinde ac irrefragabilem, $-z = C^{-A}$ incastigato Indice A, advertat, quaeso, eam sensus expertem, quinimmo et absurdam nihil aliud complecti, praeterquam $-z = C^{-\infty}$ Plusquam-Harmonico $= 0$ contra Negativorum hypo-

$$\begin{aligned}
 \text{pothesin. Maximo fane iure, quum ob Seriem Divergentem } 1 &= \frac{A}{1} + \frac{A^2}{1.2} \\
 &+ \frac{A^3}{1.2.3} + \frac{A^4}{1.2.3.4} + \frac{A^5}{1.2.3.4.5} + \frac{A^6}{1.2.3.4.5.6} + \frac{A^7}{1.2.3.4.5.6.7} + \\
 &\frac{A^8}{1.2.3.4.5.6.7.8} + \frac{A^9}{1.2.3.4.5.6.7.8.9} + \&c.; \text{ facto } A = \infty \text{ Plusquam-Harm}^\circ. (*), \\
 &\text{Residua}
 \end{aligned}$$

(*) Confule num^{um}. 56.

Residua Infinitae Magnitudinis sus deque haberi nequeant, nec ideo Series

illa ad instar aliarum Convergentium, quarum Limes est $1 - \frac{(\infty \text{ Harm.})^1}{1} +$

$$\frac{(\infty \text{ Harm.})^2}{1.2} - \frac{(\infty \text{ Harm.})^3}{1.2.3} + \frac{(\infty \text{ Harm.})^4}{1.2.3.4} - \frac{(\infty \text{ Harm.})^5}{1.2.3.4.5} + \frac{(\infty \text{ Harm.})^6}{1.2.3.4.5.6} -$$

$$\frac{(\infty \text{ Harm.})^7}{1.2.3.4.5.6.7} + \frac{(\infty \text{ Harm.})^8}{1.2.3.4.5.6.7.8} - \frac{(\infty \text{ Harm.})^9}{1.2.3.4.5.6.7.8.9} + \&c: = C^{-\infty \text{ Harm.}} = 0, \text{ in}$$

C^{-A} verti possit. Impar est ergo significationi Negativorum Exponentialia C^x vel B^x , cui succedit, recte instituta Calculi emendatione, $-C^x$ aut $-B^x$; ita, ut unica methodo pateat Basis illa Logarithmorum Realium ad Negativos Numeros pertinentium, quam frustra quaesivit Leibnítius, et irreperitam signum esse eorum impossibilitatis luculentissimum autumavit (*). Aliter in

aperto locatur absurditas expressionis ipsius $-z = C^{-A} = \left(1 - \frac{A}{\infty}\right)^\infty =$

$\left(1 - \frac{\infty'}{\infty}\right)^\infty = (-\infty'')^\infty$ (*), quae postrema Formula vagum inanemque

praesefert valorem, nisi ad Calculi iussa, vice $\tau \approx C^{-A}$, purioris et ad rem

apti Exponentialis $-C^E$ delectus haberi velit.

393. Mira itaque argumentorum concordia sancitum videtur consequentium Theorematum ius integerrimum.

I°. „*Ex Analytica simul, ac Geometrica Continuitate dimanant Logarithmi Negativorum iidem atque illi Positivorum magnitudine aequalium* „.

II°. „*Terminus Generalis cuiusvis Progressionis Geometricae formam induit $\pm(a^x)$, id iubente Continuitate* „; adeo, ut tametsi gratia exempli 0, &c.: $2^{-7}, 2^{-6}, 2^{-5}, 2^{-4}, 2^{-3}, 2^{-2}, 2^{-1}, 1, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7$, &c: satis superque faciat strictis Legibus Arithmeticis, nihilo tamen fecius sit eius inseparabile complementum quoad Geometriam et Analytin Algebraicam Series 0, &c.: $-2^{-7}, -2^{-6}, -2^{-5}, -2^{-4}, -2^{-3}, -2^{-2}$

(*) Praesertim ad pag^{as} . 288^{am} . 296^{am} . 303^{am} . 304^{am} . 307^{am} . 312^{am} . Voluminis II.
Commerci &c: nuper citati.

(**) Ex numeri 54^{ti} . considerationibus.

$2^{-2}, -2^{-1}, -1, -2^1, -2^2, -2^3, -2^4, -2^5, -2^6, -2^7, \&c:$
uti pars altera, atque vicissim.

III°. Definiri sic posset, nec incompete quidem, Logistica. „ *Curva est Transcendens, Cuspidata, sed Cuspide ad Infinitam distantiam sito, ideoque invisibili, Diametrum in Asymptoto, vel melius Pseudo-Asymptoto habens, et Aequatione praedita* $x = L(\pm y)$, *eadem nimirum cum altera* $C^{2x} = y^2$ „.

Praeter duos Ramos Reales considerare fas erit ex abundantia in Logarithmica Linea innumeros Imaginarios; quam affectionem Crurum sine numero, dum Impossibiles quoque censui subiiciantur Ordinatae, Transcendentibus universis Curvis communem fore speculatores affatim norunt Geometrae.

Hoc Aequatio ipsamet Curvae statuit ita comparata, $y = Lx = S \frac{dx}{x}$

$= \infty \text{Harm}^\circ \pm n\pi\sqrt{-1}$ (*), utpote quae tam pro $+x$, tam pro $-x$, unicum Reale Crus in hypothese $\tau \approx n \approx 0$, et alia sine numero Imaginaria ornatissime satis indigitat propter valores innumeros istius $\pm n$ Coefficientis. Aequalitas igitur a Foncenexio generalissime, ut ait, instituta $y = mC^x$ (**), seu potiore iure, ob Parametrum m de Curva in Curvam variabilem, $y = zC^x$ non respicit Logisticarum Ramos, *Incontinuos* omnes, ad sensum sagacissimi Auctoris, nulloque coniunctos vinculo Analytico: bina etenim quaelibet Crura, eadem Parametro sed inverso Signo expressa, ad modum $y = \pm aC^x$ constituunt ipsummet Locum Analyticae Continuum, atque Geometrice. Qui Cruribus illis Imaginariis analogos Reales, aeque innumeros, contemplari vellet in Curva, Exponentialem eius Aequationem

componeret sic elaboratam $\pm m(1)^{x\sqrt{-1}} = y$, aut $m^2(1)^{2x\sqrt{-1}} = y^2 = 0$. Profecto cuilibuerit Abscissae x respondent Ordinatum - Adplicatae

$\pm mC^{\pm n\pi \cdot x}$, universae nimirum Reales; nec non altero in Loco pariter

(*) Constat ex numeris 182^{do} . 294^{to} . 391^{mo} .

(**) Est τ idem ac Systematis Logarithmici Proto - Numerus. Vide numerum

ter Exponentiali $\pm m (-1)^{x\sqrt{-1}} = y$, five $m^2 (-1)^{2x\sqrt{-1}} = y^2$

$= 0$ innumerae quoque Ordinatae $y = \pm mC^{\pm \lambda P \cdot x}$ reperirentur (*).

Eccine exempla simpliciorum Linearum Transcendentium, quae nedum gaudent numero ac longitudine Infinitis Ramis, Asymptotoque Diametro, verum etiam sub apparente Imaginarietate Abscissarum Realem Curvam concludunt. Quinimmo Theorice istam obiter amplificantes monemus, quod, positis Imaginariis Abscissis simul et Ordinatis, nihilo secius Aequatio Ex-

ponentialis $\pm C^{x\sqrt{-1}} = y$, seu $C^{2x\sqrt{-1}} = y^2 = 0$ ad Realem Locum pertineat, nempe ad Logisticam: fit siquidem haec Imaginaria $y = \pm$

$$\left(1 + \frac{(x\sqrt{-1})^1}{1} + \frac{(x\sqrt{-1})^2}{1.2} + \frac{(x\sqrt{-1})^3}{1.2.3} + \frac{(x\sqrt{-1})^4}{1.2.3.4} + \frac{(x\sqrt{-1})^5}{1.2.3.4.5} \right. \\ \left. + \frac{(x\sqrt{-1})^6}{1.2.3.4.5.6} + \frac{(x\sqrt{-1})^7}{1.2.3.4.5.6.7} + \&c. \right), \text{ ideoque, conversa Abscissa } x\sqrt{-1} \text{ in}$$

Realem z , emergit $y = \pm C^z$. Tanta praeditum est veritate celeberrimum *Effatum* illud penes Logometriae Imaginariae cultores „ quod, dum in Circulo esse putant, reapse sint in Hyperbola (**).

394. Seiunctae partes $y = C^x$, $y = -C^x$ Aequationis ad Logarithmicam Paradoxorum venatoribus ansam forte fortuna praebere possent perinde ac si, praeter duo Curvae Crura haecenus propugnata, innumera super utrosque Ramos, pleonasmii ad instar, Coniugata Puncta iacerent. Quoties etenim fuerit $x = \pm \frac{m}{2n}$ minimis expressum Numeris, totidem, nempe Infinities, discreta Puncta in Crure opposito locari videntur, quum sit

tam $y = \pm \sqrt[n]{C^{\pm \frac{m}{n}}}$, tam $y = \mp \sqrt[n]{C^{\pm \frac{m}{n}}}$. Ex adverso, si vere adforent ista Puncta, liquido constaret ab Aequatione, quae statuit Positionem Tangentis,

I i i i i

videlicet

(*) Haec Theoria prostat apud numeros 299^{um} . 300^{um} . 306^{um} .

(**) Sic exposuimus in numeris potissimum 172^{do} . 178^{vo} . 192^{do} . 194^{to} . 371^{mo} .

videlicet $\frac{ydx}{dy} = \frac{\pm C^x dx}{\pm C^x dx}$; quandoquidem ob Puneta Curvae perimetro adhaerentia plures ea complecti deberet Radices, atque offerre, uti omnibus innotescit (*), $\frac{dx}{dy} = \frac{0}{0}$, $\frac{dx^2}{dy^2} = \frac{0}{0}$ &c.; et haudquaquam $\frac{dx}{dy} = \frac{1}{y}$, quemadmodum supra. Perspicuum est autem, nondum pro Transcendentibus Curvis tutiore methodo contemplandis aptius quiddam repertum fuisse prae *Quadraturis*. Castigata itaque *Exponentialitate*, commentis gignendis maxime proclivi, ac traducta Aequatione Logisticae $y^2 = C^{2x}$ in alteram $y = S \frac{dx}{x}$, praesumpta Paradoxi species confestim aufertur.

395. Doctrinae superius latae de Logarithmis Negativorum non refragantur, quae habuit Eulerus. Praestantissimus siquidem Auctor eo rem ingeniose perduxit, ut, facta testata vera Basi *autonomastica* C, Negativos Numeros — $y = y. — 1 = C^{Ly} . C^{\pm \lambda P \sqrt{-1}}$ hoc Exponentiali $C^{Ly \pm \lambda P \sqrt{-1}}$ repraesentaret, et ideo effingeret $\tau \delta L (— y) = Ly \pm \lambda P \sqrt{-1}$, nempe Imaginarium; quod certe fieri posse inter innumera Logometriae Systemata nihil prohibet (**). Veruntamen Negativorum significatio pari, ac fortasse puriore

(*) Sagacissimus Bragelongnius in *Commentariis Academiae Parisiensis* annorum MDCCXXX^{mi}. ac XXXI^{mi}., fundamenta ponens Theoriae Linearum quarti Ordinis, huiusmodi Puneta Perimetro adhaerentia, seu *Triplicia-Invisibilia* primus consideravit. Quum illa sint geminae speciei, nimirum aut orta ab evanescencia *Ovalium* aut *Lemniscatarum* adhaerentium, hinc fit, ut prior casus ope Aequationis tertii Gradus ad Sub-Tangentem inveniendam natae, cuius trium Radicum una tantum Realis emergat et ceterae Imaginariae, ab altero, in quo ternae Radices simul Reales exstent atque invicem aequales, aptissime distinguatur. In Lineis supra Ordinem quartum multo magis Foliorum seu Plexuum illarum Perimetris adhaerentium, et in Puneta abeuntium difficultas, numerusque Radicum excrescet. Quotusnam foret in Coniugatis Logisticae Punctis, scilicet Curvae Ordinis Infinitecupli, vel omnes Ordines Transcendentis?

(**) Rursus perlustrare calcem numⁱ. 141^{mi}. ne pigeat.

puriori religione obtinetur hac methodo — $y = C^{Ly}$, et ex ea profuit $\tau\tilde{x}$ — y Logarithmus Realis. Eulerianum est Imaginariam Logisticam

sibi ob oculos ponere Aequatione praeditam $C^{x \pm \lambda P \sqrt{-1}} \rightarrow y = 0$, seu

brevius $C^{x \rightarrow A \sqrt{-1}} \rightarrow y = 0$: meum est Aequationis delectus $C^x \rightarrow y = 0$,

vel potius $C^{2x} \rightarrow y^2 = 0$, Realisque Logisticae in Positivo-Negativis

confimillimus usus. Digrediens Eulerus ab Aequatione $(1 + \frac{v}{\infty})^\infty \rightarrow 1 = 0$,

seu $C^v \rightarrow 1 = 0$, quae singularis casus est oecumenicae $C^v \pm y = 0$, aut

$(1 + \frac{v}{\infty})^\infty \pm y = 0$, five $C^{2v} \rightarrow y^2 = 0$, vel potius $(1 + \frac{2v}{\infty})^\infty \rightarrow y^2 = 0$,

hanc postremam vere *Continuam*, atque universaliorē neglexit, Logarithmis simul Positivorum et Negativorum satis superque facientem. Mirum autem,

quod non persenserit, eodem saltem iure, quo $L(-y) = \frac{(y-1)^1}{1}$

$-\frac{(y-1)^2}{2} + \frac{(y-1)^3}{3} - \frac{(y-1)^4}{4} + \frac{(y-1)^5}{5} - \frac{(y-1)^6}{6} + \frac{(y-1)^7}{7} - \&c:$

$\pm \lambda P \sqrt{-1}$ ex proprio Systemate in Logistica Imaginaria statuitur, futurum etiam, ut, dempto $\pm \lambda P \sqrt{-1}$, penitusque servata Arithmetica Pro-

gressione, sit idem $L(-y) = \frac{(y-1)^1}{1} - \frac{(y-1)^2}{2} + \frac{(y-1)^3}{3} - \frac{(y-1)^4}{4} +$

$\frac{(y-1)^5}{5} - \frac{(y-1)^6}{6} + \frac{(y-1)^7}{7} - \&c: = L(+y)$ in Reali Logistica.

Composito igitur, nisi mea me fallit sententia, litis discrimine, plus equidem subtilitatis ingenique in ea positum, quam rei, aperte constabit. Sed bonarum Artium cultores habent, quod aeternum admirari possint, secretae illius cognitionis vinculum, quo continentur *Binomium* Neutoni, et universa *Logometria*. Vixdum etenim in comperto fuit Coefficientium *Binomii* Lex, ob quam simul collecti Numeri omnes in iis singillatim multiplicantes $x, x^2, x^3, x^4, x^5, \dots, x^n$ &c., aliasve $\tau\tilde{x}$ Indicis x Potestates ita sibi respondeant, ut hisce Summis proprio ordine numeratis $S', S'', S''', S^{iv}, S^v,$

.....

$$\dots\dots, S^v \text{ \&c: emergant } S'' = \frac{(S')^2}{1.2}, S''' = \frac{(S')^3}{1.2.3}, S^{iv} = \frac{(S')^4}{1.2.3.4}, S^v = \frac{(S')^5}{1.2.3.4.5}, \dots\dots, S^v = \frac{(S')^n}{1.2.3\dots n} \text{ \&c: } (*) \text{ lux illico affulfit, et inopina faci-}$$

litas. Tanti interest saepenumero vel obsoleta rimari, primasque rerum cognoscere causas.



CAPUT X.

(*) Hoc tribuit num^{us}. 46^{tus}. si cum 44^{to} et 49^{no}. comparetur.

C A P U T X.

CALCULI INTEGRALIS PARADOXON ENODATUM.

396. **N**ihil frequentius in Mathematicum Scriptis occurrit quam cacoethes Algebraicam Analyfin redarguendi falsitatis, seu, veluti non pauci iactitant, impotentiae, dum Summatoriam $\int \frac{Mdx}{x}$ exhibere coacta offert $M \cdot \frac{1}{0}$ iuxta usitatam Potestatum Fluxione Radicis affectarum Integrationem. Hoc quasi Oraculi responsum irrident nonnulli; sed alii, Paradoxorum amatores, e-contra amplectuntur libentissime, et animo exultantes, penes quos obscuriora minusque intellecta maioris sunt dignitatis, atque elegantiae. Quanto etenim plausu, ut similia similibus componam, Veritatis Mathematicae obrectatores, aut nubem saepius pro Iunone salutare sueti, Fractionem indeterminatam $\frac{0}{0}$ receperunt, varia in Matheosos puritatem loquentes, antequam expressionem illam contra Rollium recte interpretati nodum solvissent feliciter Ioannes Bernoullius anno MDCCIV., Saurinius vertentibus annis MDCCII. III. XVI. XXIII., et Guisneius sub anni lapsum MDCCVI., de universa Tangentium, ac praecipue Punctorum-Multiplicum theorie optime meriti! Nec, meo saltem iudicio, rem acu tetigit vel ipse Montucla (*) de Subtangentibus ad Ramorum cuiusvis Curvae intersectiones pertinentibus agens tum, quum Algebram nihil respondere ait inani, sensusque experte Fractione $\frac{0}{0}$, quam ego eloquii ingeniique plenissimam non unicus censeo. Eodem pacto, quum $\int \frac{dx}{x} = \int x^{-1} dx = \frac{x^0}{0} = \frac{1}{0} = \infty$ Aream Asymptoticam

K k k k k k

cam

(*) Pag^a. 107. Voluminis II. *Historiae Mathematicum*.

cam Hyperbolae Aequilaterae Apolloniana, vel Infinitum, quod vocant Harmonicum, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$

$+ \frac{1}{11} + \&c: \pm X$ Magnitudinem vel Functionem a Centrali x Abscissa pendentem significet, nescio quomodo aliter se gerere, praeter eloquentissimum datum responsum $\frac{x^0}{0}$, debuisset Analysis. Id mente volutantes

moderatiores Geometrae non amplius fallaciae, attamen virium defectus Algebram insimulant, quia $\frac{x^0}{0}$, sive $\frac{1}{0}$ quodlibet Infiniti genus, non speciale illud Harmonicum, uti isto in casu par esset, indigitet. Iure hoc didicent, an iniuria, suscipiam paucis evidentissime demonstrandum (*).

397. Potestates Variabilium vulgari modo qui versent, meliorem hoc in rei Summatoriae periculo frustra quaerunt fortunam. Exponentialibus vero

in subsidium vocatis res secus eveniet. Quandoquidem habetur $x^m = 1 + \frac{(mLx)^1}{1} + \frac{(mLx)^2}{1.2} + \frac{(mLx)^3}{1.2.3} + \frac{(mLx)^4}{1.2.3.4} + \frac{(mLx)^5}{1.2.3.4.5} + \frac{(mLx)^6}{1.2.3.4.5.6} +$

$\frac{(mLx)^7}{1.2.3.4.5.6.7} + \&c: ;$ ita, ut $\frac{x^m}{m} = \frac{1}{m} + Lx + \frac{m(Lx)^2}{2} + \frac{m^2(Lx)^3}{6} +$

$\frac{m^3(Lx)^4}{24} + \frac{m^4(Lx)^5}{120} + \frac{m^5(Lx)^6}{720} + \frac{m^6(Lx)^7}{5040} + \&c: ,$ et ideo $\frac{x^0}{0}$, vel

$\frac{x^{-1+1}}{-1+1}$, aut $S \frac{dx}{x}$, sive $S x^{-1} dx = \frac{1}{0} + Lx = \infty + Lx$, quae

postrema Formula votum adimplet, et ∞^{um} non indeterminatum, sed ex conditione Problematis definitum, ac semper Harmonicum vel Logarithmicum

statuit. Profecto, sit ex natura Problematis resolvendi $S \frac{dx}{x} = 0$ dum $x = 0:$

(*) Additio Constantis nequidem Infiniti Speciem determinat. Fit namque $\frac{1}{0} + A = \infty$.

$= 0$: proveniet ea in hypothefi $\tau \delta \infty = - L0 = -$

$$\left(- \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \&c: \right) \right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \&c:; \text{quamob-}$$

$$\text{rem } S \frac{dx}{x} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$$

$$\&c: + Lx = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \&c:$$

$$+ \frac{(x-1)^1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \frac{(x-1)^7}{7}$$

$$- \frac{(x-1)^8}{8} + \frac{(x-1)^9}{9} - \frac{(x-1)^{10}}{10} + \&c:, \text{ ubi omnia in mensura, ac pon-}$$

dere sunt, et ad unguem determinata. Quinimmo, etiam universalius, sup-

ponatur $S \frac{dx}{x} = 0$ dum $x = n$, eritque $\infty + Ln + B = 0$, nimirum

B Constans addenda $= -\infty - Ln$, et idcirco $S \frac{dx}{x} = \infty - \infty +$

$$Lx - Ln = Lx - Ln = L \left(\frac{x}{n} \right) = \frac{\left(\frac{x-n}{n} \right)^1}{1} - \frac{\left(\frac{x-n}{n} \right)^2}{2} +$$

$$\frac{\left(\frac{x-n}{n} \right)^3}{3} - \frac{\left(\frac{x-n}{n} \right)^4}{4} + \frac{\left(\frac{x-n}{n} \right)^5}{5} - \frac{\left(\frac{x-n}{n} \right)^6}{6} + \frac{\left(\frac{x-n}{n} \right)^7}{7} - \&c:, \text{ quemad-}$$

modum exigunt Integralis Calculi canones Proto-Numerum n respicientes.

Quicumque ergo fuerit $\tau \delta n$ Proto-Numeri valor hypotheticus ab 0 ad ∞ ,

Integratio elargietur omnibus determinatam absolutamque numeris Magnitu-

dinem, siquidem etiam in Limite $n = \infty$ esset $S \frac{dx}{x} = L \left(\frac{x}{\infty} \right) = Lx$

$$- L\infty = Lx - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \right)$$

$$+ \&c:), \text{ uti supra.}$$

398. Liceat paullisper in re non parvi momenti immorari. Integralis Aequatio oecumenica $S \frac{dx}{x} = L \left(\frac{x}{n} \right)$ Summatoriam peculiarem, de qua

initio actum, restituit. Fiat enim $n = 0$, statimque fluat $S \frac{dx}{x} = L \left(\frac{x}{0} \right)$

$$= Lx - L0 = Lx + L\infty = \frac{(x-1)^1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \frac{(x-1)^7}{7} - \&c: + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} +$$

$\frac{1}{7} - \&c:$ prouti superius inventum. Igitur, vel Infiniti Nota haec ∞ delea-

tur ope adiunctae Constantis, quod posterius factum, aut maneat Infiniti eadem Nota, ut in priore methodo neglectae Constantis, emergit abscondita Infiniti Logarithmici species frequentissimo in casu, quo Integrale nullefcatur quum $x = 1$, perfectamque recipit determinationem. Amoenitate autem re-

ferta quammaxime mihi videtur Aequalitas exinde deducta $\frac{(x-1)^1}{1} - \frac{(x-1)^2}{2}$

$$+ \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \frac{(x-1)^7}{7} - \&c: + 1 - \frac{1}{2} +$$

$$\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \&c: = \left(\frac{x}{0} \right)^1 - \left(\frac{x}{0} \right)^2 + \left(\frac{x}{0} \right)^3 -$$

$$\left(\frac{x}{0} \right)^4 + \left(\frac{x}{0} \right)^5 - \left(\frac{x}{0} \right)^6 + \left(\frac{x}{0} \right)^7 - \&c:, \text{ cuius Membrum Compa-}$$

rationis, suapte natura Indeterminatum, propter suppetias iam traditas ab indefinita emergit Infinitate, Confectarium non inelegans praeterea dima-

$$\text{nat, nimirum, } \left(x \cdot \frac{1}{0} \right)^1 - \left(x \cdot \frac{1}{0} \right)^2 + \left(x \cdot \frac{1}{0} \right)^3 - \left(x \cdot \frac{1}{0} \right)^4 + \left(x \cdot \frac{1}{0} \right)^5 -$$

$$\left(x \cdot \frac{1}{0} \right)^6 + \left(x \cdot \frac{1}{0} \right)^7 - \&c: = \frac{x^1 \cdot \infty^1}{1} - \frac{x^2 \cdot \infty^2}{2} + \frac{x^3 \cdot \infty^3}{3} - \frac{x^4 \cdot \infty^4}{4} +$$

$$\frac{x^5 \cdot \infty^5}{5} - \frac{x^6 \cdot \infty^6}{6} + \frac{x^7 \cdot \infty^7}{7} - \&c: = L\infty + Lx; \text{ quam veritatem inter}$$

Infiniti

Infiniti iocos non minimam confirmant vulgares Logometriae Canones, quum fit $L\infty + Lx = L(x.\infty)$; et haec est altera methodus, aequae prompta ac tutissima, Infiniti speciem illam petitam determinandi.

399. Rerum interim affinitate favente liquido constat, quantum a resolutionis concepta spe aberraverit Leibnitius Quadratorum reciprocorum Differentiam, aut Summam $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \dots$ inquirens. Ipse enim in Epistola ad Ioannem Bernoullium Hanoverae scripta anno MDCXCVI. Aequationem Logarithmicam nactus universalis-

simam $\frac{1}{e+1} x^{e+1} L(1+x) - \frac{1}{e+1} S \frac{x^{e+1} dx}{1+x}$ affirmavit, verso Indice

e in -1 , factaque post Integrationem variabili $x=1$, obtineri tunc posse quaesitum Seriei illius Infinitae valorem. Id adprobasse omnino videtur Bernoullius, cuius analytici tentaminis fusiolem historiam servant fideliter Epistolae XXXVIII^a . ac XXXIX^a . *Commercii Philosophici et Mathematici*. Ve-

rumtamen mea methodus exhibet $\frac{1}{-1+1} x^{-1+1} = \infty + Lx$: quamobrem singularis Summatoria, cui investigandae operam dedit Leibnitius, erit qualem optabat, devicta Infinitorum molestia, $(\infty + Lx)$.

$L(1+x) - S \frac{(\infty + Lx) dx}{1+x} + B$, videlicet $\infty L(1+x) + Lx(L(1+x)) - \infty . L(1+x) - S \frac{(Lx) dx}{1+x} + B$, nempe, elisis feliciter Infinitis, si Aequationem identicam nulliusve usus nolumus ad Summationem Seriei adipiscendam,

$Lx(L(1+x)) - S \frac{(Lx) dx}{1+x} + B = \frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \frac{x^5}{5^2} - \frac{x^6}{6^2} + \frac{x^7}{7^2} - \frac{x^8}{8^2} + \frac{x^9}{9^2} - \dots$, five tandem $Lx(L(1+x)) - S \frac{L(x-1) dx}{x} + B$, aut $Lx(L(1+x)) - S \frac{L(1-x) dx}{x} + B$, seu $Lx.$

$(L(1+x)) + \left(\frac{(x+1)^1}{1^2} + \frac{(x+1)^2}{2^2} + \frac{(x+1)^3}{3^2} + \frac{(x+1)^4}{4^2} + \frac{(x+1)^5}{5^2} + \frac{(x+1)^6}{6^2} + \dots \right)$

$$\frac{(x+1)^6}{6^2} + \frac{(x+1)^7}{7^2} + \frac{(x+1)^8}{8^2} + \frac{(x+1)^9}{9^2} + \&c:) + B = \frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \frac{x^5}{5^2} - \frac{x^6}{6^2} + \frac{x^7}{7^2} - \frac{x^8}{8^2} + \frac{x^9}{9^2} - \&c: \text{ in Infinitum.}$$

Quod Integrale *completum* Seriem Infinitam complectitur, cuius aequae difficultis Summa non constat; deindeque, cum ex Problematis caractere nulescere illud debeat simul ac evanuerit x , praebabit $-(0. \infty) + B$, vel potius

$$-n + B(*), \text{ aut unicam Siglam } E = \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \right.$$

$$\left. \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \&c: \right) = -2 \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \right.$$

$$\left. \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} - \&c: \right); \text{ unde evidens fit adeo comparatam fore}$$

Constantem, ut eandem Seriei *summandae* difficultatem involvat, quae resolvenda proponebatur. Hic autem eventus in causa est, cur forma inutiliter elaboratae Integralis, quae tota ad Logometriam Hyperbolicam pertinere videbatur, nihil imminuat Theorematis veritatem decernentis Naturalium Quadratorum reciprocorum Summas a Circuli Tetragonismo pendere.

400. Aliis quamplurimis Formulis enucleandis, ubi occurrat $\frac{x^{m+1}}{m+1}$

in hypothesi $m = -1$, usui esse poterunt speculationes hactenus proditae; nec nimium ab instituto deflectam itinere, si omnigenam Exponentialium prae communi Potestatum Calculo universalitatem novo tentamine locupletaverim. Iam in numero 338. absolutissimam mihi contigit invenisse Fluxionum cuiusvis Ordinis Potestatum x^m expressionem adiumento Seriei Exponentialis; adeo, ut nihil perficiendum superfit, praeterquam Fluents $\sum x^m dx$, cognitis dumtaxat Exponentialibus ipsis, aequae moliri. Tunc etenim, si minus poterunt usitata Potestates evolvendi Systemata, quae aliquando nodos solvere, veluti

(*) Constans n ex demonstratis in numeris 128. 129. est verum Nihilum; sed, cedo, huiusce symptomatis in ista Summatione nullus fiat equidem usus.

luti superius visum, non valent, nec ultra confinia se porrigunt Indicium non variabilium, abundantius ex adverso Formulae Exponentialium in Calculo Finitorum, et Infinitorum generalitatem curabunt. Quod adhuc ergo deerat ita compleo brevissime, sequente Typo disposito.

$$\begin{aligned}
 S^m x &= S dx \left(1 + \frac{m^1(Lx)^1}{1} + \frac{m^2(Lx)^2}{1.2} + \frac{m^3(Lx)^3}{1.2.3} + \frac{m^4(Lx)^4}{1.2.3.4} + \frac{m^5(Lx)^5}{1.2.3.4.5} + \frac{m^6(Lx)^6}{1.2.3.4.5.6} + \&c: \right) \\
 &= x + S \frac{m^1}{1} (Lx)^1 dx + S \frac{m^2}{1.2} (Lx)^2 dx + S \frac{m^3}{1.2.3} (Lx)^3 dx + S \frac{m^4}{1.2.3.4} (Lx)^4 dx + S \frac{m^5}{1.2.3.4.5} (Lx)^5 dx + \&c: \\
 &= x + \frac{m^1}{1} x (Lx)^1 + \frac{m^2}{1.2} x (Lx)^2 + \frac{m^3}{1.2.3} x (Lx)^3 + \frac{m^4}{1.2.3.4} x (Lx)^4 + \frac{m^5}{1.2.3.4.5} x (Lx)^5 + \&c: \\
 &\quad - m^1 x - \frac{m^2}{1} x (Lx) - \frac{m^3}{1.2} x (Lx)^2 - \frac{m^4}{1.2.3} x (Lx)^3 - \frac{m^5}{1.2.3.4} x (Lx)^4 - \&c: \\
 &\quad + m^2 x + \frac{m^3}{1} x (Lx) + \frac{m^4}{1.2} x (Lx)^2 + \frac{m^5}{1.2.3} x (Lx)^3 + \&c: \\
 &\quad - m^3 x - \frac{m^4}{1} x (Lx) - \frac{m^5}{1.2} x (Lx)^2 - \&c: \\
 &\quad + m^4 x + \frac{m^5}{1} x (Lx) + \&c: \\
 &\quad - m^5 x - \&c: \\
 &= \frac{1}{m+1} \cdot x + \frac{m^1}{m+1} \cdot \frac{x(Lx)^1}{1} + \frac{m^2}{m+1} \cdot \frac{x(Lx)^2}{1.2} + \frac{m^3}{m+1} \cdot \frac{x(Lx)^3}{1.2.3} + \frac{m^4}{m+1} \cdot \frac{x(Lx)^4}{1.2.3.4} + \frac{m^5}{m+1} \cdot \frac{x(Lx)^5}{1.2.3.4.5} + \&c: \\
 &= \frac{x}{m+1} \left(1 + \frac{(mLx)^1}{1} + \frac{(mLx)^2}{1.2} + \frac{(mLx)^3}{1.2.3} + \frac{(mLx)^4}{1.2.3.4} + \frac{(mLx)^5}{1.2.3.4.5} + \frac{(mLx)^6}{1.2.3.4.5.6} + \&c: \right) \\
 &= \frac{x}{m+1} x^m = \frac{x^{m+1}}{m+1}.
 \end{aligned}$$

Desiderata fortassis haecenus foret, aut ob Calculi perplexitatem non tam simpliciter extricanda mirabilis ista Integralium per diversa itinera arcessitorum aequipollentia, nisi diagonalibus nunc primum Infinitae Seriei Lineis simul collectis Lectorum oculis obversaretur.

401. Ratio hinc quoque, caussaue intima in comperto est Paradoxî, si velis, illius, a quo primum sumus digressi. Formula siquidem $\frac{x^{m+1}}{m+1}$

valorem finitum praeseferebat non Logarithmicum, ideo Infinita, ac Logarithmica evadit vixdum abeat m in -1 , quia hoc unico in casu Diagonales Coefficientium Lineae singulae induunt *Parallelarum* Serierum formam cum successione Signorum. Profecio

$$1 - m^1 + m^2 - m^3 + m^4 - m^5 + m^6 - \&c: = 1 + 1 + 1 + 1 + 1 + 1 + 1 + \&c: = \infty$$

$$\frac{m^1}{1} - \frac{m^2}{1} + \frac{m^3}{1} - \frac{m^4}{1} + \frac{m^5}{1} - \frac{m^6}{1} + \&c: = - (1 + 1 + 1 + 1 + 1 + 1 + 1 + \&c:) = -\infty$$

$$\frac{m^2}{1.2} - \frac{m^3}{1.2} + \frac{m^4}{1.2} - \frac{m^5}{1.2} + \frac{m^6}{1.2} - \&c: = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \&c: = \frac{\infty}{2} = \infty$$

$$\frac{m^3}{1.2.3} - \frac{m^4}{1.2.3} + \frac{m^5}{1.2.3} - \frac{m^6}{1.2.3} + \&c: = - \left(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \&c: \right) = - \frac{\infty}{6} = -\infty$$

$$\frac{m^4}{1.2.3.4} - \frac{m^5}{1.2.3.4} + \frac{m^6}{1.2.3.4} - \&c: = \frac{1}{24} + \frac{1}{24} + \frac{1}{24} + \frac{1}{24} + \frac{1}{24} + \frac{1}{24} + \frac{1}{24} + \&c: = \frac{\infty}{24} = \infty$$

$$\frac{m^5}{1.2.3.4.5} - \frac{m^6}{1.2.3.4.5} + \&c: = - \left(\frac{1}{120} + \frac{1}{120} + \frac{1}{120} + \frac{1}{120} + \frac{1}{120} + \frac{1}{120} + \frac{1}{120} + \&c: \right) = - \frac{\infty}{120} = -\infty$$

fine ullo Limite. Sed in tanta Infinitorum copia, et Signorum alternatione veram pervestigare expressionis naturam hoc opus, hic labor est. Ad inlustrandum eo magis *parallelarum* Serierum usum (*) divitemque Analyseos penum, quantum potero, amplificandam mearum virium periculum faciam, tametsi obiter, parciusque, ne ad finem properans hiantem in re levissima orationem conficere videar.

402. Profecuta itaque hypothefi Indicis $m = -1$, non parum artificii requirit ad utilitatem facta resolvendi Problematis enodatio. Ita fortasse, fin aliter nescio, Analyfi cedit. Cumulatis Diagonalibus terminis habetur ∞x

$$\frac{\infty x}{1}$$

(*) Eorum non obliviscere, quae ad hasce Series efficiendas momentosiores supra enarravimus in num°. 265.

$$\frac{\infty x (Lx)^1}{1} + \frac{\infty x (Lx)^2}{1.2} - \frac{\infty x (Lx)^3}{1.2.3} + \frac{\infty x (Lx)^4}{1.2.3.4} - \frac{\infty x (Lx)^5}{1.2.3.4.5} + \&c:$$

$$= S x^{-1} dx; \text{ scilicet, } \infty x.C^{-Lx} + B, \text{ aut } \frac{\infty x}{C^{Lx}} + B = \frac{\infty x}{x} + B = \infty x^{\frac{1}{\infty}} + B =$$

$$\infty (1 + (x-1))^{\frac{1}{\infty}} + B = S x^{-1} dx. \text{ Praeterea, si evolvatur Binomii}$$

Potestas ad morem Neutoni iuxta numeros 59. 294., aliosve, proveniet B +

$$\infty \left(1 + \frac{(x-1)^1}{\infty} - \frac{(x-1)^2}{2.\infty} + \frac{(x-1)^3}{3.\infty} - \frac{(x-1)^4}{4.\infty} + \frac{(x-1)^5}{5.\infty} - \frac{(x-1)^6}{6.\infty} \right. \\ \left. + \frac{(x-1)^7}{7.\infty} - \&c: \right) = \infty + Lx + B = S x^{-1} dx = \frac{x^0}{0} + B; \text{ qua me-}$$

thodo latens in Sinu Infiniti Functio Logarithmica τx Variabilis x emergit feliciter, et in aperto locatur. Directa igitur Integratio expressionis differentialis $\frac{dx}{x}$ facili ratione obtinebitur, absque eo quod vi numeri 335. meminerimus

esse $d(\infty x^{\frac{1}{\infty}}) = \frac{dx}{x}$. Nec difficultatem moveat character Infinitatis ∞ , quam

inita Serierum Parallelarum supputatio aequalem statuit $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \&c:$, quandoquidem cunctis in Problematum casibus e-medio tollitur ab adiicienda *Constante* B , quae indolem opportunam recipiat. Fac etenim speciminis instar summatoriam Magnitudinem simul cum x evanescentem; erit-

$$\text{que } \infty + B + Lo = 0, \text{ sive } \infty + B = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$$

$$\frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \&c:, \text{ et Integrale Incompletum } S x^{-1} dx = 1 + \frac{1}{2} + \frac{1}{3} +$$

$$\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c: + Lx, \text{ perinde ac in praecedente num}^{\circ}. 397.; \text{ unde Infinitum prodit Logarithmicum occasione data peculia-}$$

ris Problematis, quod illud evocet, enodandi.

403. Consimile proposuit ad Aequationem summandam $dx = \frac{dy}{y}$ ar-

M m m m m m

tificium

tificium in primo *Miscellaneorum Taurinensium* Volumine solertissimus Fontenexius. Nihil moror additionem Moduli (a), cum per se satis liqueat

fore $\int \frac{ady}{y} = a \int \frac{dy}{y}$, ideoque in unam coalescere geminae Formulae

Integrationem. Dum autem adhibitas methodos quilibet experiatur, et invicem conferat, meam brevitatem, atque nitore aliis praecellere existimaturum confido. Ceteris enim omisissis, quamplurimum sagacitatis, industriae, ac, propemodum dixeris, formae Integralis investigandae praenotionem aliunde

adquisitam exigit gradus accessio facta ab $y = (m^{\circ} + o.x)^{\frac{1}{o}}$ ad $y^x =$

$(m^{\circ} + o.x)^{\frac{1}{o.x}}$, quem transitum nulla Calculi opportunitate aut necessitate ducere paratum, et divinationi pene proximum placuisse Auctori obstupesco, quippe quum iisdem insistens principiis optatam Aequationem Exponentialem ab prius posita Formula $x = \frac{y^{\circ}}{o} + A$ statim deducere potuisset. Quod

dictum velim non ut eximium Scriptorem vel minimum reprehendam, prouti neque alios, a quorum placitis non semel superius discessi. Hos etenim quamquimaxime colo, eorumque doctrina, et inventis nedum profecisse, sed quantulacunque sint haecenus explicata debere me omnia fateor peringeniosis illorum elucubrationibus. Saepius fortunae est ab aliis reperta perpolire, ac reddere simpliciora: Apollineae siquidem mentes inventionis aestu fervidae viam quasi sternunt minus in sublimia actis Matheseos cultoribus, ut sese in iis perficiendis exerceant.

404. Ista dum scribo, nec paucis principibus in re Mathematica Viris, nec pene innumeris Elementa Doctrinae non praetergressis utilia censeo futura. Multo minus dicare operam laboremque fuit animus quibus, in hac praesertim temporum et pseudo-doctorum iniquitate, vel ignobile verbum, immo et comma quiddam omisissum, in summo quoque ac gravissimo Veritatis studio displiceat. Ordinem, quem poteram, meliorem curae habui; at quaedam incompta me invito, nec satis concinne disposita Exercitationis huiusce nunc lento, nunc celeri cursu, ac per temporis, locique intervalla promotae, mentis itinerumve perturbationibus circumseptam originem redolebunt.

Gloria

Gloria enim potissimum mihi fuit, semperque erit gratissima unici BENEFICENTISSIMI DOMINI iussibus obsequium praestare, obedientiamque quam maximam. Tanta porro in Calculorum copia, ac prolixitate errores fortasse manent nonnulli, quos facile castigaturum quemque mihi polliceor, Geometrarum dumtaxat iudicium expectans de Commentarii totius fundamentis, de praecipuarum Methodorum novitate atque usu, non de singulis perlevioribus huc illuc, et aliquando carptim dissertis, neque hoc priore molimine optimum in systema coactis. Mendis idcirco, si forte aliqua irrepserint, nunc in calce Operis libenti animo non iniquos saltem Iudices opto: mihi enim nefas tam longe meritis impari, excusationis causa, naevorum exempla praetexere, quos etiam Europae Ingeniorum miracula Galilaeus, Cartesius, Leibnitius, Neutonus, semel aut bis Matthesi in ipsa dormitantium ad instar susque deque habuerunt. Quaenam autem heic Nomina dixi Palladi, Musisque omnibus sacra! *Ignescit erat illis vigor, et coelestis origo!*



TABULA OPERIS SYNOPTICA

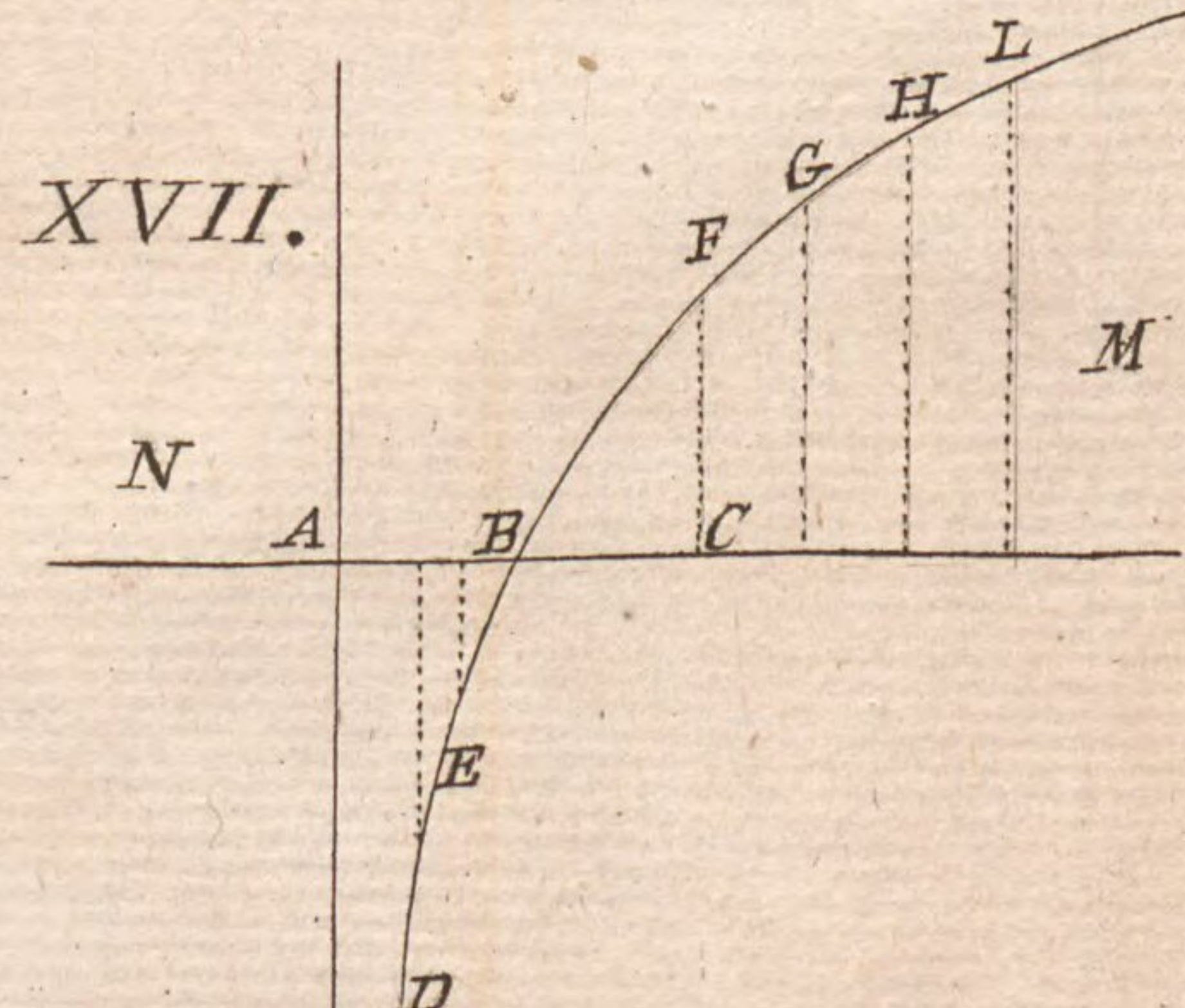
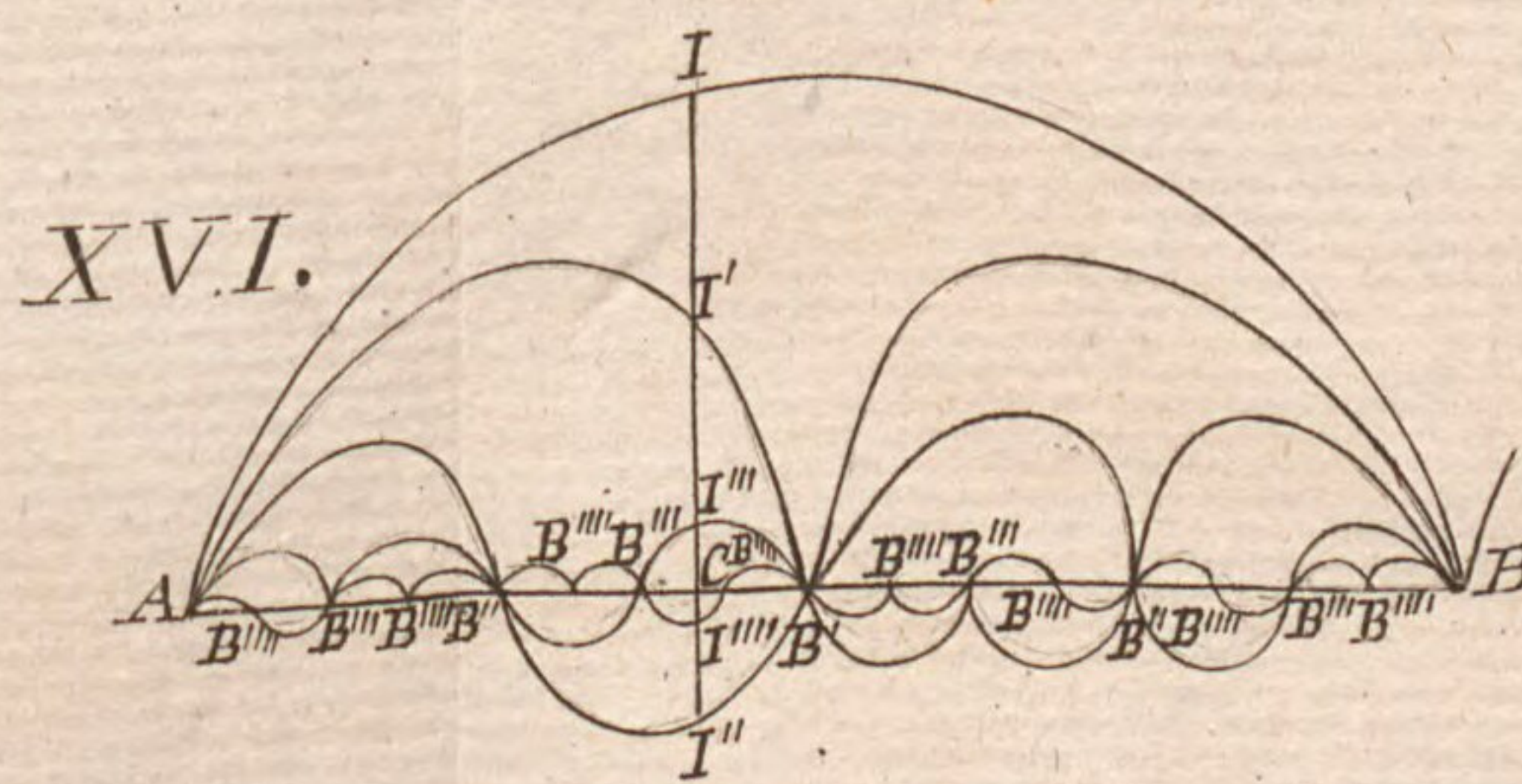
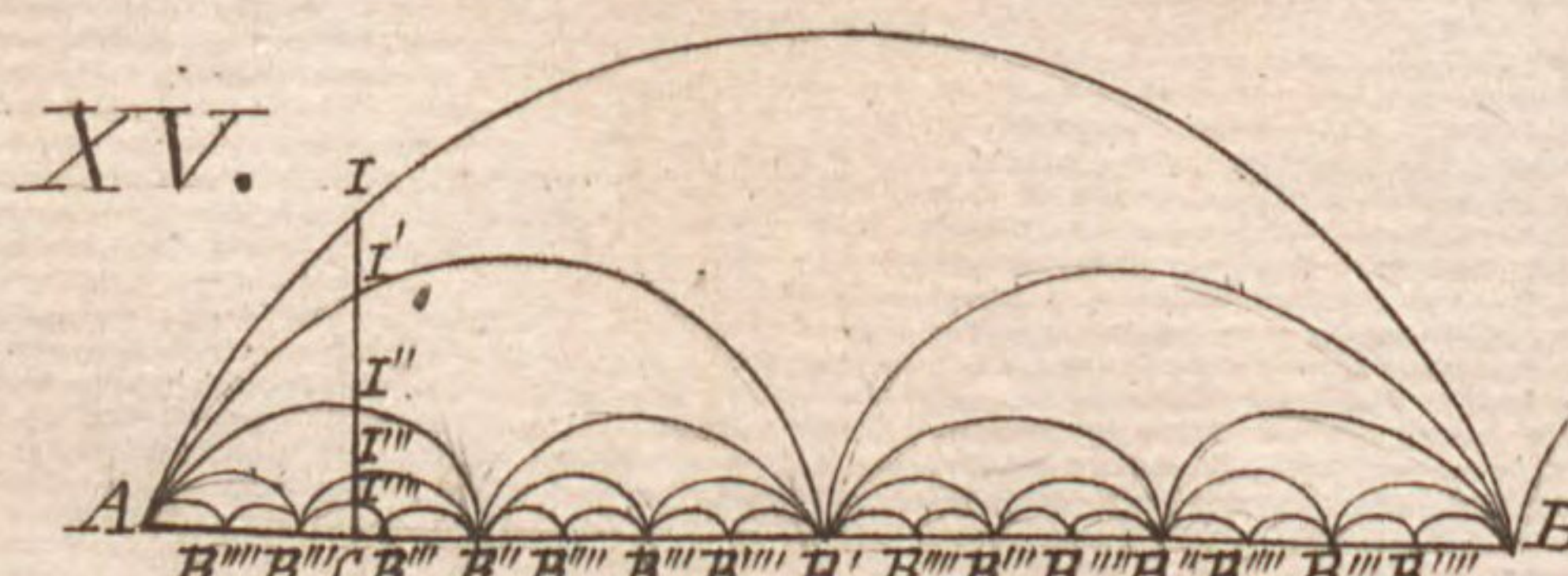
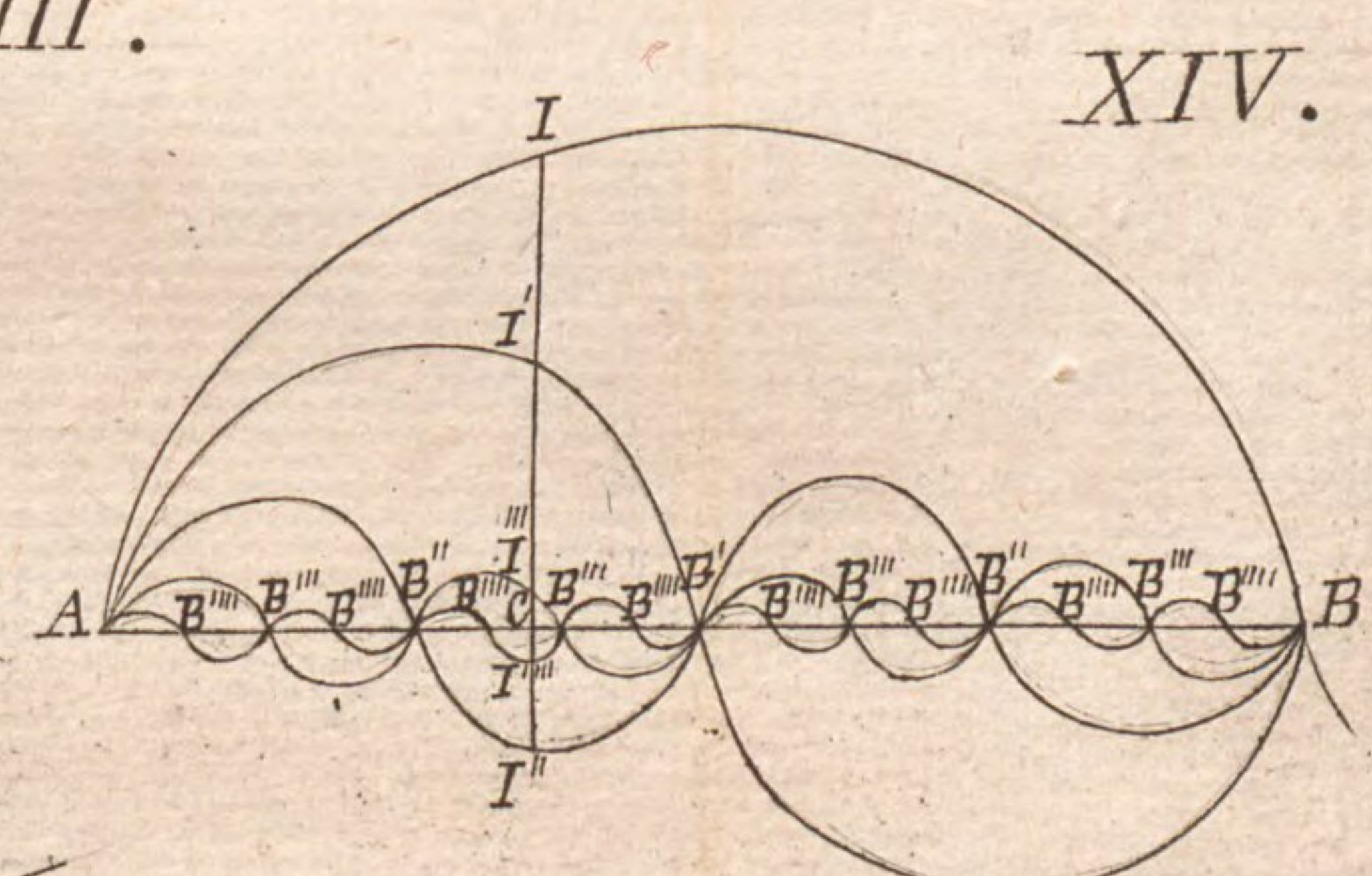
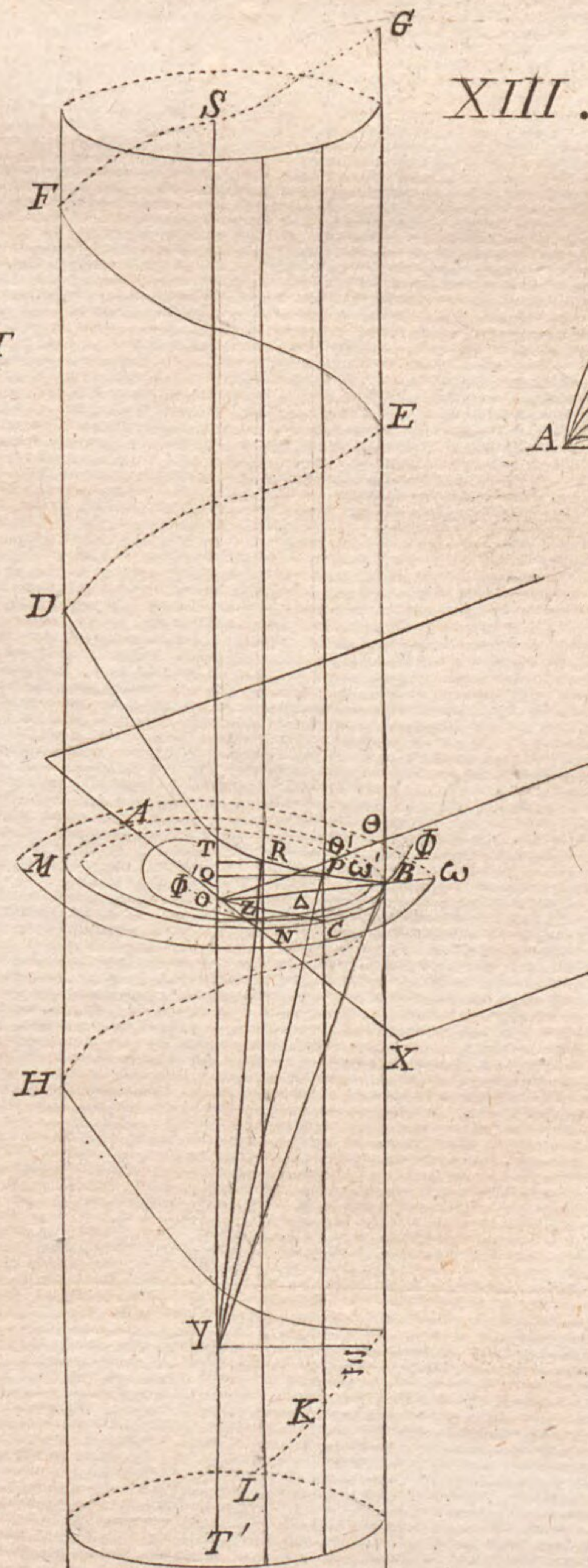
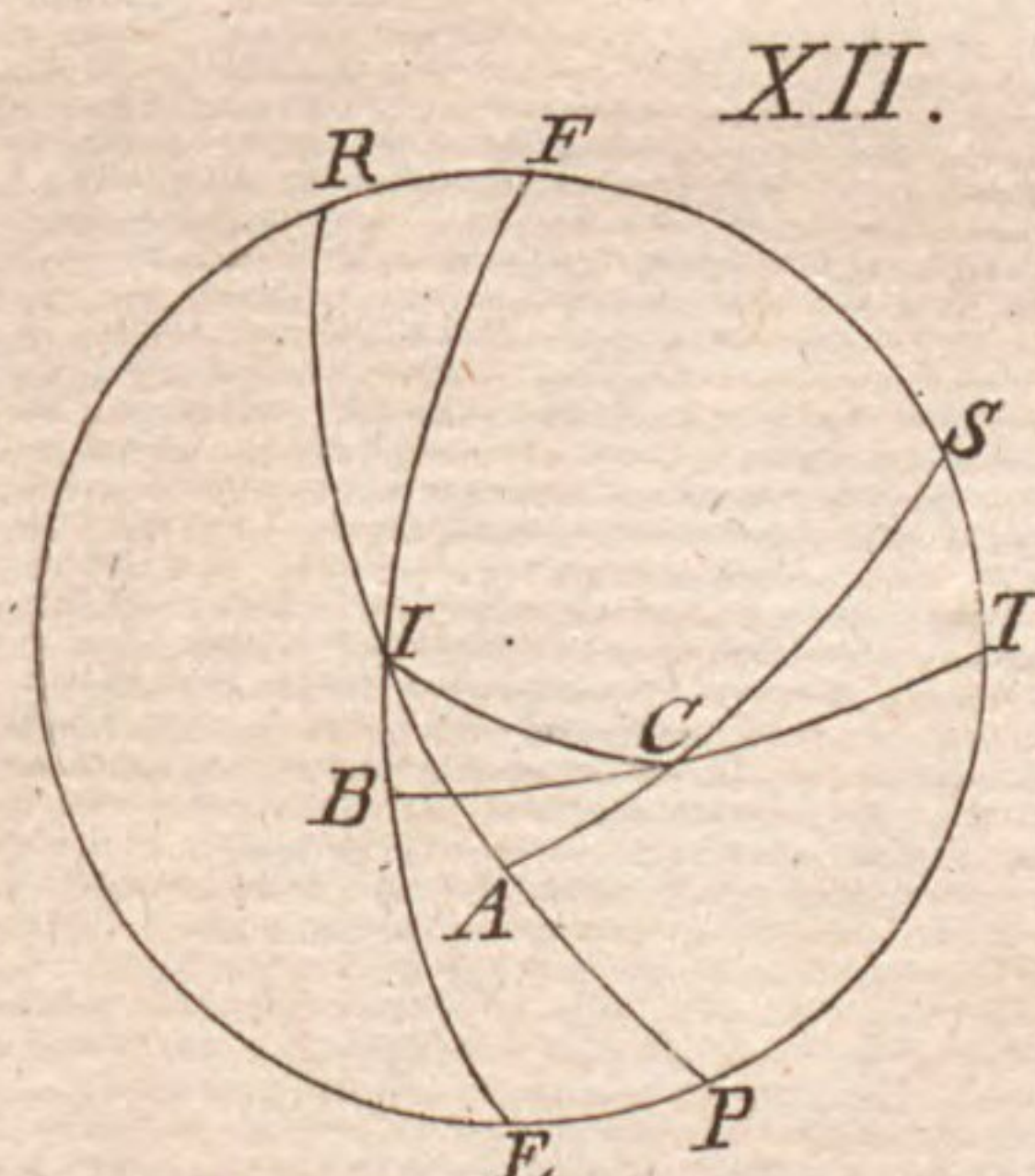
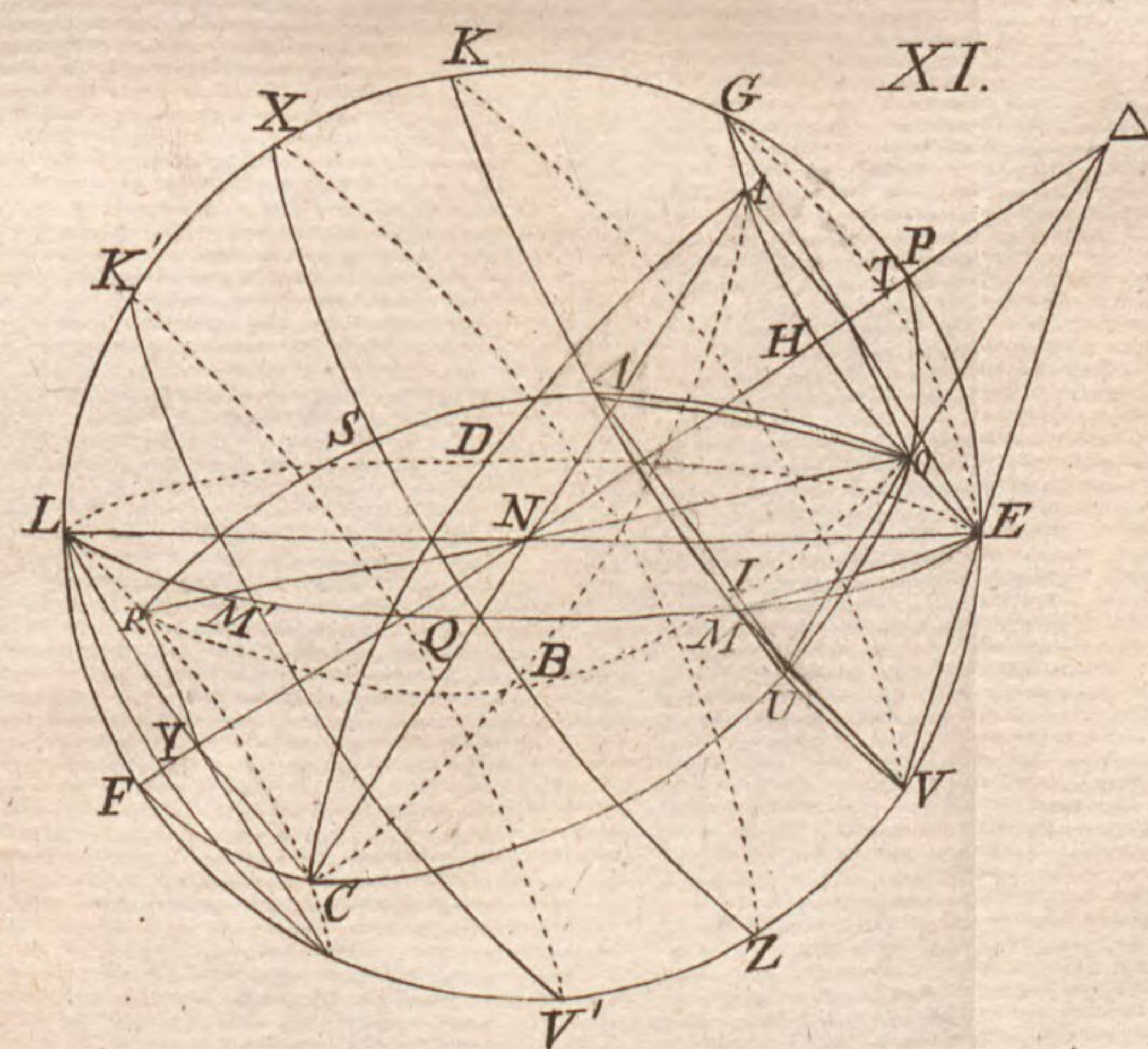
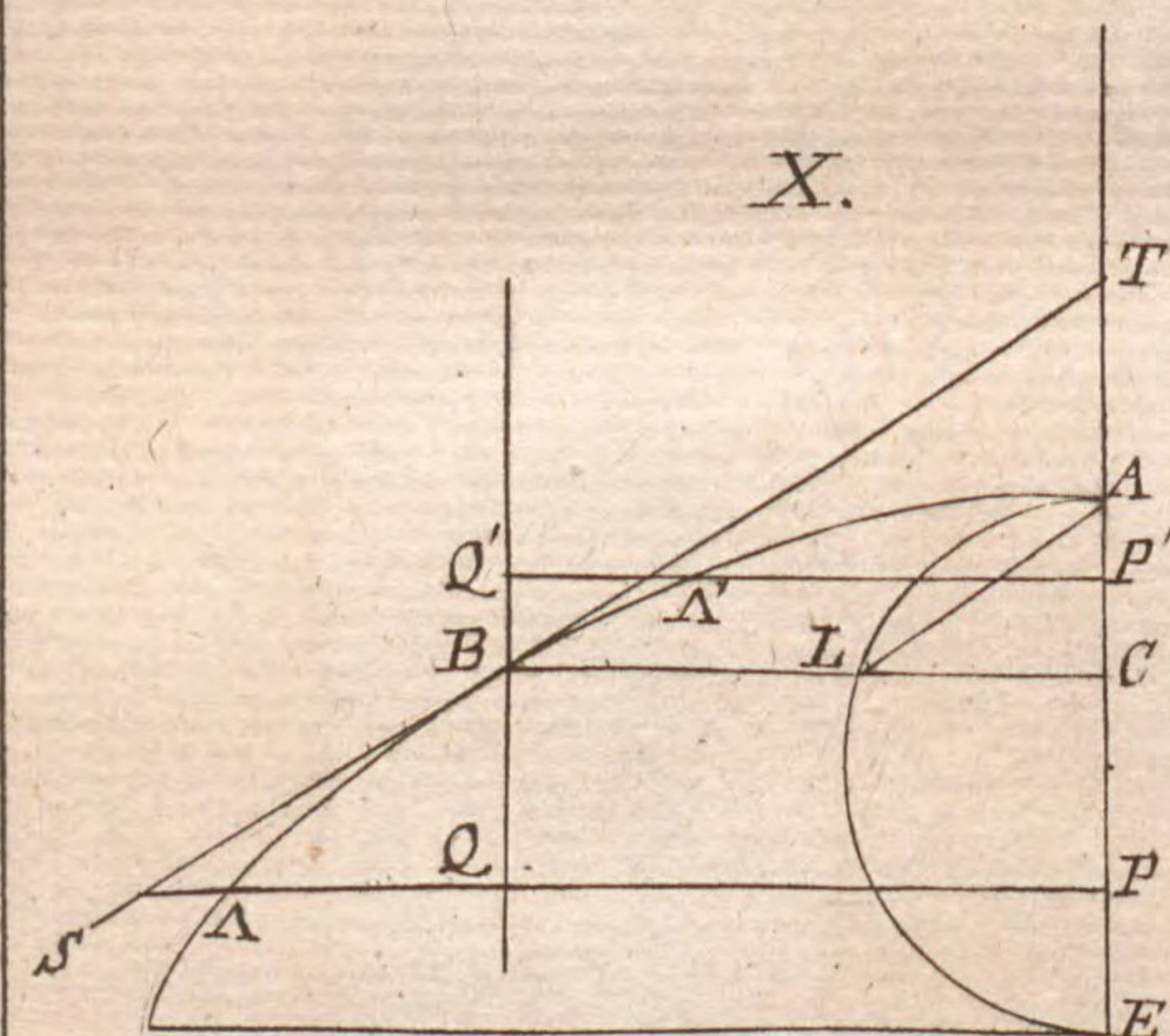
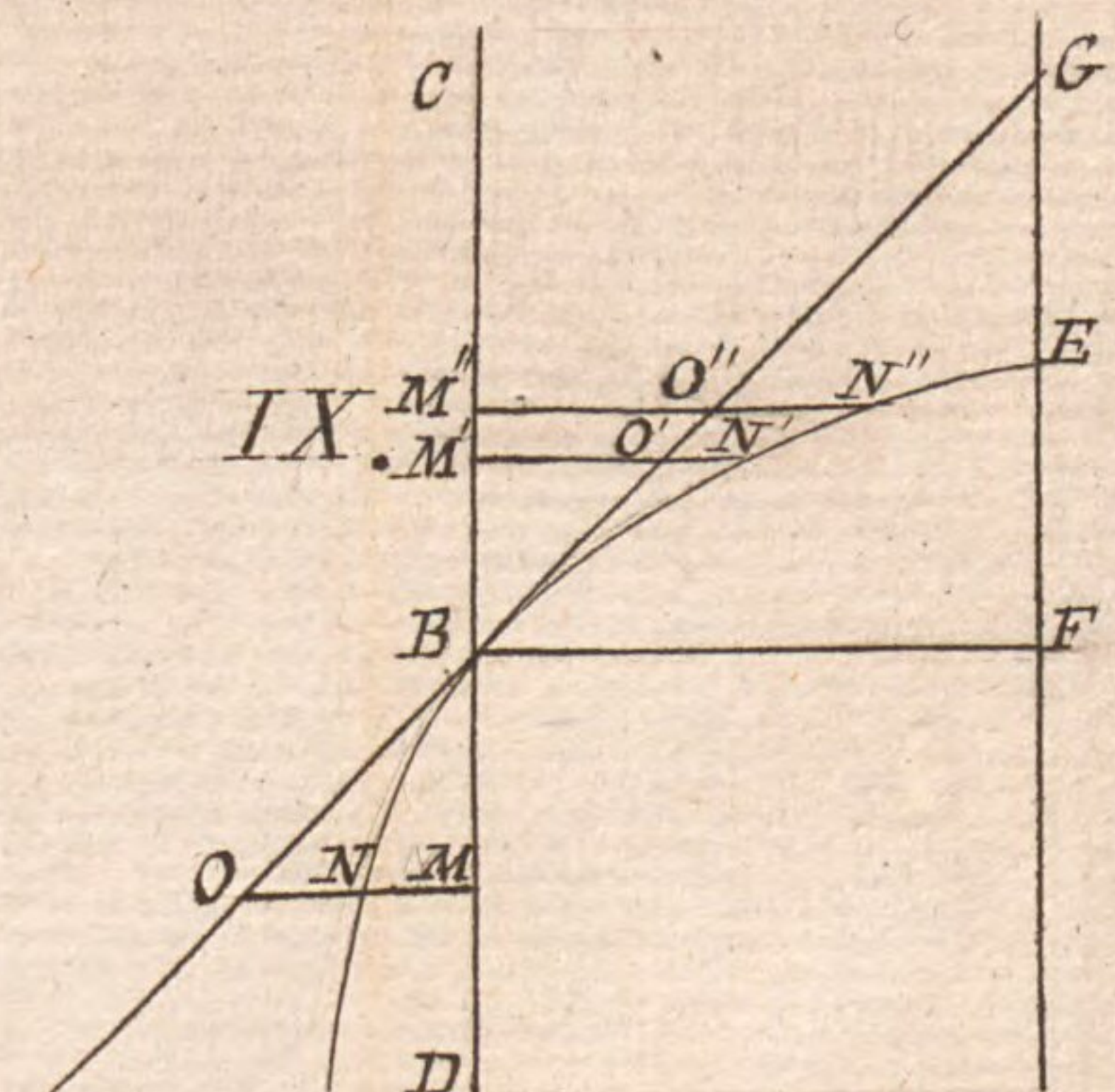
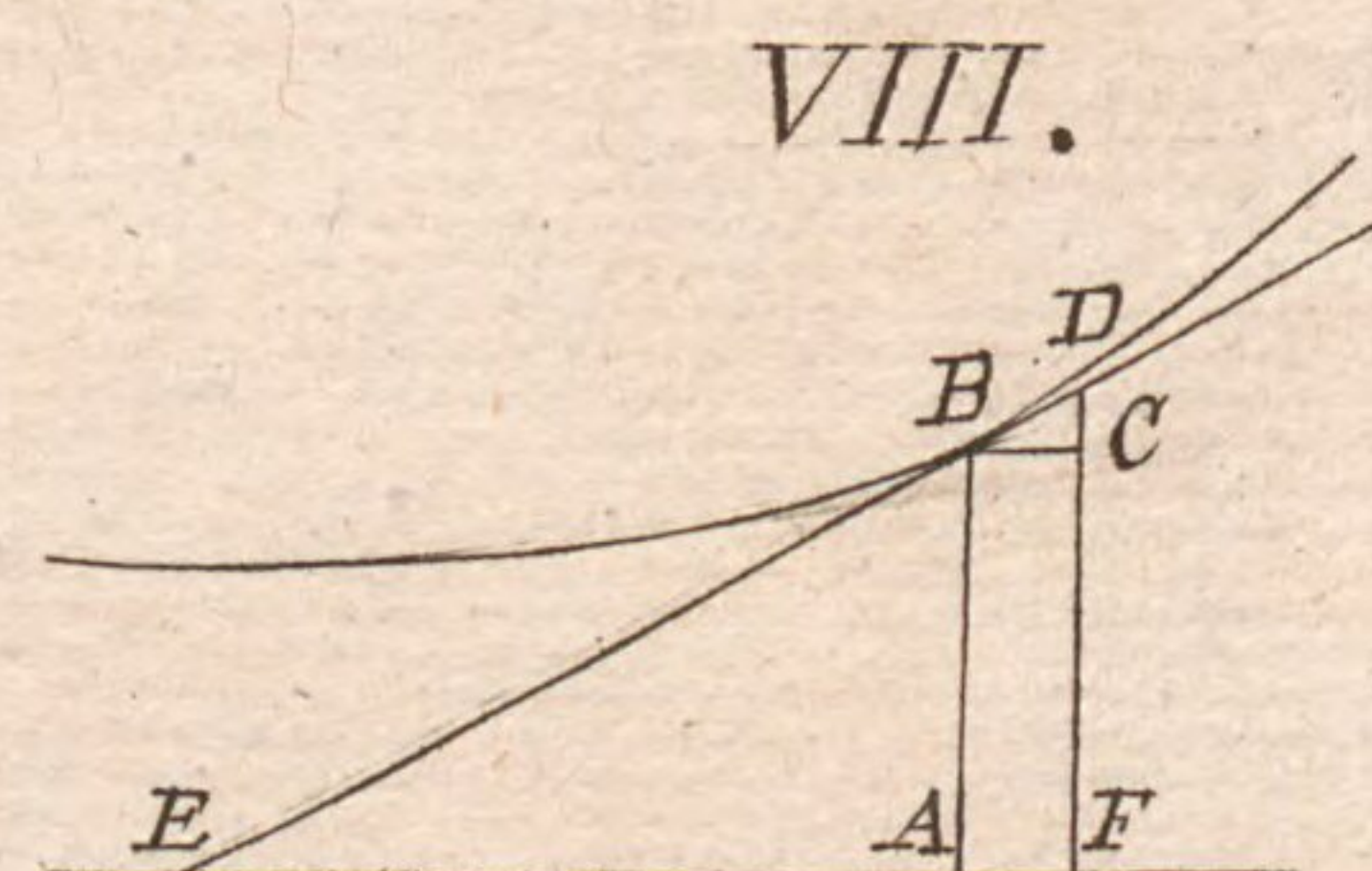
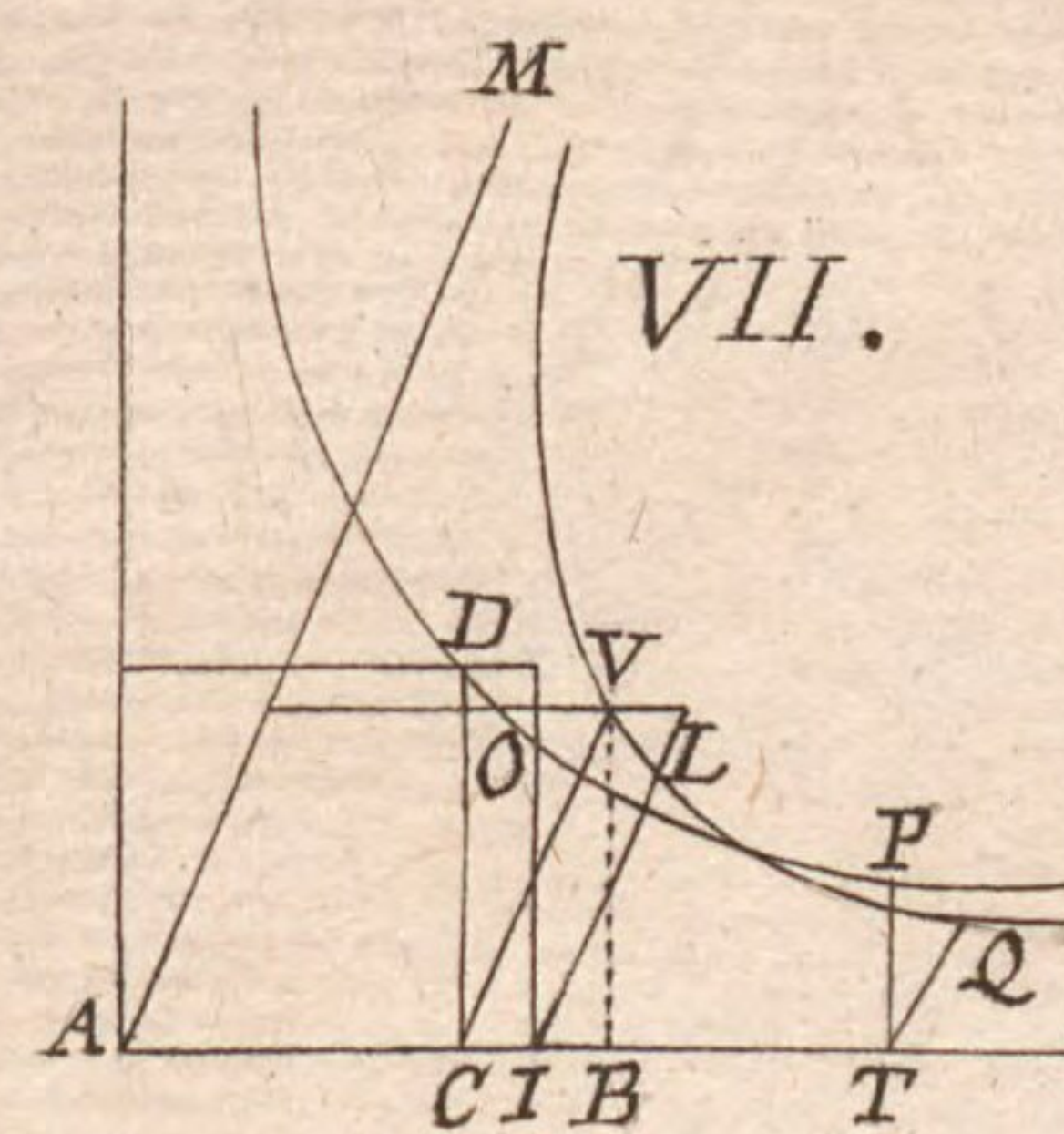
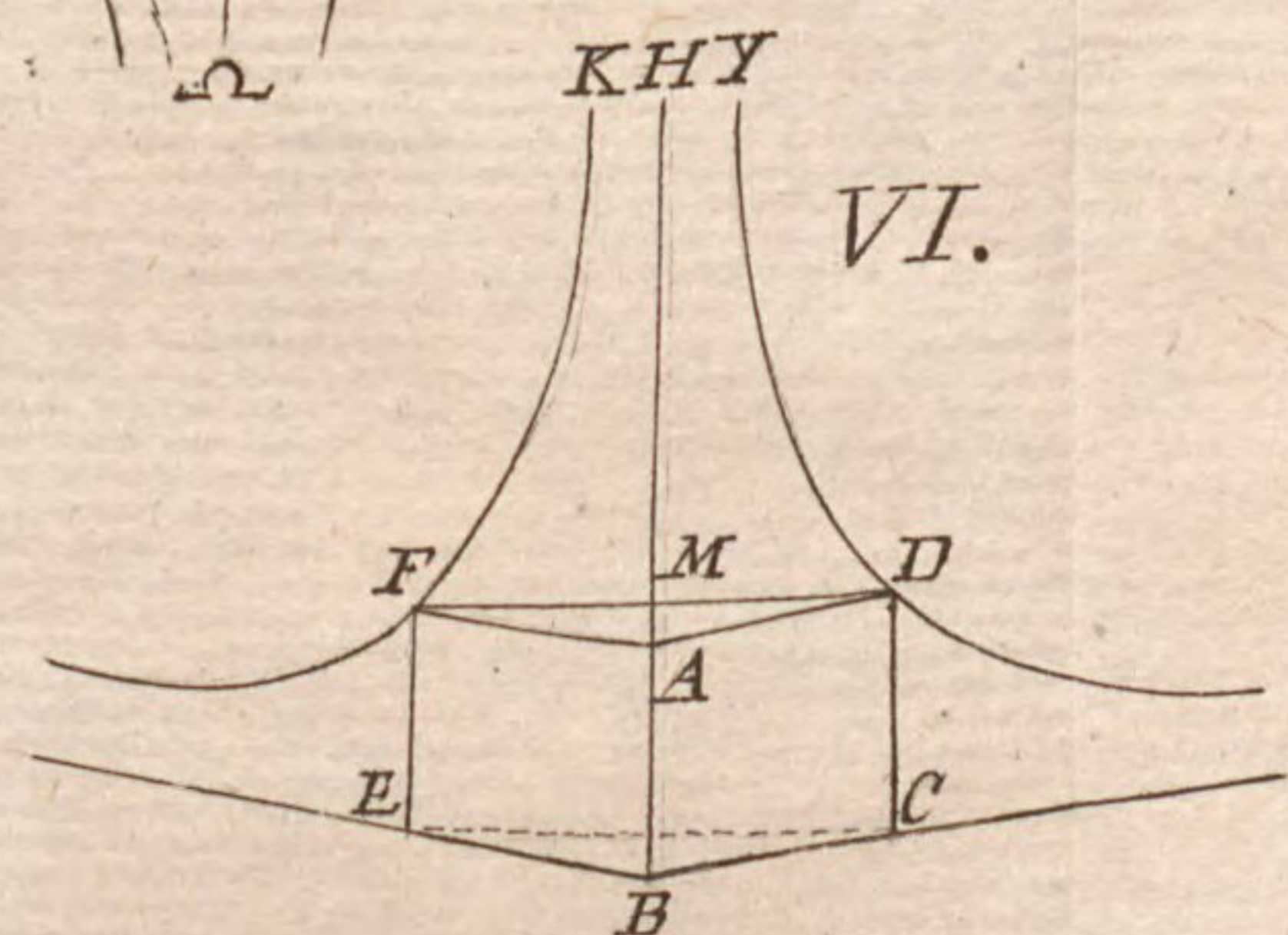
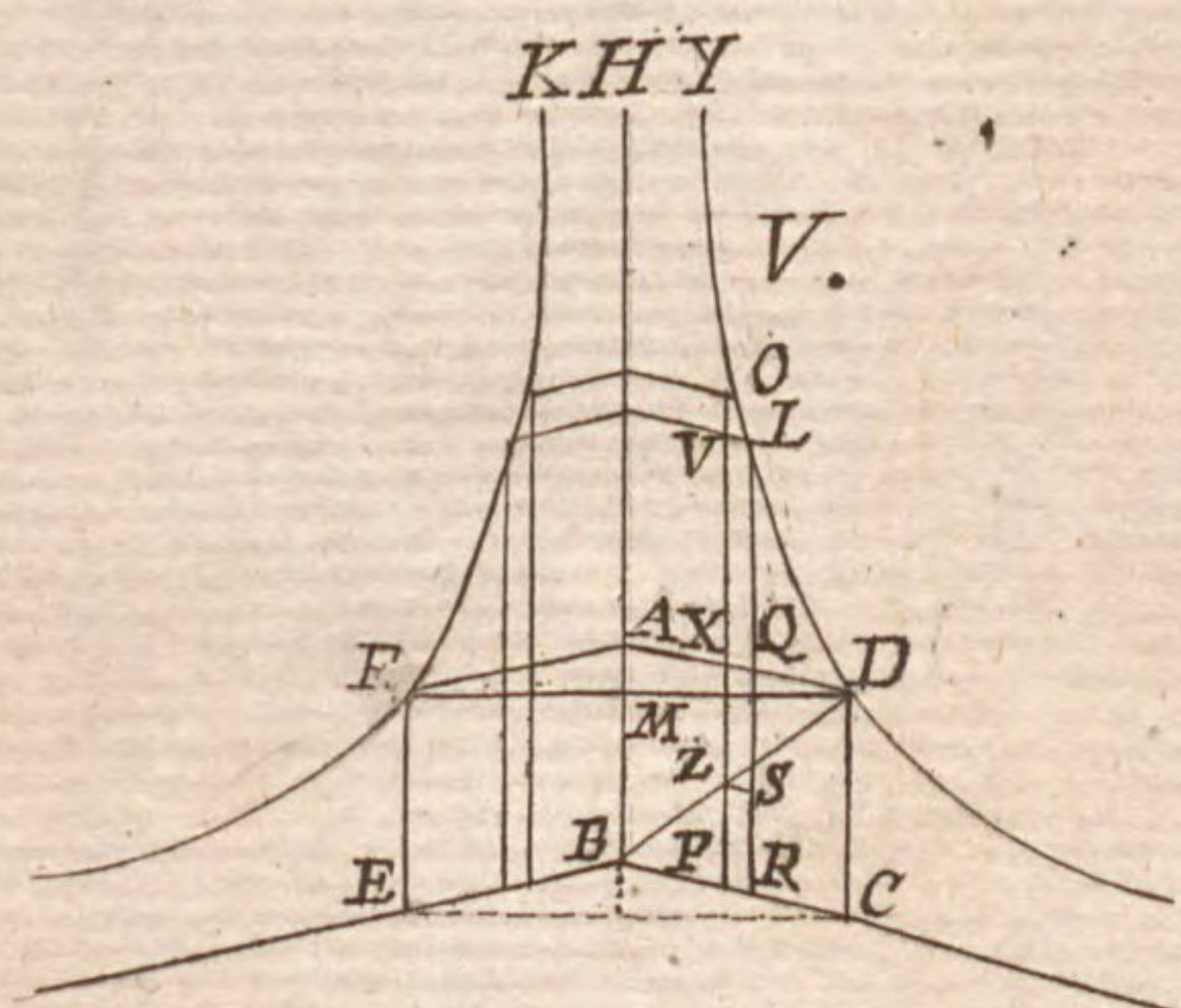
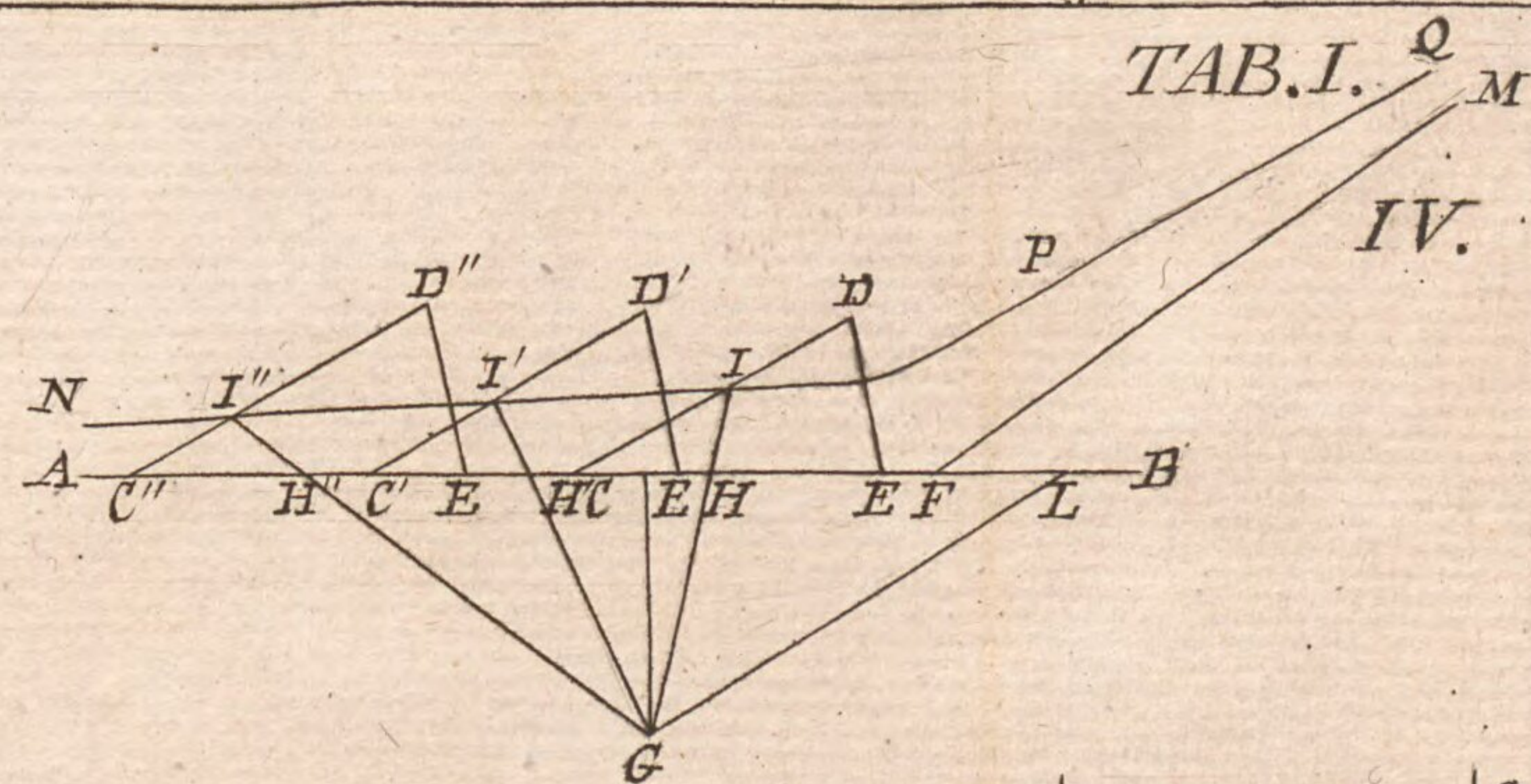
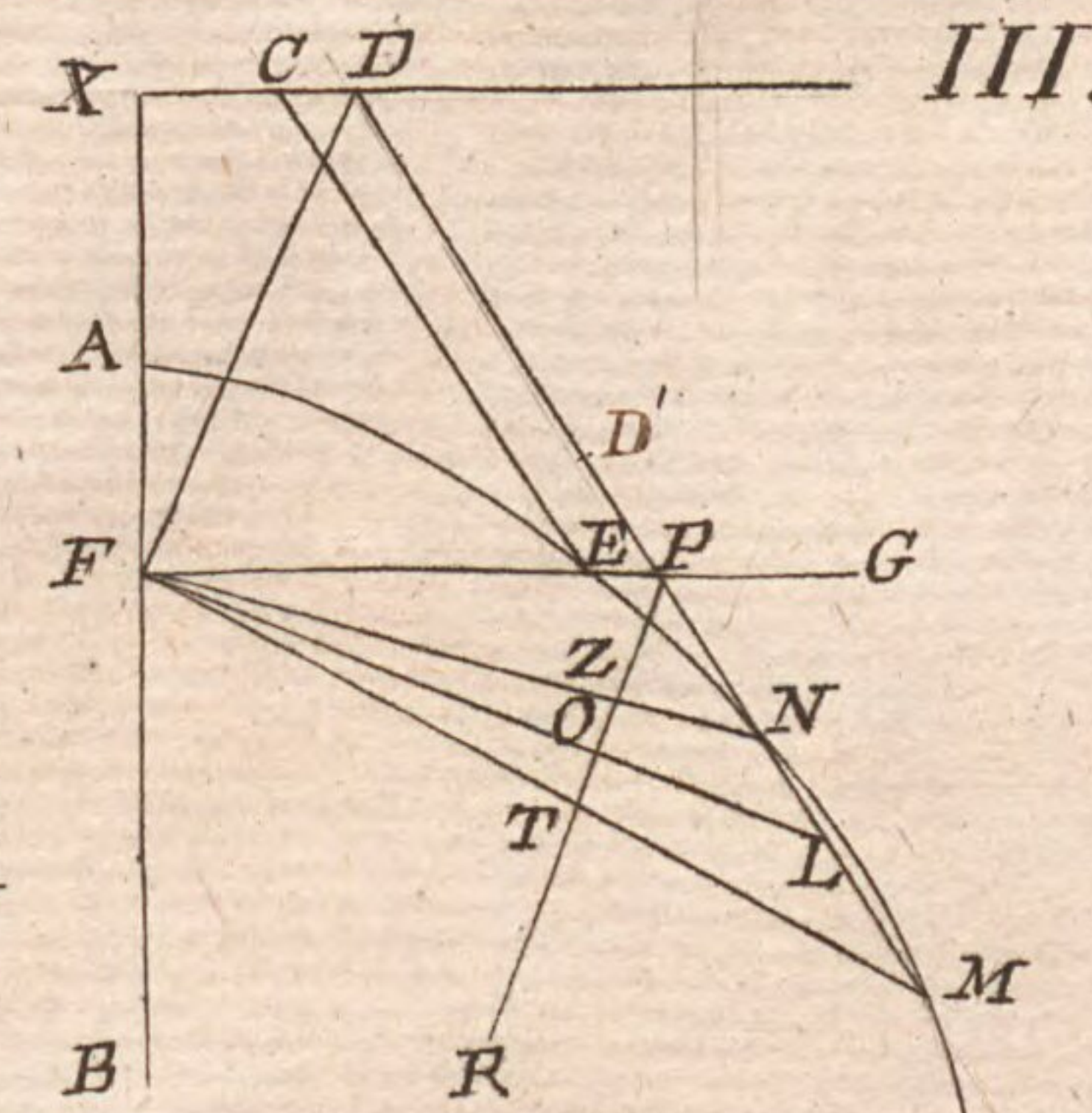
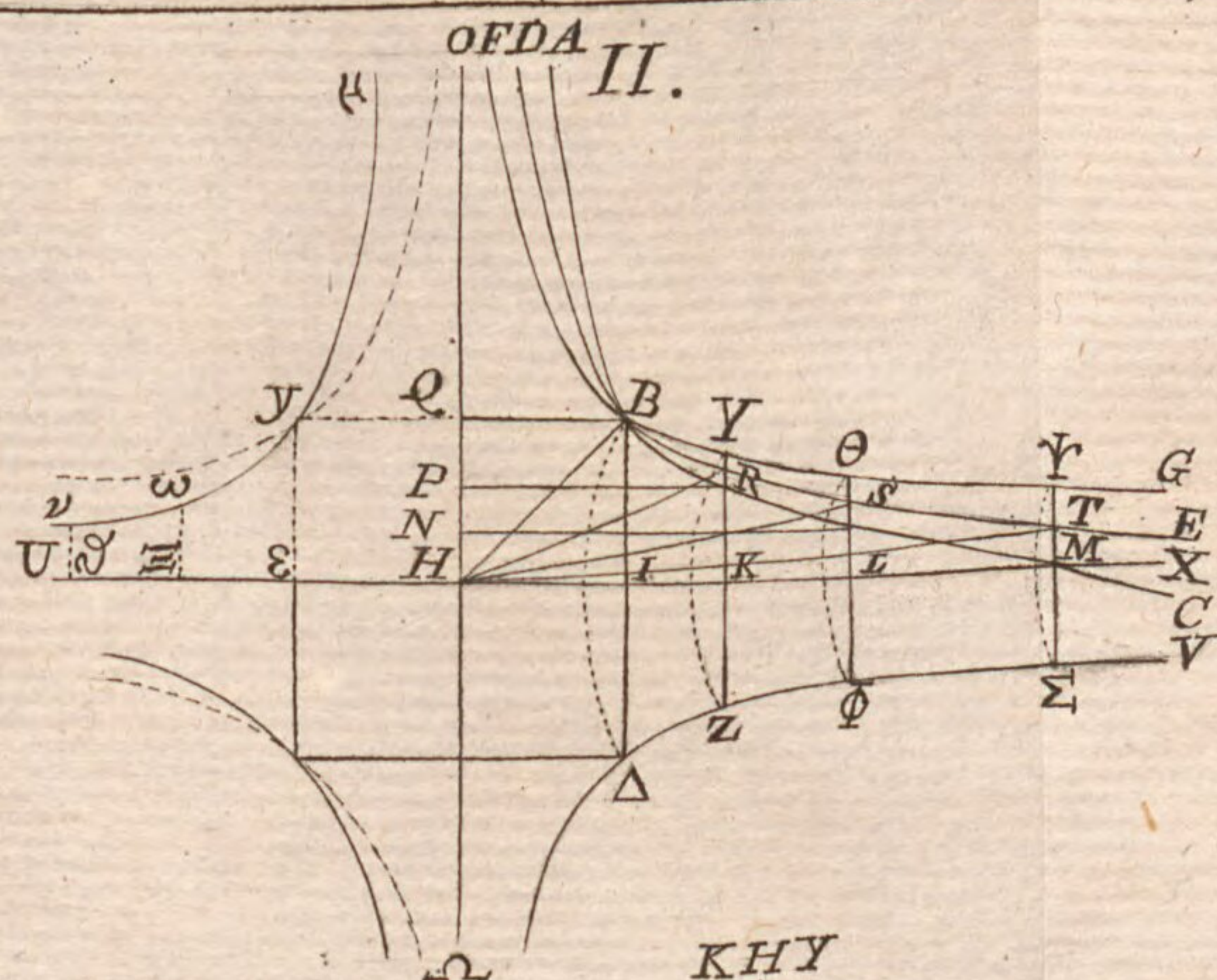
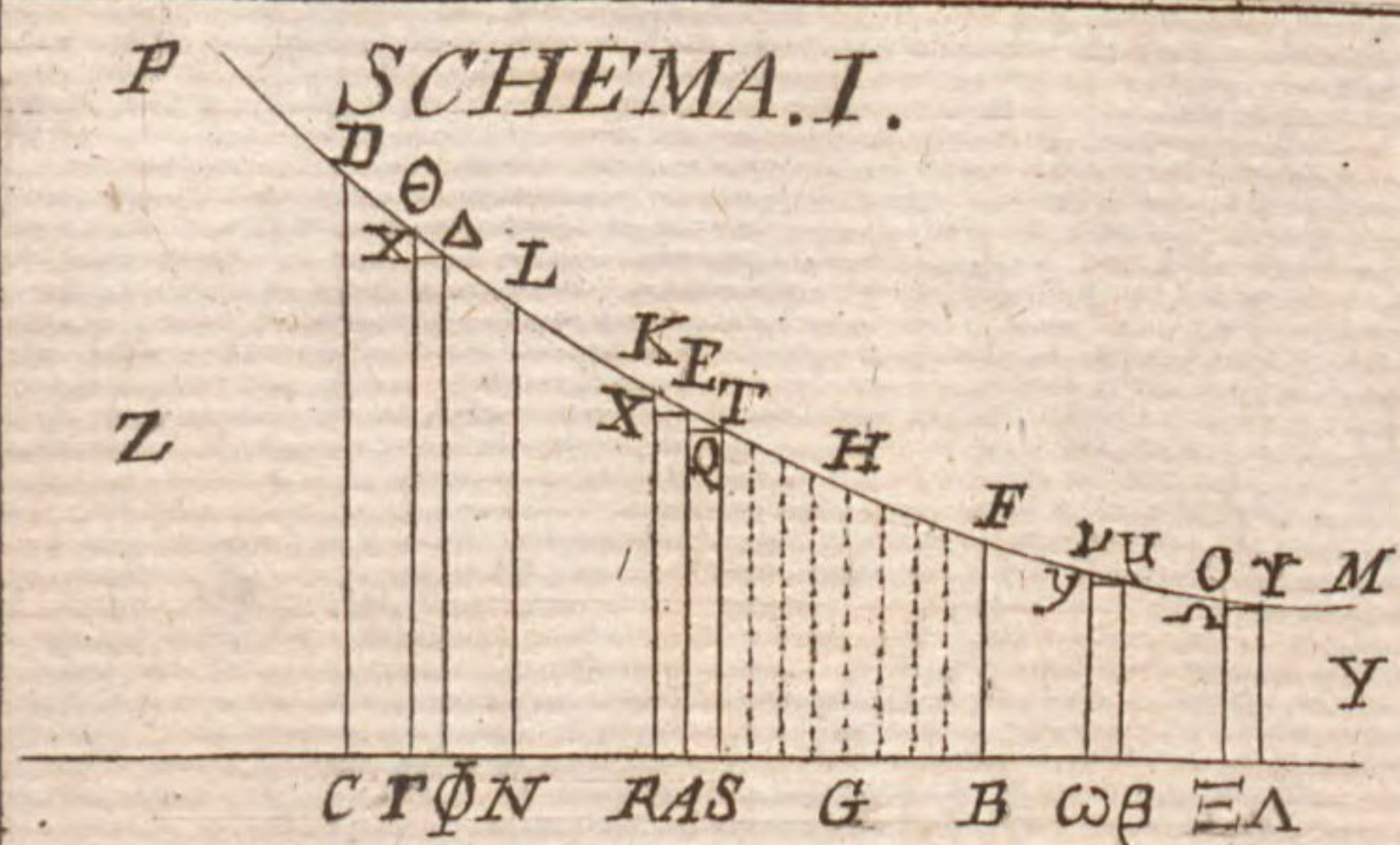
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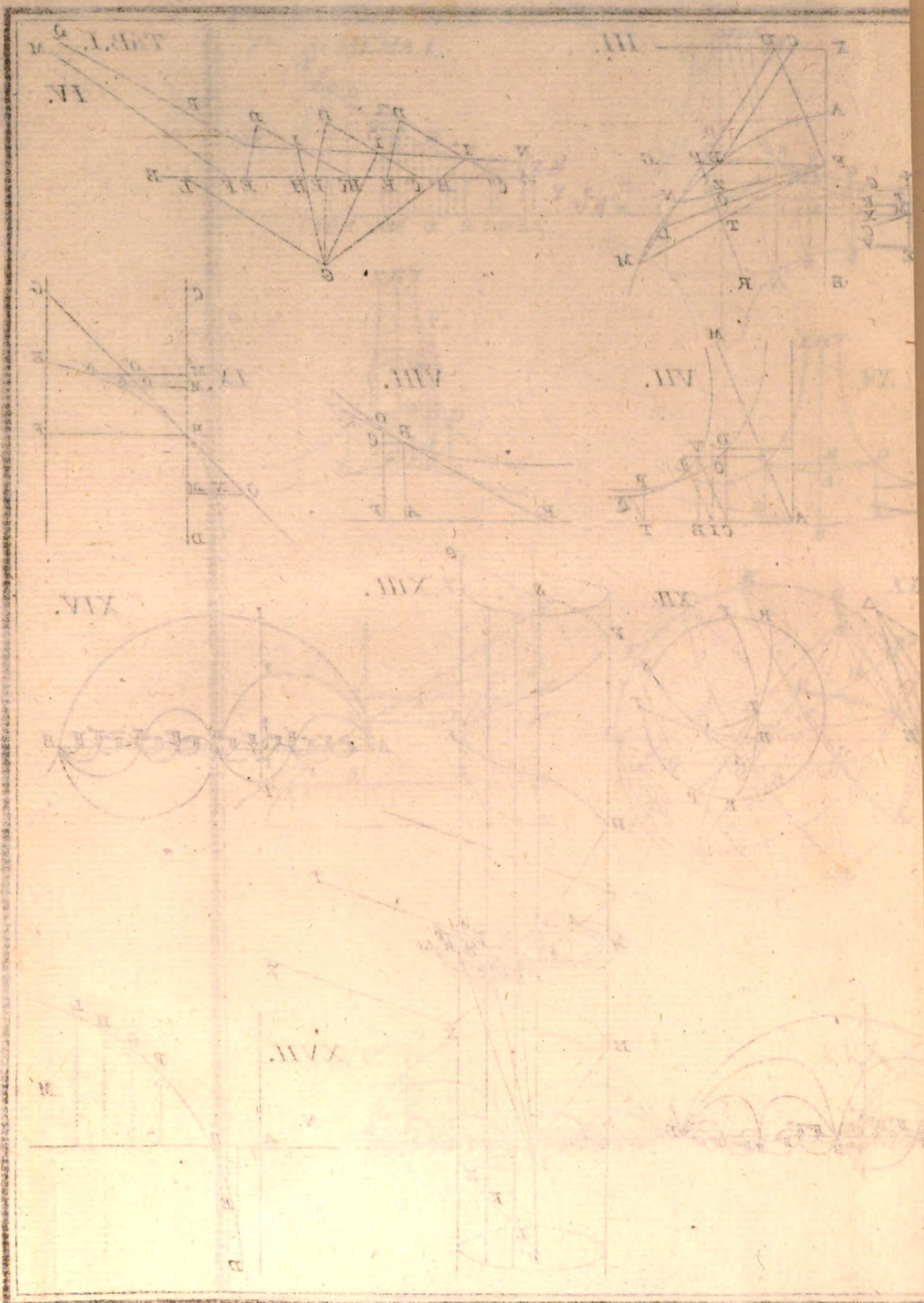
 Nalyfis Argumentorum.	Pag ^a . IX ^a .
Prolegomena Historico-Mathematica.	a Pag ^a . XI ^a . ad LXVI ^{am} .
CAPUT I ^{um} . De Binomio Neutroniano.	Pag ^a . 1.
II ^{um} . De Exponentialibus.	40.
III ^{um} . De Logarithmis.	101.
IV ^{um} . Communia Exponentialium et Logarithmorum.	164.
V ^{um} . De Sublimi Trigonometria.	192.
VI ^{um} . De Quadratura Circuli et Hyperbolae.	336.
VII ^{um} . De Radicibus Unitatis.	398.
VIII ^{um} . Applicatio Calculi Infinitesimalis.	493.
IX ^{um} . De Logarithmis Negativorum.	566.
X ^{um} . Paradoxon Calculi Summatorii.	a Pag ^a . 601. ad 611 ^{am} .

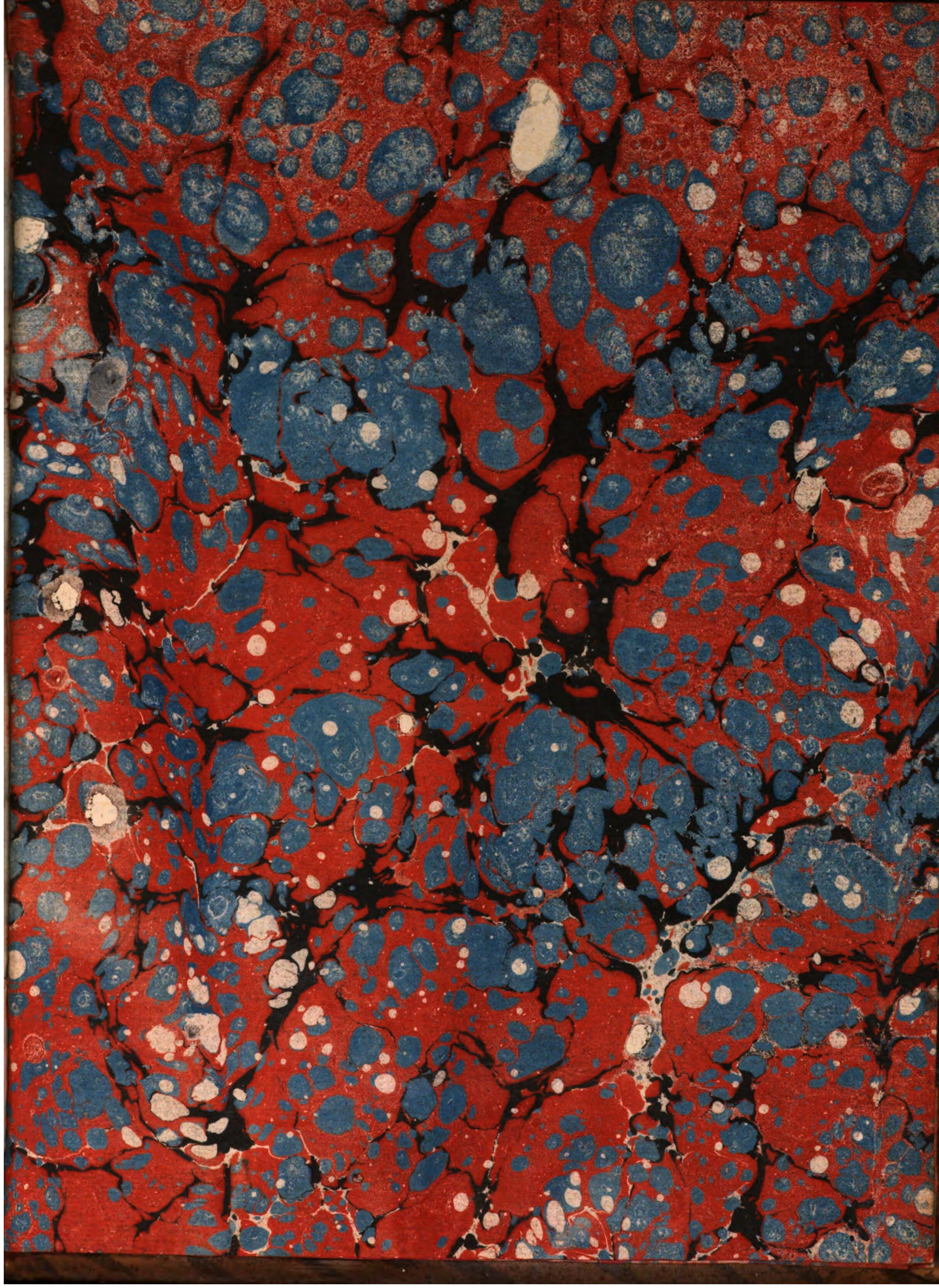
O B S E R V A T I O.

TYporum menda aut nulla exstant, vel, nisi oculos fugerint in censu perspicaciores, haudquaquam eos, qui analytice norint, perturbant; sed latebunt fortasse in vocum nonnullarum caesura, diphtongis, aliisve adeo levissima, ut grammatices, rectae interpunctionis, atque orthographiae gnarus Lector vix habeat aliquid castigandum.

Quae autem vocabula, ac dicendi, scribendive rationes promiscui usus videntur penes optimos Romani, Graecique Idiomatum cultores, veluti *Typographeum Typographia*, *Auctor Author*, *Prelum Praelum*, *Tyro Tiro*, *Asymptoti Asymptotae*, *Littera Litera*, *Autographus Authographus*, *Proemium Prooemium*, *Boiobemia Bobemia*, *Percrebruit Percrebuit*, *Priori Priore*, *Athenaeum Atheneum*, *Quaternarius Quadernarius* - - - - - &c: &c: &c: mihi vitio non dandum si varie adhibuerim, Modumque praeterea elegantissimum Potentialem *Velim*, *Sileam*, *Praebeam* &c:, qui Coniunctivis pro Indicativo tantopere delectatur, non semel adamaverim.









Ferroni, Pietro

Magnitudinum exponentialium Logarithmorum ... theoria

Florentia 1782

4 Math.p. 118

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